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Random conductance models on $\mathbb{Z}^d$

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1 Introduction

In the notes, we will summarize recent results for random walks on random conductance models on $\mathbb{Z}^d$ and their scaling limits. The notes are extracted from my lecture notes [24], and some recent progresses are added. We note that there is also a very nice survey by M. Biskup [12] on random conductance models.

Consider $\mathbb{Z}^d$, $d\geq 2$ and let E_d be the set of non-oriented nearest neighbor bonds, and (for simplicity) let the conductance $\{\mu_e : e\in E_d\}$ be i.i.d. that takes non-negative values. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space that governs the randomness of the conductance. For each $\omega\in\Omega$, let $\{X_n^{\omega}\}_{n\geq 0}$ be a discrete time Markov chain whose transition probability is given by $P_{\omega}(X_{n+1}=y|x_n=x)=\mu_{xy}/\mu_x$, where $\mu_x:=\sum_{y\sim x}\mu_{xy}$. Here and in the following we write $x\sim y$ if and only if $\{x,y\}\in E_d$. This model is called the random conductance model (RCM for short). Note that random walk on RCM is a special case of random walk in random environment (RWRE) in the sense $\{X_n^{\omega}\}_{n\geq 0}$ is reversible. The subject of RWRE has a long history; we refer to [32] for overviews of this field.

We will consider continuous time Markov chain. In fact, depending on time paramitrizations, there are two natural ones.

1. Constant speed random walk (CSRW): the holding time at x is $\exp(1)$ for all x.
2. Variable speed random walk (VSRW): the holding time at x is exponential distributed with mean μ_x^{-1}.

The corresponding discrete Laplace operators are

$$\mathcal{L}_C f(x)=\frac{1}{\mu_x}\sum_y (f(y)-f(x))\mu_{xy},\quad \mathcal{L}_V f(x)=\sum_y (f(y)-f(x))\mu_{xy}.$$

Let ν be such that $\nu(x)=1$, $\forall x\in\mathbb{Z}^d$. Then, for each finite supported f,g,

$$-(\mathcal{L}_V f,g)_{\nu}=-(\mathcal{L}_C f,g)_{\mu}=\frac{1}{2}\sum_{x,y}(f(x)-f(y))(g(x)-g(y))\mu_{xy},$$

where $(f,g)_\theta = \sum_x f(x)g(x)\theta_x$ for $\theta = \nu$ or μ. As we see, the two Markov chains are mutually a time change of the other. Note that the long time behavior of the discrete time Markov chain is similar to that of CSRW. Let $(\{Y_t\}_{t\geq 0}, \{P_\omega^x\}_{x\in\mathbb{Z}^d})$ be either the CSRW or VSRW and define

$$q_t^\omega(x,y) = P_\omega^x(Y_t = y)/\theta_y$$

be the heat kernel of $\{Y_t\}_{t\geq 0}$ where θ is either ν or μ.

If $p_+ := \mathbb{P}(\mu_e > 0) < p_c(\mathbb{Z}^d)$ where $p_c(\mathbb{Z}^d)$ is the critical probability for bond percolation on $\mathbb{Z}^d$, then $\{Y_t\}_{t\geq 0}$ is confined to a finite set $\mathbb{P}\times P_\omega^x$-a.s., so we consider the case $p_+ > p_c(\mathbb{Z}^d)$ throughout the notes. Under the condition, there exists unique infinite connected components of edges with strictly positive conductances, which we denote by $\mathcal{C}_\infty$. Typically, we will consider the case where $0 \in \mathcal{C}_\infty$, namely we consider $\mathbb{P}(\cdot|0\in\mathcal{C}_\infty)$. We note that the random walk on supercritical percolation cluster is a special case of RCM. Indeed, in that case μ_e is the Bernoulli random variable; $\mathbb{P}(\mu_e = 1) = p, \mathbb{P}(\mu_e = 0) = 1-p$ where $p > p_c(\mathbb{Z}^d)$.

We are interested in the long time behavior of $\{Y_t\}_{t\geq 0}$, especially we are interested in the following two questions:

(Q1) Long time heat kernel estimates for $q_t^\omega(\cdot,\cdot)$.

