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New Family of Integral Operators of Meromorphic Functions

Aabed Mohammed[1] and Maslina Darus[2]

[1,2]School of Mathematical Sceinces,
Faculty of Science and Technology,
Universiti Kebangsaan Malaysia
43600 Bangi, Selangor D. Ehsan,
Malaysia
[1]aabedukm@yahoo.com
[2]maslina@ukm.my

Abstract. We define here an integral operator $I_n(f_i, g_i)(z)$ for meromorphic functions in the punctured open unit disk. Some properties for this operator are derived.

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[2]-corresponding author

1 Introduction

Let Σ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the punctured open unit disk

$$\mathbb{U}^* = \{z \in \mathbb{C} : 0 < |z| < 1\} = \mathbb{U} \backslash \{0\}, \tag{1.2}$$

where $\mathbb{U}$ is the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$.

We say that a function $f \in \Sigma$ is meromorphic starlike of order $\delta\,(0 \leq \delta < 1)$, and belongs to the class $\Sigma^*(\delta)$, if it satisfies the inequality

$$-\Re\left(\frac{zf'(z)}{f(z)}\right) > \delta. \tag{1.3}$$

A function $f \in \Sigma$ is a meromorphic convex function of order $\delta\,(0 \leq \delta < 1)$, if f satisfies the following inequality

$$-\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \delta, \tag{1.4}$$

and we denote this class by $\Sigma_k(\delta)$.

For $f \in \Sigma$, Wang et al. [13] (see also [14]) introduced and studied the subclass $\Sigma_N(\lambda)$ of Σ consisting of functions $f(z)$ satisfying

$$-\Re\left(\frac{zf''(z)}{f'(z)}+1\right) < \lambda \qquad (\lambda > 1,\ z \in \mathbb{U}).$$

In the literature, several integral operators of meromorphic functions in the punctured open unit disk have been investigated and studied by many authors (cf., e.g., [1-11]).

For $i = 1, 2, \cdots, n$, $c > 0$, and $\alpha_i, \gamma_i \geq 0$, we now, introduce a generalized integral operator $I_n(f_i, g_i)(z) : \Sigma^n \to \Sigma$ as follows

$$I_n(f_i, g_i)(z) = \frac{c}{z^{c+1}} \int_0^z u^{c-1} \prod_{i=1}^{n} (uf_i(u))^{\alpha_i} \left(-u^2 g_i'(u)\right)^{\gamma_i} du, \tag{1.5}$$

where $f_i, g_i \in \Sigma$. Indeed, by varying the parameters c, α_i and γ_i, the operator $I_n(f_i, g_i)$ reduces to the following well-known integral operators.

(i) for $\gamma_i = 0$, we obtain the integral operator

$$H(z) = I_n(f_i)(z) = \frac{c}{z^{c+1}} \int_0^z u^{c-1} \prod_{i=1}^{n} (uf_i(u))^{\alpha_i} du, \tag{1.6}$$

introduced by Frasin [8].

(ii)For $c = 1$ and $\gamma_i = 0$, we obtain the integral operator

$$\mathcal{H}_n(z) = I_n(f_i)(z) = \frac{1}{z^2} \int_0^z \prod_{i=1}^{n} (uf_i(u))^{\alpha_i} du, \tag{1.7}$$

introduced by Mohammed and Darus [9].

(iii)For $c = 1$ and $\alpha_i = 0$, we obtain the integral operator

$$\mathcal{H}_{\gamma_1,\ldots,\gamma_n}(z) = I_n(g_i)(z) = \frac{1}{z^2} \int_0^z \prod_{i=1}^{n} \left(-u^2 g_i'(u)\right)^{\gamma_i} du, \tag{1.8}$$

introduced by Mohammed and Darus [10]

(iv)If $n = 1$, $\alpha_1 = 1$, $f_1 = f$ and $\gamma_1 = 0$ we have the integral operator

$$I_c(f)(z) = \frac{c}{z^{c+1}} \int_0^z u^{c-1} f(u) du,$$

which was studied by many authors (cf., e.g., [1, 2, 6]).

For the starlikeness of the integral operator $I_n(f_i, g_i)$, we have to recall here the following Lemma.

