数理解析研究所講究録
第 1821 巻 2013 年 136-142

Duality without constraint qualification in nonsmooth multiobjective programming

Do Sang Kim and Kwan Deok Bae

Department of Applied Mathematics
Pukyong National University,
Busan 608-737, Republic of Korea
E-mail : dskim@pknu.ac.kr

1 Introduction and Preliminaries

Nonsmooth phenomena in mathematics and optimization problem occurs naturally and frequently. The Clarke subdifferential method has been proved to be a powerful tool in many nonsmooth optimization problems([2], [3]). The field of multiobjective programming, has grown remarkably in different directional in the setting of optimality conditions and duality theory since 1980s.

Duality results of e.g. Wolfe [12] and Schechter [10] were shown to be differentiable and nondifferentiable cases of the duality formulation. Some duality results of Mond and Weir [9] with generalized convex were introduced as special cases of the previous dual formulations. In the nondifferentiable case, Kim and Bae [5] formulated the dual problem and established duality theorems for nondifferentiable multiobjective programs involving the support functions of a compact convex sets and linear functions.

Weir and Mond [11] have given dual problems for the convex multiobjective programming problem and established duality without a constraint qualification. And Egudo et al. [4] have formulated dual problems for differentiable multiobjective programming problems where a constraint qualification is not assumed.

Recently, Kim and Schaible [7] introduced nonsmooth multiobjective programming problems involving locally Lipschitz functions. They obtained sufficient optimality conditions and established duality relation. Based on this results, Kim and Lee [6] gave two types of optimality conditions by using generalized convexity and certain regularity conditions and proved duality theorems.

Very recently, Nobakhtian [8] present necessary and sufficient conditions and derived duality theorems for a class of nonsmooth multiobjective programming problems without constraint qualification. Our aim of this paper has two viewpoints. One is to formulate mathematical models such as primal and dual problems. Another is to establish duality theorems.

In this paper, we formulate the nonsmooth multiobjective programming involving locally Lipschitz functions and support functions. In section 2, we formulate Wolfe type and Mond-Weir type dual problem and establish weak and strong duality theorems under suitable generalized convexity conditions. Finally, we give special cases of our duality results.

We consider the following multiobjective nonsmooth programming problem,

$$\begin{array}{lll} \text{(MP)} & \text{Minimize} & f(x)+s(x|D) \\ & \text{subject to} & g(x) \leqq 0, \\ & & x \in C, \end{array}$$

where C is a convex set and $D_i,\ i=1,\cdots,p$ is a compact convex sets of $\mathbb{R}^n$. $f_i,\ g_j,\ i=1,\cdots,p,\ j=1,\cdots,m$ are real valued locally Lipschitz functions defined on C. The index sets are $P=\{1,2,\cdots,p\}$, $M=\{1,2,\cdots,m\}$. We denote the feasible set $\{x\in C|g_j(x)\leqq 0,\ j=1,\cdots,m\}$ by F. Let $I(x^*)=\{j\in M|g_j(x^*)=0\}$ denote the index set of active constraints at x^*.
The minimal index set of active constraints for F is denoted by

$$I^{=}=\{j\in M|x\in F\rightarrow g_j(x)=0\}.$$

We also denote

$$I^{<}(x^*) = I(x^*) - I^{=} = \{j \in I(x^*) | x_i \in F \text{ such that } g_j(x) < 0\}.$$

For a fixed $r \in P$ and $x^* \in \mathbb{R}^n$, we denote

$$M^r = M \setminus \{r\},$$

$$F^r(x^*) = \{x | f_i(x) + s(x|D_i) \leqq f_i(x^*) + s(x^*|D_i),\ i \in M^r\},$$

$$M^r(x^*) = \{i \in M^r | f_i(x) + s(x|D_i) = f_i(x^*) + s(x^*|D_i), \forall x \in F^r(x^*)\}.$$

We denote $C^* = \{u \in \mathbb{R}^n | u^T x \geqq 0, \forall x \in C\}$ for the polar set of an arbitrary set $C \in \mathbb{R}^n$.

For a nonempty subset C of $\mathbb{R}^n$, we denote by $co(C), cone(C)$, and C^* the convex hull of C, the cone generated by C, and the dual cone of C, respectively.

Further, $N_C(x^*)$ denotes the normal cone to C at x^* defined by

$$N_C(x^*) = \{d \in \mathbb{R}^n | < d, x - x^* > \leq 0,\ \forall\ x \in C\},$$

clearly, $(C - x^*)^* = -N_C(x^*)$.

