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The Solutions to The Radial Schrödinger Equation of The Hydrogen Atom by Means of N- Fractional Calculus Operator

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Abstract

In this article, the solutions to the radial Schrödinger equation of the Hydrogen atom (in the Coulomb field)

$$\varphi_2 \cdot x^2 + \varphi_1 \cdot 2x + \varphi \cdot \{-(1/4)x^2 + \nu x - l(l+1)\} = 0$$

are discussed by means of N-fractional calculus operator.

A particular solution to the equation above is shown as follows for example.

$$\varphi = \varphi_{[1]} = x^l e^{x/2}(e^{-x} \cdot x^{\nu-(l+1)})_{l+\nu}$$

(fractional differintegrated form)

$$= (e^{i\pi})^{l+\nu} e^{-x/2} x^{\nu-1} \ {}_2F_0(-l-\nu, l+1-\nu; -1/x)$$

$$(|-1/x|<1)$$

where

${}_pF_q(\cdots\cdots)$; Generalized Gauss hypergeometric functions.

§ 0. Introduction (Definition of Fractional Calculus)

(I) Definition. (by K. Nishimoto) ([1] Vol. 1)

Let $D=\{D_-, D_+\}$, $C=\{C_-, C_+\}$,

C_- be a curve along the cut joining two points z and $-\infty+i\,\mathrm{Im}(z)$,

C_+ be a curve along the cut joining two points z and $\infty+i\,\mathrm{Im}(z)$,

D_- be a domain surrounded by C_-; D_+ be a domain surrounded by C_+.

(Here D contains the points over the curve C .)

Moreover, let $f=f(z)$ be a regular function in D $(z\in D)$,

$$f_\nu=(f)_\nu={}_C(f)_\nu=\frac{\Gamma(\nu+1)}{2\pi i}\int_C\frac{f(\zeta)}{(\zeta-z)^{\nu+1}}d\zeta \qquad (\nu\notin Z^-), \tag{1}$$

$$(f)_{-m}=\lim_{\nu\to-m}(f)_\nu \qquad (m\in Z^+), \tag{2}$$

where $-\pi\le\arg(\zeta-z)\le\pi$ for C_-, $\quad 0\le\arg(\zeta-z)\le2\pi$ for C_+,

$\zeta\ne z$, $z\in C$, $\nu\in R$, Γ; Gamma function,

then $(f)_\nu$ is the fractional differintegration of arbitrary order ν (derivatives of order ν for $\nu>0$, and integrals of order $-\nu$ for $\nu<0$), with respect to z, of the function f, if $|(f)_\nu|<\infty$.

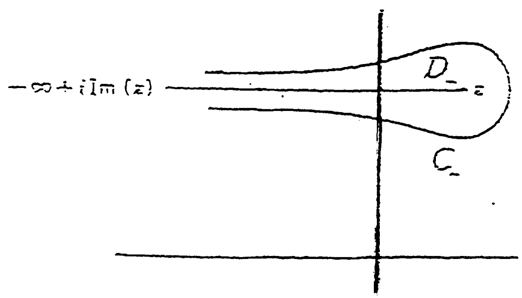

Fig. 1.

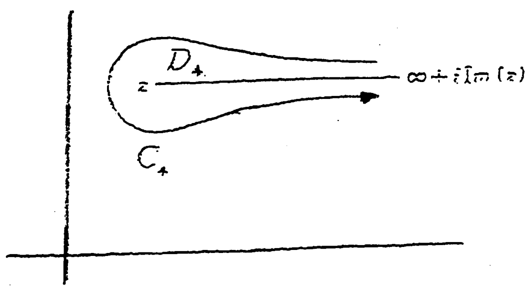

Fig. 2.

Notice that (1) is reduced to Goursat's integral for $\nu=n$ $(\in Z^+)$ and is reduced to the famous Cauchy's integral for $\nu=0$. That is, (1) is an extention of Cauchy's integral and of Goursat's one, conversely Cauchy's and Goursat's ones are special cases of (1).

Moreover, notice that (1) is the representation which unifies the derivatives and integrations.

