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ON SPECIAL VALUES OF TENSOR PRODUCT L-FUNCTIONS OF AN INNER FORM OF GSP(4) AND GL(2)

KAZUKI MORIMOTO
(GRADUATE SCHOOL OF SCIENCE, OSAKA CITY UNIVERSITY)

ABSTRACT. We consider the Rankin-Slebrg integral which represents degree 8 tensor product L-functions for quaternion unitary groups and GL_2. Using this integral representation, we prove the algebriacity of special values.

1. SET UP

Let F be a number field and E a quadratic extension. For each $n\in\mathbb{N}$, we define the similitude unitary group $G_n=\mathrm{GU}(n,n)$:

$$G_n(F)=\{g\in\mathrm{GL}(2n,E)\mid {}^tg^\sigma J_ng=\lambda_n(g)J_n,\ \lambda_n(g)\in F^\times\}$$

where σ is non-trivial element in $\mathrm{Gal}(E/F)$ and

$$J_n=\begin{pmatrix}0_n & 1_n\\ -1_n & 0_n\end{pmatrix}.$$

Let $E\subset D$ be a quaternion algebra over F. For $x\in D$, we mean the canonical involution by $\bar{x}$. For a matrix $A=(a_{ij})$ with entries in D, we denote the matrix $(\overline{a_{ij}})$ by $\bar{A}$.

Let us define the quaternion similitude unitary group H_D by

$$H_D(F)=\left\{g\in\mathrm{GL}(2,D)\mid {}^t\bar{g}\begin{pmatrix}0&1\\1&0\end{pmatrix}g=\lambda(g)\begin{pmatrix}0&1\\1&0\end{pmatrix},\ \lambda(g)\in F^\times\right\}.$$

When $D\simeq M_2(F)$, we have an isomorphism

$$H_D(F)\simeq\mathrm{GSp}(4,F)=G_2(F)\cap\mathrm{GL}(4,F).$$

We note that we can take $\varepsilon\in F^\times$ such that

$$D\simeq\left\{\begin{pmatrix}a&\varepsilon b\\ b^\sigma & a^\sigma\end{pmatrix}\mid a,b\in E\right\}.$$

Thus we may suppose that $D\subset\mathrm{Mat}_{2\times2}(E)$, so that we can consider H_D as a subgroup of $\mathrm{GL}(4,E)$. In fact, H_D can be embedded into G_2, and we fix it . Let us define a subgroup H of $G_1\times G_2$ by

$$H=\{(g_1,h_2)\in G_1\times H_D\mid \lambda_1(g_1)=\lambda_2(h_2)\},$$

and we regard H as a subgroup of G_3 by the following embedding

$$H\ni\left(\begin{pmatrix}a&b\\c&d\end{pmatrix},\begin{pmatrix}A&B\\C&D\end{pmatrix}\right)\hookrightarrow\begin{pmatrix}a&0&b&0\\0&A&0&B\\c&0&d&0\\0&C&0&D\end{pmatrix}\in G_3.$$

2. Global integral

Let $P = MN$ denote the Siegel parabolic subgroup of G_3 where

$$M(F) = \left\{ \begin{pmatrix} g & 0 \\ 0 & \lambda \cdot ({}^t g^{\sigma})^{-1} \end{pmatrix} \mid g \in \mathrm{GL}_3(E), \lambda \in F^{\times} \right\},$$
$$N(F) = \left\{ \begin{pmatrix} 1_3 & X \\ 0 & 1_3 \end{pmatrix} \mid {}^t X^{\sigma} = X \in \mathrm{Mat}_{3\times 3}(E) \right\}.$$

Let ν be a character of $\mathbb{A}_E^{\times}/E^{\times}$ and τ a character of $\mathbb{A}_F^{\times}/F^{\times}$. Then we define a character $\nu \otimes \tau$ of $P(\mathbb{A}_F)$ by

$$(\nu \otimes \tau) \left[\begin{pmatrix} g & 0 \\ 0 & \lambda \cdot ({}^t g^{\sigma})^{-1} \end{pmatrix} \begin{pmatrix} 1_3 & X \\ 0 & 1_3 \end{pmatrix} \right] = \nu(\det g) \cdot \tau(\lambda).$$

