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# Whittaker new vectors for discrete series representations of real Lie group $U(2,1)$ *

Yoshi-hiro Ishikawa
石川佳弘

**Introduction**

By definition, zeta integrals "interpolates" automorphic $L$-functions to deduce their some analytic properties, say meromorphic continuation. But to proceed into deeper arithmetic investigation, like as study of special values, we can not avoid the ramified factors of integrals. In this past decade, several nice works have been sprung out. However, the satisfactorily developed theories are essentially limited to the cases of $GL(2)$ and $GSp(4)$.

In this note we treat the Gelbart Piatetski-Shapiro integral for generic cusp forms on $U(3)$, which are recalled in §1. In §2, we report on Whittaker new vector for archimedean component of the integral. That is there exists a unique (up to constant) $K_\infty$-finite vector in Whittaker model of discrete series representation whose integral gives the Langlands $L$-factor.

## 1 Zeta integral and its $p$-adic factors

Note that we can obtain the same result without any loss of generality, even if we formulate the problem over an arbitrary totally real algebraic number field. So we take $\mathbb{Q}$ for our ground field and denote its adèle ring by $\mathbb{A}$.

**<Group structure>**

Let $E$ be an imaginary quadratic extension of $\mathbb{Q}$ and denote the non-trivial element of its Galois group by $\bar{\cdot}$. Put

$$G \ := \ \{g \in GL(3,E) \mid {}^t\bar{g} \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix} g = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}\}.$$

This defines a quasi-split unitary group of three variables over $\mathbb{Q}$. Let

$$G \ = \ BK, \qquad \text{with} \quad B \ = \ NM$$

*In the workshop, a part of talk was devoted to review the $H$-period investigation as an motivating introduction. But the main result is archimedean stuff, and we just report on it with this title, different from the one of talk.

be the Iwasawa decomposition of $G$. Then each subgroups are expressed as

$$N = \{\begin{pmatrix} 1 & b & z \\ & 1 & -\bar{b} \\ & & 1 \end{pmatrix} \in G \mid b, z \in E,\ z + \bar{z} = -|b|_E^2\},$$

$$M = \{\begin{pmatrix} \alpha & & \\ & \beta & \\ & & \bar{\alpha}^{-1} \end{pmatrix} \in G \mid \alpha \in E^\times, \beta \in E^{(1)}\}$$

and

$$K = G \cap M_3(\mathcal{O}_E),$$

where $\mathcal{O}_E$ is the ring of integers in $E$.

We need a subgroup

$$H := \mathrm{Img}\Big(\iota : U(1,1) \ni \begin{pmatrix} \star & \star \\ \star & \star \end{pmatrix} \mapsto \begin{pmatrix} \star & & \star \\ & 1 & \\ \star & & \star \end{pmatrix} \in G\Big)$$

as the Euler subgroup for a Rankin-Selberg integral. The Iwasawa decomposition of $H$ is

$$H = B_H K_H, \qquad \text{with } B_H = Z_N A, \quad K_H = K \cap H,$$

where

$$Z_N = \{\begin{pmatrix} 1 & & z \\ & 1 & \\ & & 1 \end{pmatrix} \in G \mid z \in \mathbb{R}\},$$

$$A = \{\begin{pmatrix} a & & \\ & 1 & \\ & & a^{-1} \end{pmatrix} \in G \mid a \in \mathbb{Q}^\times\}.$$

**<The standard $L$-function>**

For a cuspidal automorphic representation $\pi = \otimes_v \pi_v$ of $G(\mathbb{A}) = U(3)_{\mathbb{A}}$ and a Hecke character $\xi$ of $E$, the $\xi$-twisted $L$-function is defined by a local way as an Euler product

$$L(s; \pi \otimes \xi) := \prod_v L_v(s; \pi_v \otimes \xi_v).$$

When $\xi_p$ is unramified and $\pi_p$ is the unramified component of unramified principal series $\mathrm{Ind}_{B_p}^{G_p}(\chi)$, the unramified factor is given by

$$L_p(s; \pi_p \otimes \xi_p) := L_{E,p}(s; \xi_p) L_p(2s; \xi_p \chi) L_p(2s; \xi_p/\chi).$$

Here $\chi$ is a representation of the Borel subgroup $B_p = N_p M_p$ given by

$$\chi : n.\mathrm{diag}(\alpha, \beta, \bar{\alpha}^{-1}) \mapsto \chi_E(\alpha) \in \mathbb{C}^\times,$$

and $\chi_E$ is a character of $E_p^\times$ with conductor $\mathcal{O}_{E_p}^\times$.