(II) On the fractional calculus operator N^{ν} [3]

Theorem A. *Let fractional calculus operator (Nishimoto's operator) N^{ν} be*

$$N^{\nu}=\left(\frac{\Gamma(\nu+1)}{2\pi i}\int_{C}\frac{d\zeta}{(\zeta-z)^{\nu+1}}\right) \qquad (\nu \notin Z^{-}) \quad [\text{Refer to (1)}] \tag{3}$$

with
$$N^{-m}=\lim_{\nu\to -m} N^{\nu} \qquad (m\in Z^{+}), \tag{4}$$

and define the binary operation $\circ$ *as*

$$N^{\beta}\circ N^{\alpha}f=N^{\beta}N^{\alpha}f=N^{\beta}(N^{\alpha}f) \qquad (\alpha,\beta\in R), \tag{5}$$

then the set

$$\{N^{\nu}\}=\{N^{\nu}|\nu\in R\} \tag{6}$$

is an Abelian product group (having continuous index ν) which has the inverse transform operator $(N^{\nu})^{-1}=N^{-\nu}$ to the fractional calculus operator N^{ν}, for the function f such that

$f\in F=\{f\,|\,0\neq|f_{\nu}|<\infty,\ \nu\in R\}$, *where $f=f(z)$ and $z\in C$. (viz. $-\infty<\nu<\infty$).*

(For our convenience, we call $N^{\beta}\circ N^{\alpha}$ as product of N^{β} and N^{α}.)

Theorem B. *The "F.O.G. $\{N^{\nu}\}$" is an "Action product group which has continuous index ν" for the set F.* (*F.O.G.* ; Fractional calculus operator group) [3]

Theorem C. *Let be*

$$S:=\{\pm N^{\nu}\}\cup\{0\}=\{N^{\nu}\}\cup\{-N^{\nu}\}\cup\{0\} \quad (\nu\in R). \tag{7}$$

Then the set S is a commutative ring for the function $f\in F$, when the identity

$$N^{\alpha}+N^{\beta}=N^{\gamma} \qquad (N^{\alpha},N^{\beta},N^{\gamma}\in S) \tag{8}$$

holds. [5]

(III) **Lemma.** We have [1]

(i) $\quad ((z-c)^{\beta})_{\alpha}=e^{-i\pi\alpha}\dfrac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)}(z-c)^{\beta-\alpha} \qquad \left(\left|\dfrac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)}\right|<\infty\right),$

(ii) $\quad (\log(z-c))_{\alpha}=-e^{-i\pi\alpha}\Gamma(\alpha)(z-c)^{-\alpha} \qquad (|\Gamma(\alpha)|<\infty),$

(iii) $\quad ((z-c)^{-\alpha})_{-\alpha}=-e^{i\pi\alpha}\dfrac{1}{\Gamma(\alpha)}\log(z-c) \qquad (|\Gamma(\alpha)|<\infty),$

where $z-c\neq 0$ in (i), and $z-c\neq 0,1$ in (ii).

(iv) $\quad (u\cdot v)_{\alpha}:=\displaystyle\sum_{k=0}^{\infty}\frac{\Gamma(\alpha+1)}{k!\,\Gamma(\alpha+1-k)}u_{\alpha-k}v_{k} \qquad \begin{pmatrix} u=u(z), \\ v=v(z) \end{pmatrix}.$

§ 1. Preliminary

(I) The theorem below is reported by the author already (cf. J.F.C.Vol. 27, May (2005),83 - 88.) [31]

Theorem D. *Let be*

$$P = P(\alpha,\beta,\gamma) := \frac{\sin\pi\alpha\cdot\sin\pi(\gamma-\alpha-\beta)}{\sin\pi(\alpha+\beta)\cdot\sin\pi(\gamma-\alpha)} \qquad (|P(\alpha,\beta,\gamma)| = M < \infty) \qquad (1)$$

and

$$Q = Q(\alpha,\beta,\gamma) := P(\beta,\alpha,\gamma), \quad (|P(\beta,\alpha,\gamma)| = M < \infty) \qquad (2)$$

When $\alpha, \beta, \gamma \notin Z_0^+$, *we have*

(i) $$((z-c)^\alpha\cdot(z-c)^\beta)_\gamma = e^{-i\pi\gamma}P(\alpha,\beta,\gamma)\frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)}(z-c)^{\alpha+\beta-\gamma}, \qquad (3)$$

$$(\mathrm{Re}(\alpha+\beta+1)>0, \quad (1+\alpha-\gamma)\notin Z_0^-),$$

(ii) $$((z-c)^\beta\cdot(z-c)^\alpha)_\gamma = e^{-i\pi\gamma}Q(\alpha,\beta,\gamma)\frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)}(z-c)^{\alpha+\beta-\gamma}, \qquad (4)$$

$$(\mathrm{Re}(\alpha+\beta+1)>0, \quad (1+\beta-\gamma)\notin Z_0^-),$$

(iii) $$((z-c)^{\alpha+\beta})_\gamma = e^{-i\pi\gamma}\frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)}(z-c)^{\alpha+\beta-\gamma}, \qquad (5)$$

where

$$z-c\neq 0, \quad \left|\frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)}\right| < \infty .$$

Then the inequalities below are established from this theorem.