Let δ_P denote the modulus character of $P(\mathbb{A}_F)$. Then let $I(s, \nu \otimes \tau)$ denote the normalized degenerate principal series representation $\mathrm{Ind}_{P(\mathbb{A}_F)}^{G(\mathbb{A}_F)}((\nu \otimes \tau) \cdot \delta_P^s)$ of $G(\mathbb{A}_F)$. Here we employ the normalized induction so that $I(s, \nu \otimes \tau)$ is unitarizable when $\mathrm{Re}(s) = 0$. Then for a holomorphic section $f^{(s)}$ of $I(s, \nu \otimes \tau)$ we have the Siegel Eisenstein series defined by

$$E(g, f^{(s)}) = \sum_{\gamma \in P(F)\backslash G(F)} f^{(s)}(\gamma g).$$

This series is absolutely convergent in the right half plane $\mathrm{Re}(s) > \frac{1}{2}$ (Langlands [5]).

Let σ be an irreducible cuspidal representation of $\mathrm{GL}_2(\mathbb{A}_F)$ and let χ be a character of $\mathbb{A}_E^{\times}/E^{\times}$ such that

$$\chi|_{\mathbb{A}_F^{\times}} = \omega_{\sigma} \tag{2.0.1}$$

where ω_{σ} denotes the central character of σ. Since we have the isomorphism

$$G_1(F) \simeq \left(\mathrm{GL}(2, F) \times E^{\times}\right) / \{(a, a^{-1}) \mid a \in F^{\times}\},$$

we can regard $\sigma \boxtimes \chi$ as the irreducible cuspidal automorphic representation of $G_1(\mathbb{A}_F)$ and we denote it by π. Let V_{π} be the space of automorphic forms for π.

Let (Π, V_{Π}) be an irreducible cuspidal automorphic representation of $H_D(\mathbb{A}_F)$. Let ω_{Π} denote the central character of Π. Then we study a global integral defined by

$$Z(f^{(s)}, \phi, \Phi) = \int_{Z(\mathbb{A}_F)H(F)\backslash H(\mathbb{A}_F)} E(f^{(s)}, h)\Psi(g_1)\Phi(h_2)dh \tag{2.0.2}$$

for $f^{(s)} \in I(s, \nu \otimes \tau), \Psi \in V_{\pi}$ and $\Phi \in V_{\Pi}$, where $Z = Z_G \cap H$, Z_{G_3} denotes the center of G_3, and $h = (g_1, h_2) \in H$. Here in order for the integral (2.0.2) to be well-defined, we assume that

$$\omega_{\Pi} \cdot \omega_{\sigma} \cdot \tau^2 \cdot (\nu|_{\mathbb{A}_F^{\times}})^3 = 1.$$

Proposition 2.1. *For $Re(s) \gg 0$, we have*

$$Z(f^{(s)}, \Psi, \Phi) = \int_{S(\mathbb{A}_F)\backslash H(\mathbb{A}_F)} f^{(s)}(\eta h) W_{\Psi}(g_1) B_{\Phi}(h_2) dh$$

where B_Φ is the Bessel model of Φ with respect to a non-split torus and W_Ψ is the Whittaker model of Ψ, and S is defined as follows: Let us define the Bessel subgroups R of H_D by

$$R(F) = \left\{ \begin{pmatrix} a^\sigma & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a^\sigma & 0 \\ 0 & 0 & 0 & a \end{pmatrix} \begin{pmatrix} 1 & 0 & \varepsilon b & c \\ 0 & 1 & c^\sigma & b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in G_2(F) \mid a \in E^\times, b \in F, c \in E \right\}.$$

Then a subgroup S of H is defined by

$$S = \{(\varphi(r), r) \mid r \in R\}$$

where we denote

$$\varphi\left[\begin{pmatrix} a^\sigma & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a^\sigma & 0 \\ 0 & 0 & 0 & a \end{pmatrix} \begin{pmatrix} 1 & 0 & \varepsilon b & c \\ 0 & 1 & c^\sigma & b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right] = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix}.$$

Remark. Our integral representation is a generalization to the similitude quaternion unitary case of Saha's interpretation [11] of Furusawa's integral [2]. Note that we unfold the Rankin-Selberg integral involving the Siegel Eisenstein series on G_3 directly without recourse to the Klingen Eisenstein series on G_2. Thus even when $H_D \simeq \mathrm{GSp}(4)$, our local integral is totally different from Saha's.