Definition 1.1 *A feasible solution x^* for (MP) is efficient for (MP) if and only if there is no other feasible x for (MP) such that*

$$f_{i_0}(x) + s(x|D_{i_0}) < f_{i_0}(x^*) + s(x^*|D_{i_0}) \text{ for some } i_0 \in P,$$

$$f_i(x) + s(x|D_i) \leqq f_i(x^*) + s(x^*|D_i)\ \forall\ i \in P.$$

Definition 1.2 *Let D be a compact convex set in $\mathbb{R}^n$. The support function $s(x|D)$ is defined by*

$$s(x|D) := max\{x^T y : y \in D\}$$

The support function $s(x|D)$, being convex and everywhere finite, has a subdifferential, that is, there exists z such that

$$s(y|D) \geq s(x|D) + z^T(y - x) \text{ for all } y \in D.$$

Equivalently,

$$z^T x = s(x|D).$$

The subdifferential of $s(x|D)$ is given by

$$\partial s(x|D) := \{z \in D : z^T x = s(x|D)\}.$$

Definition 1.3 *[2, 3] Let X be an open subset of $\mathbb{R}^n$. The generalized Clarke directional derivative of a locally Lipschitz function f at x in the direction d is defined by*

$$f^c(x; d) = limsup_{y \to x, t \to 0+} \frac{f(y + td) - f(y)}{t}.$$

The Clarke generalized subgradient of a locally Lipschitz function f at x is defined by

$$\partial_c f(x) := \{\xi \in \mathbb{R}^n | f^c(x; d) \geq < \xi, d >,\ \forall d \in \mathbb{R}^n\}.$$

Lemma 1.1 *Let f be a locally Lipschitz function and $x \in domf$. Then for all $d \in \mathbb{R}^n$,*

$$f^c(x; d) = max\{< \xi, d > | \xi \in \partial_c f(x)\}$$

and $\partial_c f(x)$ is a nonempty, convex and compact set.

Definition 1.4 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function. Then*
(i) it is said to be generalized convex at x if for any y

$$f(y) - f(x) \geq < \xi, y - x >,\ \forall\ \xi \in \partial_c f(x),$$

(ii) it is said to be generalized quasiconvex at x if for any y such that $f(y) \leqq f(x)$,

$$< \xi, y - x > \leqq 0,\ \forall\ \xi \in \partial_c f(x),$$

(iii) it is said to be generalized strictly quasiconvex at x if for any y such that $f(y) \leqq f(x)$,

$$< \xi, y - x > < 0,\ \forall\ \xi \in \partial_c f(x).$$

2 Duality Theorems

We introduce Wolfe type and Mond-Weir type dual programming problems and establish weak and strong duality theorems.

$$
\begin{aligned}
\text{(WD)} \quad & \text{Maximize} \quad f(u) + u^T z + (\mu^T g(u))e \\
& \text{subject to} \quad 0 \in \sum_{i\in P} \lambda_i(\partial_c f_i(u) + z_i) + \sum_{j\in M} \mu_j \partial_c g_j(u) + N_C(u), && (1)\\
& g_j(u) = 0, \ j \in I^=, && (2)\\
& \lambda_i > 0, i = 1, \cdots, p, \ \sum_{i\in P} \lambda_i = 1, && (3)\\
& \mu_j \geqq 0, j = 1, \cdots, m. && (4)
\end{aligned}
$$

Here F_{WD} denotes the set of feasible solutions to (WD) and $g_{I^=}(\cdot)$ for $g_j(\cdot), \ j \in I^=$.

Theorem 2.1 (Weak Duality) *Suppose that $x \in F$ and $(u, z, \lambda, \mu) \in F_{WD}$. If $f_i(\cdot), \ i \in P, \ g_j(\cdot), \ j \in M$, are generalized convex functions at u. Then the following cannot hold:*

$$f_{i_0}(x) + s(x|D_{i_0}) < f_{i_0}(u) + u^T z_{i_0} + \sum_{j=1}^{m} \mu_j g_j(u), \ \textit{for some } i_0 \in P, \tag{5}$$

$$f_i(x) + s(x|D_i) \leqq f_i(u) + u^T z_i + \sum_{j=1}^{m} \mu_j g_j(u), \ \forall \ i \in P. \tag{6}$$