Corollary 1. *We have the inequalities*

(i) $$((z-c)^\alpha\cdot(z-c)^\beta)_\gamma \neq ((z-c)^\beta\cdot(z-c)^\alpha)_\gamma, \qquad (6)$$

and

(ii) $$((z-c)^\alpha\cdot(z-c)^\beta)_\gamma \neq ((z-c)^{\alpha+\beta})_\gamma, \qquad (7)$$

where

$$\alpha,\beta,\gamma\notin Z_0^+, \quad \alpha\neq\beta, \quad z-c\neq 0 .$$

Corollary 2.

(i) *When* $\alpha,\beta,\gamma\notin Z_0^+$, *and* $P(\alpha,\beta,\gamma) = Q(\beta,\alpha,\gamma) = 1$, (8)

we have

$$((z-c)^\alpha\cdot(z-c)^\beta)_\gamma = ((z-c)^\beta\cdot(z-c)^\alpha)_\gamma = ((z-c)^{\alpha+\beta})_\gamma, \qquad (9)$$

$$(\mathrm{Re}(\alpha+\beta+1)>0, \ (1+\alpha-\gamma)\notin Z_0^-, \ (1+\beta-\gamma)\notin Z_0^-),$$

(ii) *When* $\gamma = m \in Z_0^+$, *we have* ;

$$((z-c)^\alpha\cdot(z-c)^\beta)_m = ((z-c)^\beta\cdot(z-c)^\alpha)_m = ((z-c)^{\alpha+\beta})_m . \qquad (10)$$

§ 2. The Solutions to The Radial Schrödinger Equation of The Hydrogen Atom

Theorem 1. *Let be* $\varphi = \varphi(x) \in F$ *, then the homogeneous linear ordinary differential equation (Radial* Schrödinger *equation of the Hydrogen atom in the Coulomb field)*

$$L[\varphi; x; \nu, l] = \varphi_2 \cdot x^2 + \varphi_1 \cdot 2x + \varphi \cdot \{-(1/4)x^2 + \nu x - l(l+1)\} = 0 \qquad (1)$$

$$(\varphi_\alpha = d^\alpha \varphi / dx^\alpha \ \textit{for} \ \alpha > 0, \ \varphi_0 = \varphi, x \neq 0)$$

has particular solutions of the forms (fractional differintegrated forms);

Group I ;

1) $$\varphi = x^l e^{x/2}(e^{-x} \cdot x^{\nu-(l+1)})_{l+\nu} \equiv \varphi_{[1]}, \qquad \text{(denote)} \qquad (2)$$

2) $$\varphi = x^l e^{x/2}(x^{\nu-(l+1)} \cdot e^{-x})_{l+\nu} \equiv \varphi_{[2]} \qquad (3)$$

3) $$\varphi = x^l e^{-x/2}(e^{x} \cdot x^{-\nu-(l+1)})_{l-\nu} \equiv \varphi_{[3]} \qquad (4)$$

4) $$\varphi = x^l e^{-x/2}(x^{-\nu-(l+1)} \cdot e^{x})_{l-\nu} \equiv \varphi_{[4]} \qquad (5)$$

Group II ;

1) $$\varphi = x^{-(l+1)} e^{x/2}(e^{-x} \cdot x^{\nu+l})_{\nu-(l+1)} \equiv \varphi_{[5]} \qquad (6)$$

2) $$\varphi = x^{-(l+1)} e^{x/2}(x^{\nu+l} \cdot e^{-x})_{\nu-(l+1)} \equiv \varphi_{[6]} \qquad (7)$$

3) $$\varphi = x^{-(l+1)} e^{-x/2}(e^{x} \cdot x^{-\nu+l})_{-\nu-(l+1)} \equiv \varphi_{[7]} \qquad (8)$$

4) $$\varphi = x^{-(l+1)} e^{-x/2}(x^{-\nu+l} \cdot e^{x})_{-\nu-(l+1)} \equiv \varphi_{[8]} \ . \qquad (9)$$

Proof of Group I.