In order for our investigation to be non-vacuous, we assume that

$$\Pi \text{ has a Bessel model of non-split type.}$$

We note that by the result of Li [6], any irreducible cuspidal automorphic representation of $H_D(\mathbb{A})$ has a Bessel model of this type if D does not split. Moreover if $D \simeq \mathrm{Mat}_{2\times 2}(F)$, i.e., $H_D \simeq \mathrm{GSp}(4)$, Π has a Whittaker model or a Bessel model of some type. If Π is associated to a holomorphic cusp form, it is non-generic, and Pitale-Schmidt [8] shows that it does not have a Bessel model of split type. Thus such automorphic representations satisfy the above assumption.

The uniqueness of Bessel model is expected for any irreducible admissible representations of $H_D(F_v)$. However as far as the author knows, there is no reference which proves the uniqueness in general. For example, for unramified representations of $\mathrm{GSp}(4, F_v)$, Sugano [12] proves the uniqueness. Then by the uniqueness of Bessel model and Whittaker model, we obtain

$$Z(s) = \prod_{v \notin S} Z_v(W_{\Psi,v}, B_{\Phi,v}, f_v^{(s)}) \cdot Z_S(W_{\Psi,S}, B_{\Phi,S}, f_S^{(s)}).$$

Here S is a finite set of places such that any place $v \notin S$ is finite and satisfies

(1) 2 does not divide v
(2) E_v/F_v is unramified quadratic extension or $E_v \simeq F_v \oplus F_v$
(3) $\Pi_v, \pi_v, \nu_v, \tau_v$ are unramified.
(4) $D(F_v) \simeq \mathrm{Mat}_{2\times 2}(F_v)$.

Then Furusawa and Ichino computed unramified local integrals explicitly.

Proposition 2.2 (Furusawa-Ichino, Appendix in [7]). *Suppose $v \notin S$. For normalized spherical vectors W_v, B_v and $f_v^{(s)}$, we have*

$$Z_v(s) = \prod_{i=1}^{3} L\left(6s+i, \nu|_{F_v^\times} \cdot \varepsilon_{E_v/F_v}^{i+3}\right)^{-1} \cdot L\left(3s+\frac{1}{2}, \Pi \times \sigma \times (\nu|_{F^\times})^2 \times \tau\right)$$

where we normalize the measure on $H(F_v)$ suitably, and ε_{E_v/F_v} is the quadratic character of $F_v^\times$ corresponding to E_v via local class field theory.

3. Main Theorem

Assume that

$$H_D(\mathbb{R}) \simeq \mathrm{GSp}(4,\mathbb{R}) \quad \text{and} \quad F = \mathbb{Q}.$$

We possibly have $D \simeq \mathrm{Mat}_{2\times 2}(\mathbb{Q})$. We suppose that the central characters of Π and π are trivial.

Suppose that the archimedean component Π_∞ of Π is the holomorphic discrete series of $\mathrm{PGSp}(4,\mathbb{R})$ with Harish-Chandra parameter $\ell(e_1+e_2)$ with even integer ℓ where we define

$$e_i\left(\begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & t_1^{-1} & \\ & & & t_2^{-1} \end{pmatrix}\right) = t_i \quad t_i \in \mathbb{G}_m.$$

Suppose that σ is a cuspidal automorphic representation associated to a new form of weight ℓ. Then we consider an automorphic form $\Psi \in V_\sigma$ as the automorphic form on $G_1(\mathbb{A})$ by extending it trivially, i.e.

$$\Psi(ag) = \Psi(g)$$

for $a \in \mathbb{A}_E^\times$ and $g \in \mathrm{GL}(2,\mathbb{A}_\mathbb{Q})$.