(Q2) Quenched invariance principle (quenched functional central limit theorem)

Here the quenched invariance principle means $\varepsilon Y_{t/\varepsilon^2}^\omega$ converges as $\varepsilon \to 0$ to Brownian motion on $\mathbb{R}^d$ (with covariance $\sigma^2 I$) $\mathbb{P}$-a.e. ω. Note that when $\mathbb{E}\mu_e < \infty$, a weak form of convergence was already proved in the 1980s that the convergence holds in law under $\mathbb{P}\times P_\omega^0$; a milestone by Kipnis-Varadhan [23]. (Note that [23] left the possibility of $\sigma = 0$, and later $\sigma > 0$ was proved by De Masi-Ferrari-Goldstein-Wick [20].) This is sometimes referred as the annealed (or averaged) invariance principle. It took about three decades to improve the annealed invariance principle to the quenched one.

2 Random walk on the supercritical percolation cluster

Before explaining the results for percolation case let us briefly discuss the uniformly elliptic case, i.e. there exists $c \geq 1$ such that $c^{-1} \leq \mu_e \leq c$ for all $e \in E_d$, $\mathbb{P}$-a.s. (Note that in this case VSRW and CSRW do not differ essentially.) I this case, (Q1) can be answered by purely analytical result in [19]. Namely, the following both sides quenched Gaussian heat kernel estimates holds $\mathbb{P}$-a.s. for $t \geq |x-y|$:

$$c_1 t^{-d/2}\exp(-c_2|x-y|^2/t) \leq q_t^\omega(x,y) \leq c_3 t^{-d/2}\exp(-c_4|x-y|^2/t). \tag{2.1}$$

Surprisingly, the quenched invariance principle was proved only recently in [30].

Now let us discuss random walk on the supercritical percolation cluster. In this case, VSRW and CSRW do not differ essentially again.

Heat kernel estimates In this case, isoperimetric inequalities are proved in [28] (see also [29]). The following heat kernel estimates is proved in [2].

Theorem 2.1 *Let $\eta \in (0,1)$. Then, there exist constants $c_1, \cdots, c_{11} > 0$ (depending on d and the distribution of μ_e) and a family of random variables $\{U_x\}_{x\in\mathbb{Z}^d}$ with*

$$\mathbb{P}(U_x \geq n) \leq c_1 \exp(-c_2 n^\eta),$$

such that the following hold.
(a) For all $x, y \in \mathbb{Z}^d$ and $t > 0$,

$$q_t^\omega(x,y) \leq c_3 t^{-d/2}.$$

(b) For $x, y \in \mathbb{Z}^d$ and $t > 0$ with $|x-y| \vee t^{1/2} \geq U_x$,

$$q_t^\omega(x,y) \leq c_3 t^{-d/2} \exp(-c_4|x-y|^2/t) \qquad \text{if } t \geq |x-y|,$$
$$q_t^\omega(x,y) \leq c_3 \exp(-c_4|x-y|(1 \vee \log(|x-y|/t))) \qquad \text{if } t \leq |x-y|.$$

(c) For $x, y \in \mathbb{Z}^d$ and $t > 0$,

$$q_t^\omega(x,y) \geq c_5 t^{-d/2} \exp(-c_6|x-y|^2/t) \qquad \text{if } t \geq U_x^2 \vee |x-y|^{1+\eta}.$$

(d) For $x, y \in \mathbb{Z}^d$ and $t > 0$ with $t \geq c_7 \vee |x-y|^{1+\eta}$,

$$c_8 t^{-d/2} \exp(-c_9|x-y|^2/t) \leq \mathbb{E}[q_t^\omega(x,y)] \leq c_{10} t^{-d/2} \exp(-c_{11}|x-y|^2/t).$$

Quenched invariance principle In this case, the quenched invariance principle is proved in [30] for $d \geq 4$ and later extended to all $d \geq 2$ in [10, 27]. (Precise statement is given in Theorem 3.2.