Set

$$\varphi = x^\mu \phi, \qquad (\phi = \phi(x)) \qquad (10)$$

we have then

$$\varphi_1 = \mu x^{\mu-1}\phi + x^\mu \phi_1 \qquad (11)$$

and

$$\varphi_2 = \mu(\mu-1)x^{\mu-2}\phi + 2\mu x^{\mu-1}\phi_1 + x^\mu \phi_2 \ . \qquad (12)$$

Hence, yields

$$\phi_2 \cdot x^{\mu+2} + \phi_1 \cdot x^{\mu+1}(2\mu + 2) + \phi \cdot \{-(1/4)x^{\mu+2} + \nu x^{\mu+1} + x^{\mu}(\mu^2 + \mu - l^2 - l)\} = 0 \tag{13}$$

from (1), applying (10), (11) and (12).

Choose μ such that

$$\mu^2 + \mu - l^2 - l = 0 \tag{14}$$

hence

$$\mu = \begin{cases} l & (15) \\ -(l+1) & (16) \end{cases}$$

(I) Case of $\mu = l$;

In this case we have

$$\phi_2 \cdot x + \phi_1 \cdot (2l + 2) + \phi \cdot \{\nu - (1/4)x\} = 0 \tag{17}$$

from (13). Next we set

$$\phi = e^{\lambda x}\psi \qquad (\psi = \psi(x)) \tag{18}$$

we have then

$$\phi_1 = \lambda e^{\lambda x}\psi + e^{\lambda x}\psi_1 \tag{19}$$

and

$$\phi_2 = \lambda^2 e^{\lambda x}\psi + 2\lambda e^{\lambda x}\psi_1 + e^{\lambda x}\psi_2 \, . \tag{20}$$

Therefore, we obtain

$$\psi_2 \cdot x + \psi_1 \cdot (2\lambda x + 2l + 2) + \psi \cdot \{x(\lambda^2 - \tfrac{1}{4}) + 2\lambda l + 2\lambda + \nu\} = 0 \tag{21}$$

from (17), using (18), (19) and (20).

Choose

$$\lambda = \begin{cases} 1/2 & (22) \\ -1/2 & (23) \end{cases}$$

(i) Case of $\lambda = 1/2$;

In this case we have

$$\psi_2 \cdot x + \psi_1 \cdot (x + 2l + 2) + \psi \cdot (l + 1 + \nu) = 0 \ , \tag{24}$$

from (21).

Operate N^{α} (N-fractional calculus operator of order α) to the both sides of (24), yields

$$N^{\alpha}(\psi_2 \cdot x) + N^{\alpha}(\psi_1 \cdot (x+2l+2)) + N^{\alpha}(\psi \cdot (l+1+\nu)) = 0, \quad (\alpha \notin Z_0^-) \tag{25}$$

hence

$$\psi_{2+\alpha} \cdot x + \psi_{1+\alpha} \cdot (x+\alpha+2l+2) + \psi_{\alpha} \cdot (\alpha+l+1+\nu) = 0 \tag{26}$$

from (25), by Lemma (iv), since

$$N^{\alpha}(\psi_2 \cdot x) = (\psi_2 \cdot x)_{\alpha} \tag{27}$$

for example.

Choose α such that

$$\alpha = -(l+1+\nu) \equiv p \qquad \text{(denote)} \tag{28}$$

we have then

$$\psi_{2+p} \cdot x + \psi_{1+p} \cdot (x+p+2l+2) = 0 \tag{29}$$

from (26), applying (28).

Therefore, we obtain

$$u_1 \cdot x + u \cdot (x+p+2l+2) = 0 \tag{30}$$

from (29), setting

$$u = \psi_{1+p} \,. \qquad (u_{-(1+p)} = \psi) \tag{31}$$

A particular solution to this variable separable form equation (30) is given by

$$u = e^{-x} x^{-(p+2l+2)} \,. \tag{32}$$

We have then

$$\psi = u_{-(1+p)} = (e^{-x} \cdot x^{-(p+2l+2)})_{-(1+p)} \tag{33}$$

from (31), and hence

$$\phi = e^{x/2}(e^{-x} \cdot x^{-(p+2l+2)})_{-(1+p)} \tag{34}$$

from (18), applying (22) and (33).

Hence we obtain

$$\varphi = x^l e^{x/2}(e^{-x} \cdot x^{-(p+2l+2)})_{-(1+p)} \tag{35}$$

$$= x^l e^{x/2}(e^{-x} \cdot x^{\nu-(l+1)})_{l+\nu} \equiv \varphi_{[1]} \quad , \tag{2}$$

from(10) using (15) and (34).

Next we obtain

$$\varphi = x^l e^{x/2}(x^{\nu-(l+1)} \cdot e^{-x})_{l+\nu} \equiv \varphi_{[2]} \quad , \tag{3}$$

changing the order of

e^{-x} and $x^{\nu-(l+1)}$ in the parenth esis $(\;\cdot\;)_{l+\nu}$ in (2).