Proof. Let x be feasible solution for (MP) and let (u, z, λ, μ) be feasible solution for (WD). Suppose contrary to the result that (5) and (6) hold. Then

$$\sum_{i\in P} \lambda_i(f_i(x) + s(x|D_i)) < \sum_{i\in P} \lambda_i(f_i(u) + u^T z_i) + \sum_{j=1}^{m} \mu_j g_j(u) \sum_{i\in P} \lambda_i. \tag{7}$$

Since $x^T z_i \leqq s(x|D_i), \ i \in P$ and $\sum_{i\in P} \lambda_i = 1$, (7) yields

$$\sum_{i\in P} \lambda_i(f_i(x) + x^T z_i) < \sum_{i\in P} \lambda_i(f_i(u) + u^T z_i) + \sum_{j=1}^{m} \mu_j g_j(u). \tag{8}$$

By feasibility of (u, z, λ, μ), there exist $\xi_i + z_i \in \partial_c f_i(u) + z_i, \ i \in P, \ \eta_j \in \partial_c g_j(u), \ j \in M$ and $d \in -N_C(u)$ such that

$$\sum_{i\in P} \lambda_i(\xi_i + z_i) + \sum_{j\in M} \mu_j \eta_j = d.$$

Then

$$\begin{aligned}
&\sum_{i\in P}\lambda_i[(f_i(x)+x^Tz_i)-(f_i(u)+u^Tz_i)-\sum_{j=1}^m \mu_j g_j(u)]\\
&=\sum_{i\in P}\lambda_i(f_i(x)+x^Tz_i)-\sum_{i\in P}\lambda_i(f_i(u)+u^Tz_i)-\sum_{j=1}^m \mu_j g_j(u)\\
&\geqq (x-u)^T\sum_{i\in P}\lambda_i(\xi_i+z_i)-\sum_{j=1}^m \mu_j g_j(u)\\
&=-(x-u)^T\sum_{j\in M}\mu_j\eta_j-\sum_{j=1}^m \mu_j g_j(u)+(x-u)^Td\\
&\geqq \sum_{j\in M}\mu_j(g_j(u)-g_j(x))-\sum_{j=1}^m \mu_j g_j(u)\\
&=-\sum_{j\in M}\mu_j g_j(x)\\
&\geqq -\sum_{j\in M}\mu_j g_j(x)\\
&\geqq 0,
\end{aligned}$$

which is a contradiction to (8). □

Theorem 2.2 (Strong Duality) *If x^* is an efficient solution for (MP) and weak duality theorem(Theorem 2.1) holds between (MP) and (WD). $g_j(\cdot)$, $j\in P$, are generalized strictly quasiconvex at x^* and $(x^*)^Tz_i = s(x^*|D_i)$, $i\in P$, then there exist $\lambda_i^*>0$, $z_i^*\in D_i$, $i\in P$ and $\mu_j^*\geqq 0$, $j\in M$ such that $(x^*,z^*,\lambda^*,\mu^*)$ is an efficient solution for (WD) and the objective values of (MP) and (WD) are equal.*

Proof. Since x^* is efficient for (MP), then by Theorem 3.7 of [8], there exist $\lambda_i^*>0$, $z_i^*\in D_i$, $i\in P$, $\mu_j^*\geq 0$, $j\in I^<(x^*)$ and $d\in -N_C(x^*)$. By taking $\mu_j^*=0$ for $j\notin I^<(x^*)$, then $(x^*,z^*,\lambda^*,\mu^*)$ is feasible for (WD). Suppose that $(x^*,z^*,\lambda^*,\mu^*)$ is not efficient for (WD), then there exists (u,z,λ,μ) feasible for (WD) such that

$$f_{i_0}(u)+u^Tz_{i_0}+\sum_{j=1}^m\mu_j g_j(u) > f_{i_0}(x^*)+(x^*)^Tz_{i_0}^*+\sum_{j=1}^m\mu_j^* g_j(x^*),\ \text{for some } i_0\in P, \tag{9}$$

$$f_i(u)+u^Tz_i+\sum_{j=1}^m\mu_j g_j(u) \geqq f_i(x^*)+(x^*)^Tz_i^*+\sum_{j=1}^m\mu_j^* g_j(x^*),\ \forall\, i\in P. \tag{10}$$

Since $(x^*)^Tz_i^* = s(x^*|D_i)$, $i\in P$ and $\sum_{j=1}^m\mu_j^*g_j(x^*)=0$, (9) and (10) implies

$$f_{i_0}(u)+u^Tz_{i_0}+\sum_{j=1}^m\mu_j g_j(u) > f_{i_0}(x^*)+s(x^*|D_{i_0}),\ \text{for some } i_0\in P,$$

$$f_i(u)+u^Tz_i+\sum_{j=1}^m\mu_j g_j(u) \geqq f_i(x^*)+s(x^*|D_i);\ \forall\, i\in P.$$