3 Random walk on RCM

We now consider general RCM. Depending on whether the conductance is bounded from above or below, there are two cases.

$$\text{Case 1: } 0 \leq \mu_e \leq c \text{ for some } c > 0, \quad \text{Case 2: } c \leq \mu_e < \infty \text{ for some } c > 0.$$

3.1 Heat kernel estimates

Case 1 This case is treated in [11, 15, 22, 26] for $d \geq 2$. (Note that the papers [11, 15] consider a discrete time random walk and [22, 26] considers CSRW.) In [22, 11], it is proved that Gaussian heat kernel bounds do not hold in general and anomalous behavior of the heat kernel is established for d large (see also [16]). In [22], Fontes and Mathieu consider VSRW on $\mathbb{Z}^d$ with conductance given by $\mu_{xy} = \omega(x) \wedge \omega(y)$ where $\{\omega(x) : x \in \mathbb{Z}^d\}$ are i.i.d. with $\omega(x) \leq 1$ for all x and

$$\mathbb{P}(\omega(0) \leq s) \asymp s^{\gamma} \quad \text{as} \quad s \downarrow 0,$$

for some $\gamma > 0$. They prove the following anomalous annealed heat kernel behavior.

$$\lim_{t\to\infty} \frac{\log \mathbb{E}[P_\omega^0(Y_t = 0)]}{\log t} = -(\frac{d}{2} \wedge \gamma).$$

We now state the main results in [11]. Here we consider discrete time Markov chain with transition probability $\{P(x,y) : x, y \in \mathbb{Z}^d\}$ and denote by $P_\omega^n(0,0)$ the heat kernel for the Markov chain, which (in this case) coincides with the return probability for the Markov chain started at 0 to 0 at time n.

Theorem 3.1 *(i) For $\mathbb{P}$-a.e. ω, there exists $C_1(\omega) < \infty$ such that for each $n \geq 1$,*

$$P_\omega^n(0,0) \leq C_1(\omega) \begin{cases} n^{-d/2}, & d = 2,3, \\ n^{-2}\log n, & d = 4, \\ n^{-2}, & d \geq 5. \end{cases} \tag{3.1}$$

Further, for $d \geq 5$, $\lim_{n\to\infty} n^2 P_\omega^n(0,0) = 0$ $\mathbb{P}$-a.s., and for $d = 4$, $\lim_{n\to\infty} \frac{n^2}{\log n} P_\omega^n(0,0) = 0$ $\mathbb{P}$-a.s.
(ii) Let $d \geq 4$. For any increasing sequence $\{\lambda_n\}_{n\in\mathbb{N}}$, $\lambda_n \to \infty$, there exists an i.i.d. law $\mathbb{P}$ on bounded nearest-neighbor conductances with $p_+ > p_c(d)$ and $C_3(\omega) > 0$ such that for a.e. $\omega \in \{|\mathcal{C}(0)| = \infty\}$,

$$\begin{aligned} P_\omega^{2n}(0,0) &\geq C_3(\omega) n^{-2} \lambda_n^{-1} \quad \text{for } d \geq 5 \\ P_\omega^{2n}(0,0) &\geq C_3(\omega) n^{-2} (\log n) \lambda_n^{-1} \quad \text{for } d = 4. \end{aligned}$$

along a subsequence that does not depend on ω.

Note that the last result in (i) for $d = 4$ is due to [14] and the result in (ii) for $d = 4$ is due to [13]. As we can see, Theorem 3.1 shows anomalous behavior of the Markov chain for $d \geq 4$. We will give a key idea of the proof of (ii) for $d \geq 5$ here.

Suppose we can show that for large n, there is a box of side length ℓ_n centered at the origin such that in the box a bond with conductance 1 ('strong' bond) is separated from other sites by bonds with conductance $1/n$ ('weak' bonds), and at least one of the 'weak' bonds is connected to the origin by a path of bonds with conductance 1 within the box. Then the probability that the walk is back to the origin at time n is bounded below by the probability that the walk goes directly towards the above place (which costs $e^{O(\ell_n)}$ of probability) then crosses the weak bond (which costs $1/n$), spends time $n - 2\ell_n$ on the strong bond (which costs only $O(1)$ of probability), then crosses a weak bond again (another $1/n$ term) and then goes back to the origin on time (another $e^{O(\ell_n)}$ term). The cost of this strategy is $O(1)e^{O(\ell_n)}n^{-2}$ so if can take $\ell_n = o(\log n)$ then we obtain n^{-2}.

Case 2 This case is treated in [4] for $d \geq 2$. For the VSRW, it is shown that Theorem 2.1 holds.

