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Strongly starlikeness criteria for certain analytic functions

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Abstract

Let $\mathcal{U}_3(\lambda)$ be the subcless of analytic functions $f(z)$ in the open unit disk $\mathbb{U}$ which was introduced by S. Ponnusamy (Appl. Math. Lett. **24**(2011), 381 - 385). For $f(z)\in\mathcal{U}_3(\lambda)$, some condition for the domain of $|z|$ such that $f(z)$ is strongly starlike of order γ in $\mathbb{U}$.

1 Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

(1.1) $$f(z)=z+\sum_{n=2}^{\infty}a_nz^n$$

that are analytic in the open unit disk $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$, and let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of $f(z)$ which are univalent in $\mathbb{U}$.

Obradović and Ponnusamy [2] define the class $\mathcal{U}(\lambda)$ of $f(z)\in\mathcal{A}$ satisfying the condition

(1.2) $$\left|\left(\frac{z}{f(z)}\right)^2 f'(z)-1\right|<\lambda \qquad (z\in\mathbb{U})$$

for some real $\lambda>0$.

The condition (1.2) is equivalent to

$$\left|z^2\left(\frac{1}{f(z)}-\frac{1}{z}\right)'\right|<\lambda \qquad (z\in\mathbb{U}).$$

Ponnusamy [3] introduces the class $\mathcal{U}_3(\lambda)$ of function $f(z)\in\mathcal{U}(\lambda)$ for which $a_3-a_2^2=0$.

For some real $\gamma\in(0,1]$, a function $f(z)\in\mathcal{A}$ is called strongly starlike of order γ if

(1.3) $$\left|\arg\frac{zf'(z)}{f(z)}\right|<\frac{\pi\gamma}{2} \qquad (z\in\mathbb{U}).$$

We deote by $\mathcal{SS}(\gamma)$ the set of all strongly starlike functions of order γ in $\mathbb{U}$.

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Ponnusamy [3] has shown the following theorem.

Theorem 1.1 *Let* $f(z) \in \mathcal{U}_3(\lambda)$, $\gamma \in (0,1]$, *and*

$$\lambda_*(\gamma, |a_2|) = \frac{-2(1+2\cos\frac{\pi\gamma}{2})|a_2| + 2\sin\frac{\pi\gamma}{2}\sqrt{5+4\cos\frac{\pi\gamma}{2} - 4|a_2|^2}}{5+4\cos\frac{\pi\gamma}{2}}.$$

Then $f(z) \in \mathcal{SS}(\gamma)$ *for* $0 < \lambda \leq \lambda_*(\gamma, |a_2|)$.

The aim of this paper is to derive a condition for the domain of $f(z) \in \mathcal{U}_3(\lambda)$ to be in the class $\mathcal{SS}(\gamma)$.

2 Main Result

Suppose that $f \in \mathcal{U}_3(\lambda)$. Then a simple calculation shows that

$$-z\left(\frac{z}{f(z)}\right)' + \left(\frac{z}{f(z)}\right) = \left(\frac{z}{f(z)}\right)^2 f'(z) \tag{2.1}$$

$$= 1 + A_3 z^3 + \cdots = 1 + \lambda w(z), \ w(z) \in \mathcal{B}_3,$$

where $\mathcal{B}_3$ denotes the set of all analytic functions $w(z)$ in $\mathbb{U}$ such that $w(0) = w(0)' = w(0)'' = 0, w'''(0) \neq 0$ and $|w(z)| < 1$ for $z \in \mathbb{U}$. From (2.1), we easily have the following representation for $\dfrac{z}{f(z)}$:

$$\frac{z}{f(z)} - 1 = -a_2 z - \lambda \int_0^1 \frac{w(tz)}{t^2} dt. \tag{2.2}$$

Since $w(z) \in \mathcal{B}_3$, from the Schwarz lemma

$$|w(z)| \leq |z|^3 \tag{2.3}$$

holds true. Thus, we have that

$$\left|\frac{z}{f(z)} - 1\right| \leq |z|\left(|a_2| + \frac{\lambda}{2}|z|^2\right), \ z \in \mathbb{U}. \tag{2.4}$$

We have the following result.

Theorem 2.1
If $f(z) \in \mathcal{U}_3(\lambda)$, *then* $f(z) \in \mathcal{SS}(\gamma)$ *for* $|z| < \min\{1, r_o\}$, *where* $r_0 = \sqrt{R_0}$ *for the positive root* R_0 *of the equation*

$$\lambda^2\left(\frac{5}{4} + \cos\frac{\pi\gamma}{2}\right)X^3 + \lambda|a_2|\left(1 + 2\cos\frac{\pi\gamma}{2}\right)X^2 + |a_2|^2 X + \cos^2\frac{\pi\gamma}{2} - 1 = 0. \tag{2.5}$$

Proof

Suppose that $f(z) \in \mathcal{U}_3(\lambda)$. Then we can see from (2.1) and (2.3) that

$$\left| \left(\frac{z}{f(z)}\right)^2 f'(z) - 1 \right| = \lambda |w(z)| \leq \lambda |z|^3. \tag{2.6}$$