This would contradict weak duality. The objective values of (MP) and (WD) are clearly equal at their respective efficient points. □

$$
\begin{array}{lll}
\text{(MD)} & \text{Maximize} & f(u) + u^T z \\
& \text{subject to} & 0 \in \sum_{i\in P} \lambda_i(\partial_c f_i(u) + z_i) + \sum_{j\in M} \mu_j \partial_c g_j(u) + N_C(u), \quad (11) \\
& & \mu_j g_j(u) \geqq 0,\ j \in M,\ g_j(u) = 0,\ j \in I^=, \quad (12) \\
& & \lambda_i > 0, i = 1, \cdots, p, \sum_{i\in P} \lambda_i = 1, \quad (13) \\
& & \mu_j \geqq 0, j = 1, \cdots, m. \quad (14)
\end{array}
$$

Here, F_{MD} denotes the set of feasible solution to (MD).

Theorem 2.3 (Weak Duality) *Let $x \in F$ and $(u, z, \lambda, \mu) \in F_{MD}$. If $f_i(\cdot)$, $i \in P$, are generalized strictly convex functions and $g_j(\cdot)$, $j \in M$, are generalized quasiconvex at u, then the following cannot hold:*

$$f_{i_0}(x) + s(x|D_{i_0}) < f_{i_0}(u) + u^T z_{i_0},\ \textit{for some } i_0 \in P, \tag{15}$$

$$f_i(x) + s(x|D_i) \leqq f_i(u) + u^T z_i,\ \forall\ i \in P. \tag{16}$$

Proof. Let x be feasible solution for (MP) and (u, z, λ, μ) be feasible solution for (MD). Suppose contrary to the result that (15) and (16) hold. Then

$$f_{i_0}(x) + s(x|D_{i_0}) < f_{i_0}(u) + u^T z_{i_0},\ \text{for some } i_0 \in P, \tag{17}$$

and

$$f_i(x) + s(x|D_i) \leqq f_i(u) + u^T z_i,\ \forall\ i \in P. \tag{18}$$

Since $x^T z_i \leqq s(x|D_i),\ i \in P$, (17) and (18) yields

$$f_{i_0}(x) + x^T z_{i_0} < f_{i_0}(u) + u^T z_{i_0},\ \text{for some } i_0 \in P, \tag{19}$$

and

$$f_i(x) + x^T z_i \leqq f_i(u) + u^T z_i,\ \forall\ i \in P. \tag{20}$$

We suppose that $x \in F$, we have

$$g_j(x) \leqq g_j(u).$$

By assumption of $f(\cdot),\ i \in P$ and $g_j(\cdot),\ j \in M$, we have

$$\langle \sum_{i\in P} \lambda_i(\xi_i + z_i), x - u\rangle < 0,\ \forall\ \xi_i + z_i \in \partial_c f_i(u) + z_i, \tag{21}$$

$$\langle \sum_{j\in M} \mu_j \eta_j, x - u\rangle \leqq 0,\ \forall \eta_j \in \partial_c g_j(u). \tag{22}$$

By (21) and (22), we have

$$\langle \sum_{i\in P} \lambda_i(\xi_i + z_i) + \sum_{j\in M} \mu_j \eta_j, x - u\rangle < 0, \tag{23}$$

for all $\xi_i + z_i \in \partial_c f_i(u) + z_i$ and $\eta_j \in \partial_c g_j(u)$. From the constraints of (MD), it follows that for some $d \in -N_C(u),\ \xi_i + z_i \in \partial_c f_i(u) + z_i,\ \eta_j \in \partial_c g_j(u)$,

$$\langle \sum_{i\in P} \lambda_i(\xi_i + z_i) + \sum_{j\in M} \mu_j \eta_j, x - u\rangle \geqq (x - u)^T d \geq 0.$$

This is a contradiction to (23). Hence (15) and (16) cannot hold. □

Remark 2.1 *If we replace the generalized strictly convexity by the generalized strictly quasiconvexity, then above weak duality holds under the regularity condition of f_i, $i \in P$.*