3.2 Quenched invariance principle

For $t \geq 0$, let $\{Y_t\}_{t\geq 0}$ be either CSRW or VSRW and define

$$Y_t^{(\varepsilon)} := \varepsilon Y_{t/\varepsilon^2}. \tag{3.2}$$

For Case 1, the quenched invariance principle was proved in [15, 26], and for Case 2, in [4]. The following unified version (i.e. for any $\mu_e \in [0, \infty)$) is proved in [1].

Theorem 3.2 *(i) Let $\{Y_t\}_{t\geq 0}$ be the VSRW. Then $\mathbb{P}$-a.s. $Y^{(\varepsilon)}$ converges (under P_ω^0) in law to Brownian motion on $\mathbb{R}^d$ with covariance $\sigma_V^2 I$ where $\sigma_V > 0$ is non-random.*
(ii) Let $\{Y_t\}_{t\geq 0}$ be the CSRW. Then $\mathbb{P}$-a.s. $Y^{(\varepsilon)}$ converges (under P_ω^0) in law to Brownian motion on $\mathbb{R}^d$ with covariance $\sigma_C^2 I$ where $\sigma_C^2 = \sigma_V^2/(2d\mathbb{E}\mu_e)$ if $\mathbb{E}\mu_e < \infty$ and $\sigma_C^2 = 0$ if $\mathbb{E}\mu_e = \infty$.

Local central limit theorem In [5], a sufficient condition is given for the quenched local CLT to hold. Using the results, the following local CLT is proved in [4] for Case 2.

Theorem 3.3 *Let $q_t^\omega(x, y)$ be the heat kernel for VSRW for Case 2 and write $k_t(x) = (2\pi t\sigma_V^2)^{-d/2}\exp(-|x|^2/(2\sigma_V^2 t))$ where σ_V is as in Theorem 3.2 (i). Let $T > 0$, and for $x \in \mathbb{R}^d$, write $[x] = ([x_1], \cdots, [x_d])$. Then*

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}^d} \sup_{t\geq T} |n^{d/2} q_{nt}^\omega(0, [n^{1/2}x]) - k_t(x)| = 0, \quad \mathbb{P} - a.s.$$

The key idea of the proof is as follows: one can prove the parabolic Harnack inequality using Theorem 2.1. This implies the uniform Hölder continuity of $n^{d/2}q_{nt}^\omega(0, [n^{1/2}\cdot])$, which, together with Theorem 3.2 implies the pointwise uniform convergence.

For the case of simple random walk on the supercritical percolation, this local CLT is proved in [5]. Note that in general when $\mu_e \leq c$, such local CLT does NOT hold because of the anomalous behavior of the heat kernel and the quenched invariance principle.

3.3 CSRW with $\mathbb{E}\mu_e = \infty$

According to Theorem 3.2 (ii), one does not have the usual central limit theorem for CSRW with $\mathbb{E}\mu_e = \infty$ in the sense the scaled process degenerates as $\varepsilon \to 0$. A natural question is what is the right scaling order and what is the scaling limit. The answers are given in [3, 6, 17] for the case of heavy-tailed environments with $d \geq 3$. Let $\{\mu_e\}$ satisfies

$$\mathbb{P}(\mu_e \geq c_1) = 1, \quad \mathbb{P}(\mu_e \geq u) = c_2 u^{-\alpha}(1 + o(1)) \quad \text{as } u \to \infty, \tag{3.3}$$

for some constants $c_1, c_2 > 0$ and $\alpha \in (0, 1]$.

In order to state the result, we first introduce the Fractional-Kinetics (FK) process and the Fontes-Isopi-Newman (FIN) diffusion ([21]).