We have

$$\varphi_{[1]} = \varphi_{[2]} \quad \text{when} \quad (l+\nu) \in Z_0^+ \ .$$

(ii) Case of $\lambda = -1/2$;

In this case we have

$$\psi_2 \cdot x + \psi_1 \cdot (-x + 2l + 2) + \psi \cdot (\nu - l - 1) = 0 \quad , \tag{36}$$

from (21).

Therefore, in the same manner as above, we obtain

$$\varphi = x^l e^{-x/2}(e^x \cdot x^{-\nu-(l+1)})_{l-\nu} \equiv \varphi_{[3]} \tag{4}$$

from (36) and

$$\varphi = x^l e^{-x/2}(x^{-\nu-(l+1)} \cdot e^{x})_{l-\nu} \equiv \varphi_{[4]} \quad , \tag{5}$$

changing the order of

e^{x} and $x^{-\nu-(l+1)}$ in the parenthesis $(\;\cdot\;)_{l-\nu}$ in (4).

(II) Case of $\mu = -(l+1)$;

Set $-(l+1)$ instead of l in the solutions

$$\varphi_{[1]}, \quad \varphi_{[2]}, \quad \varphi_{[3]}, \quad \text{and} \quad \varphi_{[4]},$$

we have then the solutions

$$\varphi_{[5]}, \quad \varphi_{[6]}, \quad \varphi_{[7]}, \quad \text{and} \quad \varphi_{[8]},$$

respectively.

§ 3. The Familiar Forms of The Solutions in Section 2.

The familiar forms of the solutions that are obtained ones in section 2 are shown as follows.

Corollary 1. *We have*

Group I.

1) $\varphi_{[1]} = (e^{i\pi})^{l+\nu} e^{-x/2} x^{\nu-1} {}_2F_0(-l-\nu, l+1-\nu; -1/x) \qquad (|-1/x|<1),$ (1)

2) $\varphi_{[2]} = (e^{-i\pi})^{l+\nu} e^{-x/2} x^{-(l+1)} \dfrac{\Gamma(2l+1)}{\Gamma(l+1-\nu)} {}_1F_1(-l-\nu; -2l; x), \quad (|x|<1),$ (2)

$(|\Gamma(2l+1-k)/\Gamma(l+1-\nu)|<\infty)$

3) $\varphi_{[3]} = e^{x/2} x^{-(1+\nu)} {}_2F_0(-l+\nu, l+1+\nu; 1/x) \qquad (|1/x|<1),$ (3)

4) $\varphi_{[4]} = (e^{-i\pi})^{l-\nu} e^{x/2} x^{-(l+1)} \dfrac{\Gamma(2l+1)}{\Gamma(l+1+\nu)} {}_1F_1(\nu-l; -2l; -x), \quad (|-x|<1),$ (4)

$(|\Gamma(2l+1-k)/\Gamma(\nu+l+1)|<\infty)$

Group II.

1) $\varphi_{[5]} = (e^{i\pi})^{\nu-(l+1)} e^{-x/2} x^{\nu-1} {}_2F_0(l+1-\nu, -l-\nu; -1/x), \quad (|-1/x|<1),$ (5)

2) $\varphi_{[6]} = (e^{-i\pi})^{-(l+1)+\nu} e^{-x/2} x^{l} \dfrac{\Gamma(-2l-1)}{\Gamma(-l-\nu)} {}_1F_1(l+1-\nu; 2l+2; x), \quad (|x|<1),$ (6)

$(|\Gamma(-2l-1-k)/\Gamma(-\nu-l)|<\infty)$

3) $\varphi_{[7]} = e^{x/2} x^{-(1+\nu)} {}_2F_0(l+1+\nu, \nu-l; 1/x) \qquad (|1/x|<1),$ (7)

4) $\varphi_{[8]} = (e^{i\pi})^{l+1+\nu} e^{x/2} x^{l} \dfrac{\Gamma(-2l-1)}{\Gamma(\nu-l)} {}_1F_1(l+1+\nu; 2l+2; -x), \quad (|-x|<1),$ (8)

$(|\Gamma(-2l-1-k)/\Gamma(\nu-l)|<\infty)$

Where ${}_pF_q(\cdots\cdots)$ *are the generalized Gauss hypergeometric functions.*

Proof of Group I.