Theorem 3.1. *Suppose that $\ell > 6$. Let $\Phi \in V_\Pi$ and $\Psi \in V_\sigma$ be arithmetic automorphic forms in the sense of Harris [4]. Then for an integer m such that $2 < m \leq \frac{\ell}{2}-1$, we have*

$$\frac{L(m, \Pi\times\sigma)}{\pi^{4m}\langle \Psi\otimes\Phi, \Psi\otimes\Phi\rangle} \in \overline{\mathbb{Q}}$$

and

$$\left(\frac{L(m, \Pi\times\sigma)}{\pi^{4m}\langle \Psi\otimes\Phi, \Psi\otimes\Phi\rangle}\right)^\tau = \frac{L(m, \Pi^\tau\times\sigma^\tau)}{\pi^{4m}\langle \Psi^\tau\otimes\Phi^\tau, \Psi^\tau\otimes\Phi^\tau\rangle}$$

for all $\tau \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Here we define

$$\langle \Psi\otimes\Phi, \Psi\otimes\Phi\rangle = \int_{Z_H(\mathbb{A}_\mathbb{Q})H(\mathbb{Q})\backslash H(\mathbb{A}_\mathbb{Q})} |\Psi(g_1)\Phi(h_2)|^2\, dh$$

where we denote $h = (g_1, h_2) \in H(\mathbb{A}_\mathbb{Q})$, and dh is the Tamagawa measure on $H(\mathbb{A}_\mathbb{Q})$.

We can prove this by a similar way with Garrett-Harris [3]. For a detail of the proof, we refer to [7].

3.1. **Period Relation.** Let (Π, V_Π) be an irreducible cuspidal automorphic representation of $\mathrm{GSp}(4, \mathbb{A}_\mathbb{Q})$ as in Theorem 3.1. Further we assume that Π is tempered and non-endoscopic. We suppose that there exists an irreducible cuspidal automorphic representation (Π_D, V_{Π_D}) of $H_D(\mathbb{A}_\mathbb{Q})$ such that for every place v such that $H_D(\mathbb{Q}_v) \simeq \mathrm{GSp}(4, \mathbb{Q}_v)$,

$$\Pi_v \simeq \Pi_{D,v}.$$

Then Π_D satisfies the condition in Theorem 3.1. Comparing the equations in Theorem 3.1 for Π and Π_D, we obtain the following relation.

Corollary 3.1. *For any arithmetic forms* $\Phi \in V_\Pi$ *and* $\Phi_D \in V_{\Pi_D}$, *we have*

$$\langle \Phi, \Phi \rangle / \langle \Phi_D, \Phi_D \rangle \in \overline{\mathbb{Q}}$$

and

$$(\langle \Phi, \Phi \rangle / \langle \Phi_D, \Phi_D \rangle)^\tau = \langle \Phi^\tau, \Phi^\tau \rangle / \langle \Phi_D^\tau, \Phi_D^\tau \rangle$$

for any $\tau \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Here we define the pairing $\langle \Phi_D, \Phi_D \rangle$ by

$$\langle \Phi, \Phi \rangle = \int_{Z_{H_D}(\mathbb{A}_\mathbb{Q}) H_D(\mathbb{Q}) \backslash H_D(\mathbb{A}_\mathbb{Q})} |\Phi_D(h)|^2 \, dh$$

where dh is the Tamagawa measure on $H_D(\mathbb{A}_\mathbb{Q})$, and we define $\langle \Phi, \Phi \rangle$ similarly.

3.2. **Remarks on Theorem 3.1.**

3.2.1. *critical point.* The critical points in Theorem 3.1 does not cover all critical points on the right half plane $\mathrm{Re}(s) > 0$. Indeed the critical points for $s = \frac{1}{2}$ and $\frac{1}{6}$ are not included due to the analytic property of Eisenstein series.

3.2.2. *Split case.* When $H_D \simeq \mathrm{GSp}(4)$, similar results are proved by many people. Furusawa [2] discovered an integral representation of this L-function and he proved the algebriacity at the rightmost critical point for Siegel cusp forms and elliptic cusp form of full level. Pitale-Schmidt [9] extended his result with respect to the level of elliptic cusp forms, and Saha [10] extended with respect to both of levels of Siegel cusp forms and elliptic cusp form. Saha [11] also proved the algebraicity for other critical points combining the pull-back formula and differential operators. On the other hand, Böcherer-Heim [1] showed the algebraicity at all critical points in the full modular balanced mixed weight case using Heim's integral representation.

3.2.3. *Yoshida's Conjecture.* When the irreducible cuspidal automorphic representation of $\mathrm{GSp}(4, \mathbb{A}_\mathbb{Q})$ is associated to a Siegel cusp form, our result is compatible with Yoshida's calculation [13] on Deligne peirod.

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