Definition 3.4 *Let $\{B_d(t)\}$ be a standard d-dimensional Brownian motion started at* 0.
(i) For $\alpha \in (0,1)$, let $\{V_\alpha(t)\}_{t\geq 0}$ be an α-stable subordinator independent of$\{B_d(t)\}$, which is determined by $\mathbb{E}[\exp(-\lambda V_\alpha(t))] = \exp(-t\lambda^\alpha)$. Let $V_\alpha^{-1}(s) := \inf\{t : V_\alpha(t) > s\}$ be the rightcontinuous inverse of $V_\alpha(t)$. We define the fractional-kinetics process $\mathbf{FK}_{d,\alpha}$ by

$$\mathbf{FK}_{d,\alpha}(s) = B_d(V_\alpha^{-1}(s)), \quad s \in [0, \infty).$$

(ii) Let (x_i, ν_i) on $\mathbb{R} \times \mathbb{R}_+$ be an inhomogeneous Poisson point process with intensity $dx\alpha\nu^{-1-\alpha}d\nu$ and let ρ be the random discrete measure define by $\rho := \sum_i \nu_i \delta_{x_i}$. Set $\phi_\rho(t) := \int_{\mathbb{R}} \ell(t,y)\rho(dy)$ where $\ell(\cdot,\cdot)$ is the local time of the Brownian motion $\{B_1(t)\}$. We define the Fontes-Isopi-Newman (FIN) diffusion by

$$Z(s) = B_1(\phi_\rho^{-1}(s)), \quad s \in [0, \infty).$$

In other word, the FIN diffusion is a diffusion process (with $Z(0) = 0$) that can be expressed as a time change of Brownian motion with the speed measure ρ.

The FK process is non-Markovian process, which is γ-Hölder continuous for all $\gamma < \alpha/2$ and is self-similar, i.e. $\mathbf{FK}_{d,\alpha}(\cdot) \overset{(d)}{=} \lambda^{-\alpha/2}\mathbf{FK}_{d,\alpha}(\lambda\cdot)$ for all $\lambda > 0$. The density of the process $p(t,x)$ started at 0 satisfies the fractional-kinetics equation

$$\frac{\partial^\alpha}{\partial t^\alpha} p(t,x) = \frac{1}{2}\Delta p(t,x) + \delta_0(x)\frac{t^{-\alpha}}{\Gamma(1-\alpha)}.$$

This process is well-known in physics literatures, see [31] for details.

Theorem 3.5 *Let $d \geq 3$ and Let $\{Y_t\}_{t\geq 0}$ be the CSRW of RCM that satisfies (3.3).*
(i) ([3]) Let $\alpha \in (0,1)$ in (3.3) and let $Y_t^{(\varepsilon)} := \varepsilon Y_{t/\varepsilon^{2/\alpha}}$. Then $\mathbb{P}$-a.s. $Y^{(\varepsilon)}$ converges (under P_ω^0) in law to a multiple of the fractional-kinetics process $c \cdot \mathbf{FK}_{d,\alpha}$ on $D([0,\infty), \mathbb{R}^d)$ equipped with the J_1-topology.

(ii) ([17]) *Let* $d = 2$, $\alpha \in (0,1)$ *in (3.3) and let* $Y_t^{(\varepsilon)} := \varepsilon Y_{t(\log(1/\varepsilon))^{1-1/\alpha}/\varepsilon^{2/\alpha}}$. *Then the conclusion of (i) holds.*
(iii) ([17]) *Let* $d = 1$, $\alpha \in (0,1)$ *in (3.3) and let* $Y_t^{(\varepsilon)} := \varepsilon Y_{c_* c_\varepsilon t/\varepsilon}$, *where* $c_* = \mathbb{E}[\mu_e^{-1}]$ *and*

$$c_\varepsilon := \inf\{t \geq 0 : \mathbb{P}(\mu_e > t) \leq \varepsilon\} = \varepsilon^{-1/\alpha}(1 + o(1)).$$

Then, $Y^{(\varepsilon)}$ *converges in law to the FIN diffusion* $Z(t)$ *under* $\mathbb{P} \times P_0^\mu$.
(iv) ([6]) *Let* $\alpha = 1$ *in (3.3) with* $c_1 = c_2 = 1$ *and let* $Y_t^{(\varepsilon)} := \varepsilon Y_{t\log(1/\varepsilon)/\varepsilon^2}$. *Then* $\mathbb{P}$*-a.s.* $Y^{(\varepsilon)}$ *converges (under* P_ω^0*) in law to Brownian motion on* $\mathbb{R}^d$ *with covariance* $\sigma_C^2 I$ *where* $\sigma_C = 2^{-1/2}\sigma_V > 0$.