Lemma 1.1([12]). *Suppose that the function $\Psi : \mathbb{C}^2 \to \mathbb{C}$ satisfies the following condition:*

$$\Re\{\Psi(is,t)\} \leq 0, \qquad \left(s,t \in \mathcal{R};\ t \leq \frac{-(1+s^2)}{2}\right).$$

If the function $p(z) = 1 + p_1 z + \ldots$, is analytic in $\mathbb{U}$ and

$$\Re\{\Psi(p(x), zp'(x))\} > 0, \quad (z \in \mathbb{U}),$$

then

$$\Re\{p(z)\} > 0 \quad (z \in \mathbb{U}).$$

2 Main Results

In the next theorem, we place conditions for the meromorphically starlikeness of the integral operator $I_n(f_i, g_i)(z)$ which is defined in (1.5).

Theorem 2.1. *For $i = 1, 2, \ldots, n$, let $f_i, g_i \in \Sigma$, $\alpha_i, \gamma_i \geq 0$ and let $c > 0$. If $f_i \in \Sigma^*$, $g_i \in \Sigma_k$, and $\sum_{i=1}^{n}(\alpha_i + \gamma_i) = 1$, then the general integral operator $I_n(f_i, g_i)(z)$ belongs to the meromophic starlike function class.*

Proof. From (1.5) it follows that

$$z^2 I_n{}'(f_i, g_i)(z) + (c+1)\, z I_n(f_i, g_i)(z) = c \prod_{i=1}^{n} (z f_i(z))^{\alpha_i} \left(-z^2 g_i'(z)\right)^{\gamma_i} \tag{2.1}$$

Differentiating both sides of (2.1) logarithmically and multiplying by z, we obtain

$$\frac{z^2 I_n{}''(f_i, g_i)(z) + (c+3) z I_n{}'(f_i, g_i)(z) + (c+1) I_n(f_i, g_i)(z)}{z I_n{}'(f_i, g_i)(z) + (c+1) I_n(f_i, g_i)(z)}$$

$$= \sum_{i=1}^{n} \alpha_i \frac{z f_i'(z)}{f_i(z)} + \sum_{i=1}^{n} \gamma_i \left(\frac{z g_i''(z)}{g_i'(z)} + 1\right) + \sum_{i=1}^{n} \alpha_i + \sum_{i=1}^{n} \gamma_i. \tag{2.2}$$

Which is equivalent to

$$-\frac{z^2 I_n{}''(f_i, g_i)(z) + (c+2) z I_n{}'(f_i, g_i)(z)}{z I_n{}'(f_i, g_i)(z) + (c+1) I_n(f_i, g_i)(z)}$$

$$= \sum_{i=1}^{n} \alpha_i \left(-\frac{zf_i'(z)}{f_i(z)} \right) + \sum_{i=1}^{n} \gamma_i \left(-\frac{zg_i''(z)}{g_i'(z)} - 1 \right) + 1 - \sum_{i=1}^{n} (\alpha_i + \gamma_i). \tag{2.3}$$

We can write (2.3) as the following

$$-\frac{\frac{zI_n'(f_i,g_i)(z)}{I_n(f_i,g_i)(z)} \left(\frac{zI_n''(f_i,g_i)(z)}{I_n'(f_i,g_i)(z)} + c + 2 \right)}{\frac{zI_n'(f_i,g_i)(z)}{I_n(f_i,g_i)(z)} + c + 1}$$

$$= \sum_{i=1}^{n} \alpha_i \left(-\frac{zf_i'(z)}{f_i(z)} \right) + \sum_{i=1}^{n} \gamma_i \left(-\frac{zg_i''(z)}{g_i'(z)} - 1 \right) + 1 - \sum_{i=1}^{n} (\alpha_i + \gamma_i). \tag{2.4}$$

We define the regular function p in $\mathbb{U}$ by

$$p(z) = -\frac{zI_n'(f_i, g_i)(z)}{I_n(f_i, g_i)(z)}, \tag{2.5}$$

and $p(0) = 1$. Differentiating $p(z)$ logarithmically, we obtain

$$-p(z) + \frac{zp'(z)}{p(z)} = 1 + \frac{zI_n''(f_i, g_i)(z)}{I_n'(f_i, g_i)(z)}. \tag{2.6}$$

From (2.4),(2.5) and (2.6) we obtain

$$p(z) + \frac{zp'(z)}{-p(z) + c + 1} = \sum_{i=1}^{n} \alpha_i \left(-\frac{zf_i(z)'}{f_i(z)} \right) + \sum_{i=1}^{n} \gamma_i \left(-\frac{zg_i''(z)}{g_i'(z)} - 1 \right) + 1 - \sum_{i=1}^{n} (\alpha_i + \gamma_i). \tag{2.7}$$