Theorem 2.4 (Strong Duality) *If x^* is efficient for (MP) and weak duality theorem(Theorem 2.3) holds between (MP) and (MD), $g_j(\cdot)$, $j \in M$, are generalized strictly quasiconvex at x^* and $(x^*)^T z_i = s(x^*|D_i)$, $i \in P$, then there exist $\lambda_i^* > 0$, $z_i^* \in D_i$, $i \in P$ and $\mu_j^* \geqq 0$, $j \in M$ such that $(x^*, z^*, \lambda^*, \mu^*)$ is efficient for (MD) and the objective values of (MP) and (MD) are equal.*

Proof. Since x^* is efficient for (MP), then by Theorem 3.7 of [8], there exist $\lambda_i^* > 0$, $z_i^* \in D_i$, $i \in P$, $\mu_j^* \geq 0$, $j \in I^<(x^*)$ and $d \in -N_C(x^*)$. By taking $\mu_j^* = 0$ for $j \notin I^<(x^*)$, then $(x^*, z^*, \lambda^*, \mu^*)$ is feasible for (MD). Suppose that $(x^*, z^*, \lambda^*, \mu^*)$ is not efficient for (MD), then there exists (u, z, λ, μ) feasible for (MD) such that

$$f_{i_0}(u) + u^T z_{i_0} > f_{i_0}(x^*) + (x^*)^T z_{i_0}^* = f_{i_0}(x^*) + s(x^*|D_{i_0}), \text{ for some } i_0 \in P,$$
$$f_i(u) + u^T z_i \geqq f_i(x^*) + (x^*)^T z_i^* = f_i(x^*) + s(x^*|D_i), \ \forall i \in P.$$

However, since x^* is efficient for (MP), this would contradict weak duality. The objective values of (MP) and (MD) are equal at their respective efficient points. □

3 Special Cases

We give some special cases of our duality results.

(i) If $D_i = \{0\}$, $i = 1, \cdots, k$, then our dual programs (**MP**), (**WD**) and (**MD**) reduced to the problems considered in Egudo et al. [4].

(ii)If $D_i = \{0\}$, $i = 1, \cdots, k$, then (MP), (WD) and (MD) reduce to corresponding (MP), (DM) and (D2M) in Nobakhtian [8].

References

[1] V. Chankong and Y. Y. Haimes, *Multiobjective Decision Making. [Theory and Methodology]*, North-Holland Series in System Science and Engineering, Vol. 8, North-Holland, New York, 1983.

[2] F.H. Clarke, *Optimization and Non-Smooth Analysis*, Canadian Mathematical Society Series of Monographs and Advanced Texts, Wiley & Sons, New York, 1983.

[3] F. H. Clarke, Yu. S. Ledyaev, R. J. Stern and P. R. Wolenski, *Nonsmooth Analysis and Control Theory*, Graduate Texts in Mathematics, Vol. 178, Springer, New York, 1998.

[4] R. R. Egudo, T. Weir and B. Mond, *Duality without constraint qualification for multiobjective programming*, Journal of the Australian Mathematical Society. Series B: Applied Mathematics, Vol. 33, no. 4, pp. 531-544, 1992.

[5] D. S. Kim and K. D. Bae, *Optimality conditions and duality for a class of nondifferentiable multiobjective programming problems*, Taiwanese Journal of Mathematics, Vol.13, no.2B, pp.789-804, 2009.

[6] D. S. Kim and H. J. Lee, *Optimality conditions and duality in nonsmooth multiobjective programs*, Hindawi Publishing Corporation, Journal of Inequalities and Applications, Vol.2010, Article ID 939537, pp.1-12, 2010.

[7] D. S. Kim and S. Schaible, *Optimality and duality for invex nonsmooth multiobjective programming problems*, Optimization, Vol. 53, no. 2, pp. 165-176, 2004.

[8] S. Nobakhtian, *Duality without constraint qualification in nonsmooth optimization*, International Journal of Mathematics and Mathematical Sciences, Vol. 2006, Article ID 30459, pp. 1-11, 2006.

[9] B. MOND AND T. WEIR, *Generalized Concavity and Duality,*, Generalized concavity in optimization and economics, (eds. S. Schaible and W. T. Ziemba), Academic Press, New York, 1981.

[10] M. SCHECHTER, *A subgradient duality theorem*, Journal of Mathematical Analysis, Vol. 61, pp. 850-855, 1977.

[11] T. WEIR AND B. MOND, *Multiple objective programming duality without a constraint qualification*, Utilitas Mathematica, Vol. 39, pp. 41-55, 1991.

[12] P. WOLFE, *A duality theorem for non-linear programming*, Quarterly of Applied Mathematics, Vol. 19, pp. 239-244, 1961.