We have

1) $\varphi_{[1]} = x^l e^{x/2} (e^{-x} \cdot x^{\nu-(l+1)})_{l+\nu}$ (§2. (2))

$$= x^l e^{x/2} \sum_{k=0}^{\infty} \frac{\Gamma(l+\nu+1)}{k!\,\Gamma(l+\nu+1-k)} (e^{-x})_{l+\nu-k} (x^{\nu-(l+1)})_k \qquad \text{(by Lemma (iv))} \tag{9}$$

$$= x^l e^{x/2} \sum_{k=0}^{\infty} \frac{\Gamma(l+\nu+1)}{k!\,\Gamma(l+\nu+1-k)} (e^{i\pi})^{l+\nu-k} e^{-x} \{e^{-i\pi k} \frac{\Gamma(k+l+1-\nu)}{\Gamma(l+1-\nu)} x^{\nu-(l+1)-k}\} \tag{10}$$

$$= (e^{i\pi})^{l+\nu} e^{-x/2} x^{\nu-1} \sum_{k=0}^{\infty} \frac{[-l-\nu]_k\,[l+1-\nu]_k}{k!} (-x)^{-k} \tag{11}$$

$$= (e^{i\pi})^{l+\nu} e^{-x/2} x^{\nu-1} {}_2F_0(-l-\nu, l+1-\nu\,; -1/x), \quad (|-1/x|<1)\,, \tag{1}$$

since

$$\Gamma(\lambda-k) = (-1)^{-k} \frac{\Gamma(\lambda)\Gamma(1-\lambda)}{\Gamma(k+1-\lambda)} = (-1)^{-k} \frac{\Gamma(\lambda)}{[1-\lambda]_k}\,, \quad (k \in Z_0^+) \tag{12}$$

$$(e^{\lambda x})_{l+\nu-k} = \lambda^{l+\nu-k} e^{\lambda x}\,, \tag{13}$$

$$(x^{\nu-(l+1)})_k = e^{-i\pi k} \frac{\Gamma(k+l+1-\nu)}{\Gamma(l+1-\nu)} x^{\nu-(l+1)-k} \tag{14}$$

and

$[\lambda]_k = \lambda(\lambda+1)\cdots(\lambda+k-1) = \Gamma(\lambda+k)/\Gamma(\lambda)$ with $[\lambda]_0 = 1$.

(Notation of Pohhammer).

2) $$\varphi_{[2]} = x^l e^{x/2} (x^{\nu-(l+1)} \cdot e^{-x})_{l+\nu} \qquad (\text{§2. } (3))$$

$$= x^l e^{x/2} \sum_{k=0}^{\infty} \frac{\Gamma(l+\nu+1)}{k!\,\Gamma(l+\nu+1-k)} (x^{\nu-(l+1)})_{l+\nu-k} (e^{-x})_k \tag{15}$$

$$= x^l e^{x/2} \sum_{k=0}^{\infty} \frac{(-1)^k [-l-\nu]_k}{k!} \{e^{-i\pi(l+\nu-k)} \frac{\Gamma(2l+1-k)}{\Gamma(l+1-\nu)} x^{-2l-1+k}\} (e^{i\pi})^k e^{-x} \tag{16}$$

$$\left(\left| \frac{\Gamma(2l+1-k)}{\Gamma(l+1-\nu)} \right| < \infty \right)$$

$$= x^{-l-1} (e^{-i\pi})^{l+\nu} e^{-x/2} \sum_{k=0}^{\infty} \frac{(-1)^k [-l-\nu]_k (-1)^{-k}\, \Gamma(2l+1)}{k!\, \Gamma(l+1-\nu)[-2l]_k} x^k \tag{17}$$

$$= (e^{-i\pi})^{l+\nu} \frac{\Gamma(2l+1)}{\Gamma(l+1-\nu)} e^{-x/2} x^{-l-1} \sum_{k=0}^{\infty} \frac{[-l-\nu]_k}{k!\, [-2l]_k} x^k \tag{18}$$

$$= (e^{-i\pi})^{l+\nu} e^{-x/2} x^{-l-1} \frac{\Gamma(2l+1)}{\Gamma(l+1-\nu)} {}_1F_1(-l-\nu\,;\, -2l\,;\, x), \quad (|x|<1)\,, \tag{2}$$