**<Zeta integral>**

For a generic cusp form $\varphi$ belonging to generic $\pi$, Gelbart and Piatetski-Shapiro introduced the following zeta integral

$$\mathcal{Z}(s;\varphi,\xi) := \int_{H(F)\backslash H(\mathbb{A})H} \varphi|_H(h)\, E^H(s;h,\xi)\,\mathrm{d}h.$$

Here $E^H$ is an Eisenstein series on $H(\mathbb{A})$

$$E^H(s;h,\xi) \ := \sum_{\gamma\in B_H(\mathbb{Q})\backslash H(\mathbb{Q})} f_\xi^{(s)}(\gamma h),$$

where $f_\xi^{(s)}$ is a section in the principal series $\mathrm{Ind}_{B_H(\mathbb{A})}^{H(\mathbb{A})}(1_{N_H}\otimes\xi\otimes e^{2s})$, which is factorizable as $f_\xi^{(s)} = \otimes_v f_{\xi,v}^{(s)}$. By the Langlands theory of Eisenstein series the integral is continued to the whole $s$-plane.

**<Unfolding and local integrals>**

Assume the generic cusp form is localizable; $\varphi = \otimes_v\varphi_v$. By using the multiplicity one result on Whittaker models and an unfolding procedure, the Rankin-Selberg integral decomposes into a product of local integrals:

$$\mathcal{Z}(s;\varphi,\xi) = \prod_v \mathcal{Z}_v(s;W,f_\xi^{(s)}),$$

with

$$\mathcal{Z}_v(s;W,f_\xi^{(s)}) \ := \int_{Z_{N,v}\backslash H_v} W_{\varphi_v}|_{H_v}(h_v) f_\xi^{(s)}(h_v)\mathrm{d}h_v.$$

Here $Z_{N,v}$ is the center of the maximal nilpotent subgroup $N_v$ of $G_v$, $W_{\varphi_v}$ is a Whittaker vector

$$W_{\varphi_v}(g_v) \ := \ \ell_\psi(\pi_v(g_v)\cdot\varphi_v)$$

corresponding to $\varphi_v\in\pi_v$, where $\ell_\psi\in\mathrm{Hom}_{G_v}(\pi_v,\mathrm{Ind}_{N_n}^{G_v}\psi_{N_v})$ is a non-trivial functional. And $f_\xi^{(s)}$ is a special section of the principal series $\mathrm{Ind}_{B_{H,v}}^{H_v}(\xi|\cdot|^s)$ of $H_v$ induced up from its Borel subgroup $\iota(\begin{pmatrix} * & * \\ & * \end{pmatrix})$. Note that this integral vanishes unless $\varphi$ is generic.

Over the places where everything is unramified, Gelbart and Piatetski-Shapiro showed the coincidence of local factors of $L$-function and zeta integral by using the Casselman-Shalika formula.

**Proposition 1.1 ([Ge-PS] §4)** *For the unramified (i.e. $K_p$-spherical )$\pi_p$'s,*

$$\mathcal{Z}_p(s;W,f_\xi^{(s)}) \ = \ L_p(s;\pi_p\otimes\xi_p).$$

□

Next step of investigation is to analyze ramified factors. The $p$-adic case was treated by Baruch in his thesis [Ba], upon which Miyauchi succeeded to find "Whittaker new vector" by using his compact subgroup sequence. The detail would be reported in his article of this proceedings.

Apparently the big lacking is Archimedean study of the integral $\mathcal{Z}_\infty(s;W,f_\xi^{(s)})$.

## 2 Archimedean results

We consider the Archimedean component of Gelbart-PS integral. By the genericity of cuapidal representation $\pi$, the Archimedean component $\pi_\infty$ must be large. Here we treat the case of discrete series exclusively. That is $\pi_\infty \cong \pi_\Lambda$ with Harish-Chandra parameter $\Lambda = (\Lambda_1, \Lambda_2, \Lambda_3) \in \mathbb{Z}^3$ satisfying

$$\Lambda_1 \; > \; \Lambda_3 \; > \; \Lambda_2.$$

We parameterize the infinite component of Hecke character

$$\xi_\infty : \mathbb{C}^\times \ni \delta \mapsto |\delta|^{2t} \left(\frac{\delta}{|\delta|}\right)^m \in \mathbb{C}^\times,$$

$(t, m) \in \mathbb{C} \times \mathbb{Z}$ as usual. Then the Langlands factor defined by the $L$-parameter is of the form:

$$L_\infty(s; \pi_\Lambda, \xi_{(t,m)}) \; = \; \prod_{i=1}^{3} \Gamma_{\mathbb{C}}\left(s + t + |\Lambda_i| + \frac{|m|}{2}\right).$$

By the Cayley transform $\mathcal{C}$, our group is the unitary group for the Hermitian form $\mathrm{diag}(1, 1, -1)$. So the maximal compact subgroup $K_\infty$ is isomorphic to $U(2) \times U(1)$ and all the $K_\infty$-type $\tau \subset \pi_\Lambda$ can be parametrized by triple

$$\mu = [\mu_1, \mu_2; \mu_3] \; \in \; \left\{\Lambda + m[1, -1; 0] + n[1, 0; -1] \;\middle|\; m, n \in \mathbb{N}\right\},$$

where $(\mu_1, \mu_2)$ is the highest weight of $U(2)$-representation and $\mu_3$ is the parameter of $U(1)$-character.