Let us put

$$\Psi(u, v) = u + \frac{v}{-u + c + 1}. \tag{2.8}$$

From the hypositHes of Theorem 2.1, (2.7) and (2.8) we obtain

$$\Re\{\Psi(p(z), zp'(z))\} = \sum_{i=1}^{n} \alpha_i \left(-\Re\frac{zf_i'(z)}{f_i(z)} \right) + \sum_{i=1}^{n} \gamma_i \left\{ \Re \left(-\frac{zg_i''(z)}{g_i'(z)} - 1 \right) \right\} + 1 - \sum_{i=1}^{n} (\alpha_i + \gamma_i)$$

$$> 1 - \sum_{i=1}^{n} (\alpha_i + \gamma_i) = 0. \tag{2.9}$$

Now we proceed to show that

$$\Re\{\Psi(is, t)\} \le 0, \qquad \left(s, t \in \mathcal{R};\ t \le \frac{-(1 + s^2)}{2} \right).$$

Indeed, from (2.8), we have

$$\Re\{\Psi(is, t)\} = \Re \left\{ is + \frac{t}{-is + c + 1} \right\} = \frac{t\,(c + 1)}{s^2 + (c + 1)^2} \le -\frac{(1 + s^2)\,(c + 1)}{2\left[s^2 + (c + 1)^2\right]} < 0. \tag{2.10}$$

Thus,from (2.9),(2.10) and by using Lemma 1.1, we conclude that $\Re\{p(z)\} > 0$, and so

$$-\Re \left\{ \frac{zI_n'(f_i, g_i)(z)}{I_n(f_i, g_i)(z)} \right\} > 0.$$

that is $I_n(f_i, g_i)(z)$ is starlike .

Next, we place conditions for the integral operator $I_n(f_i, g_i)$ to be in the class $\Sigma_N(\lambda)$.

Theorem 2.2. *For $i = 1, 2, \ldots, n$, let $f_i, g_i \in \Sigma$, α_i, $\gamma_i \geq 0$ and let $c > 0$. If $f_i \in \Sigma^*(\delta)$, $g_i \in \Sigma_k(\delta)$, and*

$$\sum_{i=1}^{n} (\alpha_i + \gamma_i) > \frac{c+1}{1-\delta}, \tag{2.11}$$

then $I_n(f_i, g_i)(z) \in \Sigma_N(\lambda)$, $\lambda > 1$.

Proof. Equivalently, (2.3) can be written as

$$\frac{-\left(\frac{zI_n''(f_i,g_i)(z)}{I_n'(f_i,g_i)(z)}+1\right)-c-1}{\frac{(c+1)I_n(f_i,g_i)(z)}{zI_n'(f_i,g_i)(z)}+1} = \sum_{i=1}^{n} \alpha_i \left(-\frac{zf_i'(z)}{f_i(z)}\right) + \sum_{i=1}^{n} \gamma_i \left(-\frac{zg_i''(z)}{g_i'(z)} - 1\right) + 1 - \sum_{i=1}^{n} (\alpha_i + \gamma_i). \tag{2.12}$$

Therefore

$$-\left(\frac{zI_n''(f_i,g_i)(z)}{I_n'(f_i,g_i)(z)} + 1\right) = \frac{(c+1)I_n(f_i,g_i)(z)}{zI_n'(f_i,g_i)(z)} \left[\sum_{i=1}^{n} \alpha_i \left(-\frac{zf_i'(z)}{f_i(z)}\right) + \sum_{i=1}^{n} \gamma_i \left(-\frac{zg_i''(z)}{g_i'(z)} - 1\right) + 1 - \sum_{i=1}^{n} (\alpha_i + \gamma_i)\right]$$
$$+ \sum_{i=1}^{n} \alpha_i \left(-\frac{zf_i'(z)}{f_i(z)}\right) + \sum_{i=1}^{n} \gamma_i \left(-\frac{zg_i''(z)}{g_i'(z)} - 1\right) + c + 2 - \sum_{i=1}^{n} (\alpha_i + \gamma_i). \tag{2.13}$$