3) $$\varphi_{[3]} = x^l e^{-x/2}(e^x \cdot x^{-\nu-(l+1)})_{l-\nu} \qquad (\S 2.\ (4))$$

$$= x^l e^{-x/2} \sum_{k=0}^{\infty} \frac{\Gamma(l-\nu+1)}{k!\,\Gamma(l-\nu+1-k)} (e^x)_{l-\nu-k} (x^{-\nu-(l+1)})_k \qquad (19)$$

$$= x^l e^{x/2} \sum_{k=0}^{\infty} \frac{(-1)^k [\nu-l]_k}{k!} \{e^{-i\pi k} \frac{\Gamma(\nu+l+1+k)}{\Gamma(\nu+l+1)} x^{-\nu-(l+1)-k}\} \qquad (20)$$

$$= x^{-\nu-1} e^{x/2} \sum_{k=0}^{\infty} \frac{[\nu-l]_k\,[\nu+l+1]_k}{k!} x^{-k} \qquad (21)$$

$$= x^{-\nu-1} e^{x/2}\, {}_2F_0(\nu-l,\ \nu+l+1\ ;1/x), \quad (|1/x|<1)\,. \qquad (3)$$

4) $$\varphi_{[4]} = x^l e^{-x/2}(x^{-\nu-(l+1)} \cdot e^{-x})_{l-\nu} \qquad (\S 2.\ (5))$$

$$= x^l e^{-x/2} \sum_{k=0}^{\infty} \frac{\Gamma(l-\nu+1)}{k!\,\Gamma(l-\nu+1-k)} (x^{-\nu-(l+1)})_{l-\nu-k} (e^x)_k \qquad (22)$$

$$= x^l e^{-x/2} \sum_{k=0}^{\infty} \frac{(-1)^k [\nu-l]_k}{k!} \{e^{-i\pi(l-\nu-k)} \frac{\Gamma(2l+1-k)}{\Gamma(\nu+l+1)} x^{-2l-1+k}\} e^x \qquad (23)$$

$$\left(\left| \frac{\Gamma(2l+1-k)}{\Gamma(\nu+l+1)} \right| < \infty \right)$$

$$= (e^{-i\pi})^{l-\nu} x^{-l-1} e^{x/2} \sum_{k=0}^{\infty} \frac{(-1)^k\ [\nu-l]_k\ (-1)^{-k}\ \Gamma(2l+1)}{k!\ \Gamma(\nu+l+1)\,[-2l]_k} (-x)^k \qquad (24)$$

$$= (e^{-i\pi})^{l-\nu} x^{-l-1} e^{x/2} \frac{\Gamma(2l+1)}{\Gamma(\nu+l+1)}\, {}_1F_1(\nu-l;\ -2l\ ;\ -x), \quad (|-x|<1)\,. \qquad (4)$$

Proof of Group II.

We can obtain the solutions in Group II from the ones $\varphi_{[1]} \sim \varphi_{[4]}$ in Group I, setting $-(l+1)$ instead of l , respectively. (cf. § 2.)

§ 4. Commentary

[I] In the quantum mechanics, it is requested that

$$l \le (n-1) \in Z_0^+ \,;\, (l = 0, 1, 2, \cdots, n-1), \tag{1}$$

n ; The principal quantum number, $(n = \nu)$

l ; Orbital angular momentum quantum number.

Moreover, for the relationship (1), we have

(i) $$\varphi_{[1]} = \varphi_{[2]} = x^l e^{x/2} (x^{n-(l+1)} \cdot e^{-x})_{l+n} \qquad ((l+n) \in Z^+) \tag{2}$$

$$= (e^{-i\pi})^{l+n} e^{-x/2} x^{-l-1} \frac{\Gamma(2l+1)}{\Gamma(l+1-n)} {}_1F_1(-l-n\,;\,-2l\,;\,x), \quad (|x|<1), \tag{3}$$

and

(i i) $$\varphi_{[5]} = \varphi_{[6]} = x^{-(l+1)} e^{x/2} (x^{n+l} \cdot e^{-x})_{n-(l+1)} \qquad ((n-l-1) \in Z_0^+) \tag{4}$$

$$= (e^{-i\pi})^{n-l-1} x^l e^{-x/2} \frac{\Gamma(-2l-1)}{\Gamma(-n-l)} {}_1F_1(1+l-n\,;\,2l+2\,;\,x), \quad (|x|<1), \tag{5}$$

respectively.

We have the below classically.

$$\varphi_{[1]} = \varphi_{[2]} = x^l e^{x/2} D^{l+n} (x^{n-(l+1)} \cdot e^{-x}) \qquad (D = d/dx) \tag{6}$$

and

$$\varphi_{[5]} = \varphi_{[6]} = x^{-(l+1)} e^{x/2} D^{n-(l+1)} (x^{n+l} \cdot e^{-x}) \quad , \tag{7}$$

from (2) and (4), respectively.