Remark 3.6 *(i) In [7], a scaling limit theorem similar to Theorem 3.5 (i), (ii) was shown for symmetric Bouchaud's trap model (BTM) for* $d \geq 2$. *Let* $\{\tau_x\}_{x\in\mathbb{Z}^d}$ *be a positive i.i.d. and let* $a \in [0,1]$ *be a parameter. Define a random weight (conductance) by*

$$\mu_{xy} = \tau_x^a \tau_y^a \qquad \text{if } x \sim y,$$

and let $\mu_x = \tau_x$ *be the measure. Then, the BTM is the CSRW with the transition probability* $\mu_{xy}/\sum_y \mu_{xy}$ *and the measure* μ_x. *If* $a = 0$, *then the BTM is a time change of the simple random walk on* $\mathbb{Z}^d$ *and it is called symmetric BMT, while non-symmetric refers to the general case* $a \neq 0$. *(This terminology is a bit confusing. Note that the Markov chain for the BTM is reversible w.r.t.* μ *for all* $a \in [0,1]$.*)*
(ii) In [21, 8], it is proved that the scaling limit (in the sense of finite-dimensional distributions) of the BTM on $\mathbb{R}$ *is the FIN diffusion.*

3.4 Some idea of the proof of quenched invariance principle

Let us briefly overview the proof of the quenched invariance principle for VSRW. As usual for the functional central limit theorem, the key tool is 'corrector'. Let $\varphi = \varphi_\omega : \mathbb{Z}^d \to \mathbb{R}^d$ be a harmonic map, so that $M_t = \varphi(Y_t)$ is a P_ω^0-martingale. Let I be the identity map on $\mathbb{Z}^d$. The corrector is

$$\chi(x) = (\varphi - I)(x) = \varphi(x) - x.$$

It is referred to as the 'corrector' because it corrects the non-harmonicity of the position function. For simplicity, let us consider CLT (instead of functional CLT) for Y. By definition, we have

$$\frac{Y_t}{t^{1/2}} = \frac{M_t}{t^{1/2}} - \frac{\chi(Y_t)}{t^{1/2}}.$$

Since we can control φ (due to the heat kernel estimates), the martingale CLT gives that $M_t/t^{1/2}$ converges weakly to the normal distribution. So all we need is to prove

$\chi(Y_t)/t^{1/2} \to 0$. This can be done once we have (a) $P_\omega^0(|Y_t| \geq At^{1/2})$ is small and (b) $|\chi(x)|/|x| \to 0$ as $|x| \to \infty$. (a) holds by the heat kernel upper bound, so the key is to prove (b), namely sublinearity of the corrector. Note that there maybe many global harmonic functions, so we should chose one such that (b) holds.

In general, we do not have nice heat kernel estimates. In such case, we consider the following subcluster

$$\mathcal{C}_{\infty,K} := \{e \in \mathcal{C}_\infty : K^{-1} \leq \mu_e \leq K\}.$$

When K is large enough, $\mathcal{C}_{\infty,K}$ is also an infinite cluster. Consider the Markov chain traced on $\mathcal{C}_{\infty,K}$. Then one can obtain nice heat kernel estimates like Theorem 2.1 and obtain quenched invariance principle for the traced Markov chain. The desired invariance principle for the original Markov chain can be obtained by showing that the occupation time for the original Markov chain on $\mathcal{C}_\infty \setminus \mathcal{C}_{\infty,K}$ is small.

3.5 Percolation on half/square planes

The above mentioned corrector method relies on the fact that the environment is stationary and ergodic with respect to the translation on $\mathbb{Z}^d$. So the method does not work for half/square planes.

Quite recently ([18]) it is proved that the quenched invariance principle holds for random walk on the supercritical percolation cluster on $\mathbb{L} := \{(x_1, \cdots, x_d) \in \mathbb{Z}^d : x_{j_1}, \cdots, x_{j_l} \geq 0\}$ for some $1 \leq j_1 < \cdots < j_l \leq d$, $l \leq d$. The ideas of the proof are twofold. One is to make a full use of the heat kernel estimates. (In the previous work, only upper bound of Theorem 2.1 was used.) The other is to use the information of the whole space random walk (especially its quenched invariance principle), and to use methods of Dirichlet forms to analyze the behavior around the boundaries.

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