Taking real part of both sides of (2.13), we obtain

$$\begin{aligned}
-\Re\left(\frac{zI_n''(f_i,g_i)(z)}{I_n'(f_i,g_i)(z)} + 1\right) &= \Re\left\{\frac{(c+1)I_n(f_i,g_i)(z)}{zI_n'(f_i,g_i)(z)} \left[\sum_{i=1}^{n} \alpha_i \left(-\frac{zf_i'(z)}{f_i(z)}\right) + \sum_{i=1}^{n} \gamma_i \left(-\frac{zg_i''(z)}{g_i'(z)} - 1\right) + 1\right.\right. \\
&\quad \left.\left. - \sum_{i=1}^{n} (\alpha_i + \gamma_i)\right]\right\} + \sum_{i=1}^{n} \alpha_i \left(-\Re\frac{zf_i'(z)}{f_i(z)}\right) + \sum_{i=1}^{n} \gamma_i \Re\left(-\frac{zg_i''(z)}{g_i'(z)} - 1\right) + c + 2 \\
&\quad - \sum_{i=1}^{n} (\alpha_i + \gamma_i) \\
&\leq \left|\frac{(c+1)I_n(f_i,g_i)(z)}{zI_n'(f_i,g_i)(z)} \left[\sum_{i=1}^{n} \alpha_i \left(-\frac{zf_i'(z)}{f_i(z)}\right) + \sum_{i=1}^{n} \gamma_i \left(-\frac{zg_i''(z)}{g_i'(z)} - 1\right) + 1\right.\right. \\
&\quad \left.\left. - \sum_{i=1}^{n} (\alpha_i + \gamma_i)\right]\right| + \sum_{i=1}^{n} \alpha_i \left(-\Re\frac{zf_i'(z)}{f_i(z)}\right) + \sum_{i=1}^{n} \gamma_i \Re\left(-\frac{zg_i''(z)}{g_i'(z)} - 1\right) + c + 2 \\
&\quad - \sum_{i=1}^{n} (\alpha_i + \gamma_i).
\end{aligned} \tag{2.14}$$

Let

$$\lambda = \left| \frac{(c+1)I_n(f_i,g_i)(z)}{zI_n{}'(f_i,g_i)(z)} \left[\sum_{i=1}^{n} \alpha_i \left(-\frac{zf_i{}'(z)}{f_i(z)} \right) + \sum_{i=1}^{n} \gamma_i \left(-\frac{zg_i{}''(z)}{g_i{}'(z)} - 1 \right) + 1 \ - \sum_{i=1}^{n} (\alpha_i + \gamma_i) \right] \right|$$

$$+ \sum_{i=1}^{n} \alpha_i \left(-\Re \frac{zf_i{}'(z)}{f_i(z)} \right) + \sum_{i=1}^{n} \gamma_i \Re \left(-\frac{zg_i{}''(z)}{g_i{}'(z)} - 1 \right) + c + 2 - \sum_{i=1}^{n} (\alpha_i + \gamma_i).$$

Since $\left| \frac{(c+1)I_n(f_i,g_i)(z)}{zI_n{}'(f_i,g_i)(z)} \left[\sum_{i=1}^{n} \alpha_i \left(-\frac{zf_i{}'(z)}{f_i(z)} \right) + \sum_{i=1}^{n} \gamma_i \left(-\frac{zg_i{}''(z)}{g_i{}'(z)} - 1 \right) + 1 \ - \sum_{i=1}^{n} (\alpha_i + \gamma_i) \right] \right| > 0$, $f_i \in \Sigma^*(\delta)$, $g_i \in \Sigma_k(\delta)$, then we have

$$\lambda > c + 2 - (1 - \delta) \sum_{i=1}^{n} (\alpha_i + \gamma_i).$$

Then, by the hypothesis (2.11), we have $\lambda > 1$. Therefore, $I_n(f_i, g_i)(z) \in \Sigma_N(\lambda), \ \lambda > 1$.

If we set $\gamma_i = 0$ in Theorem 2.2, then we have [8, Theorem 2.6].

Further, Putting $c = 1, \gamma_i = 0$ in Theorem 2.2, we get

Corollary 2.3. *For $i = 1, 2, \ldots, n$, let $f_i \in \Sigma$, $\alpha_i \geq 0$. If $f_i \in \Sigma^*(\delta)$, and*

$$\sum_{i=1}^{n} \alpha_i > \frac{2}{1 - \delta},$$

then $\mathcal{H}_n(z) \in \Sigma_N(\lambda), \ \lambda > 1$.

In addition, taking $c = 1, \alpha_i = 0$ in Theorem 2.2, we receive

Corollary 2.4. *For $i = 1, 2, \ldots, n$, let $g_i \in \Sigma$, $\gamma_i \geq 0$. If $g_i \in \Sigma_k(\delta)$, and*

$$\sum_{i=1}^{n} \gamma_i > \frac{2}{1 - \delta},$$

then $\mathcal{H}_{\gamma_1, \ldots, \gamma_n}(z) \in \Sigma_N(\lambda), \ \lambda > 1$.

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