Therefore, it follows from (2.4) and (2.6) that

$$\begin{aligned} \left|\arg\frac{zf'(z)}{f(z)}\right| &\leq \left|\arg\left(\left(\frac{z}{f(z)}\right)^2 f'(z)\right)\right| + \left|\arg\frac{z}{f(z)}\right| \\ &\leq \arcsin(\lambda|z|^3) + \arcsin\left(|z|\left(|a_2| + \frac{\lambda}{2}|z|^2\right)\right) \\ &= \arcsin\left(\lambda|z|^3\sqrt{1-|z|^2\left(|a_2|+\frac{\lambda}{2}|z|^2\right)^2} + \sqrt{1-\lambda^2|z|^6}|z|\left(|a_2|+\frac{\lambda}{2}|z|^2\right)\right). \end{aligned} \tag{2.7}$$

Now, we have to find the range of $|z|$ for $f(z) \in \mathcal{SS}(\gamma)$ such that

$$\arcsin\left(\lambda|z|^3\sqrt{1-|z|^2\left(|a_2|+\frac{\lambda}{2}|z|^2\right)^2} + \sqrt{1-\lambda^2|z|^6}|z|\left(|a_2|+\frac{\lambda}{2}|z|^2\right)\right) < \frac{\pi\gamma}{2}, \tag{2.8}$$

which is equivalent to

$$\lambda|z|^3\sqrt{1-|z|^2\left(|a_2|+\frac{\lambda}{2}|z|^2\right)^2} + \sqrt{1-\lambda^2|z|^6}|z|\left(|a_2|+\frac{\lambda}{2}|z|^2\right) < \sin\frac{\pi\gamma}{2}. \tag{2.9}$$

Putting

$$F(X) = \lambda^2\left(\frac{5}{4} + \cos\frac{\pi\gamma}{2}\right)X^3 + \lambda|a_2|\left(1 + 2\cos\frac{\pi\gamma}{2}\right)X^2 + |a_2|^2 X - \sin^2\frac{\pi\gamma}{2} \tag{2.10}$$

and

$$G(X) = \lambda^2\left(\frac{5}{4} - \cos\frac{\pi\gamma}{2}\right)X^3 + \lambda|a_2|\left(1 - 2\cos\frac{\pi\gamma}{2}\right)X^2 + |a_2|^2 X - \sin^2\frac{\pi\gamma}{2}, \tag{2.11}$$

(2.9) can be written as $F(X)G(X) > 0$ with $X = |z|^2$. Since $F(0) < 0$ and $G(0) < 0$, $F(X)G(X) > 0$ is equivalent to $F(X) < 0$ and $G(X) < 0$. Comparing the coefficients of $F(X)$ and $G(X)$, we easily find the inequality $G(X) < F(X)$.

In order to find the condition of $|z|$ such that $f(z) \in \mathcal{U}_3(\lambda)$ to be in $\mathcal{SS}(\gamma)$, we consider the condition for $F(X) < 0$.
Since

$$F'(X) = 3\lambda^2\left(\frac{5}{4} + \cos\frac{\pi\gamma}{2}\right)X^2 + 2\lambda|a_2|\left(1 + 2\cos\frac{\pi\gamma}{2}\right)X + |a_2|^2 > 0$$

and $\lim_{X\to\infty} F'(X) = \infty$,

$F(X)$ is an increasing function for X. Thus $F(X)$ has a positive root $R_0 > 0$. Therefore, for $|z| < \min\{1, R_0\}$, inequality (2.9) holds.

Remark

Substituting $X = 1$ and solving the equation (2.5) as the equation of λ, we have $\lambda_*(\gamma, |a_2|)$ of Theorem1.1 .

3 Example

We give an example which shows the existance of r_0 satisfying Theorem 2.1 . Since $F(X)$ has a unique solution for $0 < X < 1$ if $F(1) > 0$, we consider a condition of $|a_2|$ for $F(1) > 0$.

From

$$F(1) = \lambda^2\left(\frac{5}{4} + \cos\frac{\pi\gamma}{2}\right) + \lambda|a_2|\left(1 + 2\cos\frac{\pi\gamma}{2}\right) + |a_2|^2 - \sin^2\frac{\pi\gamma}{2} > 0,$$

we have

$$\left(|a_2| + \frac{\lambda\left(1 + 2\cos\frac{\pi\gamma}{2}\right)}{2}\right)^2 > \sin^2\frac{\pi\gamma}{2}(1 - \lambda^2),$$

This gives us that

$$|a_2| > \sin\frac{\pi\gamma}{2}\sqrt{1-\lambda^2} - \frac{\lambda\left(1 + 2\cos\frac{\pi\gamma}{2}\right)}{2} \quad (0 < \lambda < 1). \tag{3.1}$$

If $|a_2|$ satisfies the condition (3.1), $F(X)$ has a positive root for $0 < X < 1$.

Let us take $\lambda = \dfrac{1}{2}$ and $\gamma = \dfrac{2}{3}$. Then $|a_2| > \dfrac{1}{4}$ from (3.1). Thus, we may take $|a_2| = 1$ and

$$F(X) = 7X^3 + 16X^2 + 16X - 12. \tag{3.2}$$

Since $F(0) = -12 < 0$, and $F(1) = 27 > 0$, $F(X)$ has a real positive root $0 < X < 1$. Actually, the root r_0 of (3.2) satisfies $0.47605 < r_0 < 0.47615$.

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