Next we have

(iii) $$\varphi_{[3]} = \varphi_{[4]} = x^l e^{-x/2} (e^x \cdot x^{-n-(l+1)})_{l-n} \tag{8}$$

$$= x^l e^{-x/2} (e^x \cdot x^{-n-(l+1)})_{-m} \qquad ((n-l) = m \in Z^+) \tag{9}$$

$$= x^l e^{-x/2} \int \cdots \int e^x x^{-(n+l+1)} (dx)^m \tag{10}$$

and

(iv) $$\varphi_{[7]} = \varphi_{[8]} = x^{-(l+1)} e^{-x/2} (e^x \cdot x^{-n+l})_{-n-(l+1)} \tag{11}$$

$$= x^{-(l+1)} e^{-x/2} (e^x \cdot x^{-n+l})_{-m} \qquad ((n+l+1) = m \in Z^+) \tag{12}$$

$$= x^{-(l+1)} e^{-x/2} \int \cdots \int e^x x^{-n+l} (dx)^m , \tag{13}$$

omitting the additional arbitraly constants of the integrations.

Note. We have (for $m, n \in Z_0^+$)

$L_n(x) = e^x D^n (e^{-x} x^n)$; Laguerre function.

$L_n^m(x) = D^m L_n(x)$; Associated Laguerre function.

[II] We have

$$\varphi_2 \cdot x^2 + \varphi_1 \cdot 2x + \varphi \cdot \{-(1/4)x^2 + nx - n(n-1)\} = 0 , \tag{14}$$

from §2. (1), setting $\nu = n$ and $l = n-1$.

And when $n = l+1$ we have

$$\varphi = \varphi_{[5]} = \varphi_{[6]} = x^{-n} e^{x/2} (e^{-x} \cdot x^{2n-1})_0 = x^{n-1} e^{-x/2} \tag{15}$$

from (4). The function shown by (15) satisfies equation (14) clrearly.

[III] The homogeneous Fukuhara equation is given by

$$\varphi_2 + \varphi_1 \cdot (a + b/x) + \varphi \cdot (p + q/x + r/x^2) = 0, \quad (x \neq 0) \tag{16}$$

that is,

$$\varphi_2 \cdot x^2 + \varphi_1 \cdot (ax^2 + bx) + \varphi \cdot (px^2 + qx + r) = 0. \tag{17}$$

Therefore, we have

$$\varphi_2 \cdot x^2 + \varphi_1 \cdot 2x + \varphi \cdot \{(-1/4)x^2 + \nu x - l(l+1)\} = 0 \qquad (\S 2.\ (1))$$

from (17), setting

$$a = 0, b = 2, p = -1/4, \ q = \nu \ \text{and} \ r = -l(l+1) . \tag{18}$$

Namely, the radial Schrodinger equation of the Hydrogen atom is a special one of the homogeneous Fukuhara equation (16).

(cf. Fractional Calculus, Vol. 5, (1996) pp.136 - 142 ; by K. Nishimoto)

[IV]. Usually the solutions to the equation so-called special differential equations such as the above ones are obtained by means of Frobenius.

Then compare the our solving manner (Method of NFCO) with that of Frobenius and others.

[V] The equation § 2. (1) is shown in Chapter 5 in the volume of Ian N. Sneddon (Special Functions of Mathematical Physics and Chemistry (1961); Oliver and Boyd, Edingburgh.) in the form

$$\frac{d^2R}{dx^2}+\frac{2}{x}\cdot\frac{dR}{dx}-\left\{\frac{1}{4}-\frac{\nu}{x}+\frac{l(l+1)}{x^2}\right\}R=0, \quad (x\neq 0) \tag{19}$$

and a solution is given by

$$R=R_{n,l}(x)=e^{-(1/2)x}x^l\, L_{n+l}^{2l+1}(x) \quad (n\geq l+1). \tag{20}$$

[VI] Indeed we have (see the Note above)

$$R=R_{n,l}(x)=e^{-(1/2)x}x^l\, D^{2l+1}(e^xD^{n+l}(e^{-x}x^{n+l})), \tag{21}$$

hence we obtain

$$R=R_{1,0}(x)=e^{-(1/2)x}D^1(e^xD^1(e^{-x}x))=-e^{-(1/2)x}, \tag{22}$$

for $l=0$ and $n=1$.

And we obtain, when $l=0$ and $\nu=n=1$,

$$\varphi_{[1]}=\varphi_{[2]}=x^le^{x/2}(e^{-x}\cdot x^{\nu-(l+1)})_{l+\nu}=e^{x/2}(e^{-x}\cdot 1)_1 \tag{23}$$

$$=-e^{-x/2} \ . \tag{24}$$

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