For a $K_\infty$-finite vector $w$ of $\pi_\Lambda$ belonging to $\tau_\mu$, we denote the corresponding Whittaker function by

$$W^{(\mu,w)}(g) \; := \; \ell_\psi(\pi_\Lambda(g).w).$$

**Definition 2.1** *We say that $K_\infty$-finite Whittaker function $W^{(\mu,w)}$ is a Whittaker new vector for the Gelbart Piatetski-Shapiro integral if the equality*

$$\mathcal{Z}_\infty(s; W, f^{(s)}_{\xi,\Phi}) \; = \; c \times L_\infty(s; \pi_\Lambda, \xi_{(t,m)})$$

*can be attained by $W^{(\mu,w)}$ alone. Here $c$ is a non-zero constant and $f^{(s)}_{\xi,\Phi}$ is the section*

$$f^{(s)}_{\xi,\Phi}(h) \; := \; \int_{\mathbb{C}^\times} \Phi\left(h^{-1}.[z, z]\right) \xi(z) \, |z|^{2s} \, \mathrm{d}^\times z \; \in \; I^H(s; \xi)$$

*constructed from a Schwartz class $\Phi \in \mathcal{S}(\mathbb{C}^2)$, called as Jacquet section.* ■

**Theorem 2.2** *If the Harish-Chandra parameter satisfies the condition $\Lambda_1 + \Lambda_3 < 0$, then the large discrete series $\pi_\Lambda$ admits a Whittaker new vector in its Whittaker model*

$$W^{(\mu^{\mathrm{good}}, w^{\mathrm{good}})} \; \in \; \mathcal{W}h_\psi(\pi_\Lambda),$$

*which is unique up to constant multiple. The $K_{H\infty}$-finite Schwartz function $\Phi$ is also uniquely determined.* □

*Sketch of Pf.*) Our task is to specify "the good" $K_\infty$-type $\tau_{\mu^{\rm good}}$ of $\pi_\Lambda$, where "the good" $K_\infty$-finite vector $w^{\rm good}$ can be found. We can carry out it by the following steps.

**Step 1**. Obtain an explicit formula for the minimal $K_\infty$-type [1] Whittaker function $W^{(\Lambda,w)}$ for each

$$w = \left| \begin{matrix} \Lambda_1, \Lambda_2 \\ k \end{matrix} \right\rangle \otimes \mathbf{1}_{\Lambda_3} \in \tau_\Lambda,$$

where $\{\left| \begin{smallmatrix} \Lambda_1, \Lambda_2 \\ k \end{smallmatrix} \right\rangle \mid \Lambda_1 \geq k \geq \Lambda_2\}$ is the Gel'fand-Zetlin basis for the $U(2)$-representation with highest weight $\Lambda_1 > \Lambda_2$ and $\mathbf{1}_{\Lambda_3}$ the base of $U(1)$-character $(u \mapsto u^{\Lambda_3})$.

$$W\left(\begin{pmatrix} y & & \\ & 1 & \\ & & y^{-1} \end{pmatrix}\right) = \sum_{\Lambda_1 \geq k \geq \Lambda_2} \gamma_k^\lambda \cdot y^{\Lambda_1 - \Lambda_2 - \frac{1}{2}} W_{0,k-\Lambda_1-\Lambda_2+\Lambda_3}(2\sqrt{b_\psi} y) \times \left( \left| \begin{matrix} \Lambda_1, \Lambda_2 \\ k \end{matrix} \right\rangle \otimes \mathbf{1}_{\Lambda_3} \right).$$

Here $b_\psi$ is a constant controlling the normalization of additive character $\psi$, and $\gamma_k^\lambda$'s are normalizing constant depending on $\lambda$ and $\psi$.

**Step 2**. Write down the recursive relations among $K_\infty$-finite Whittaker vectors coming from the rank one differential operators.

**Step 3**. Normalize the additive character $\psi$ of $N_\infty$ to get two $\Gamma_{\mathbb{C}}$ from the Mellin transform of $W^{(\mu,w)}$. That is $b_\psi = \pi^2$. This step depends on the Cayley transform $\mathcal{C}$ that is on the Hermitian form.

**Step 4**. Normalize the Schwartz function as

$$\Phi_{m_1,n_1;m_2,n_2}(z_1, z_2) := \prod_{i=1}^{2} z_i^{m_i} \overline{z_i}^{n_i} \times \exp\left( - \pi |z_i|^2 \right),$$

where $(m_1, n_1; m_2, n_2) \in \mathbb{Z}_{\geq}^4$, to get one $\Gamma_{\mathbb{C}}$ from the integral definition of $f_{\xi,\Phi}^{(s)}(h)$.

**Step 5**. Regarding the zeta integral $\mathcal{Z}_\infty(s; W, f_{\xi,\Phi}^{(s)})$ as a $K_{H\infty}$-coupling between Whittaker vector and the Jacquet section for $\Phi_{m_1,n_1;m_2,n_2}$, we obtain the constraint among the parameters;

$$n_1 - m_1 = m_\xi + \Lambda_3, \qquad n_2 - m_2 = -\Lambda_3.$$

By using the relation in Step 2, we specify "the good" $K_\infty$-type satisfying the above constraint;

$$\mu^{\rm good} = \left[ m_\xi - |\Lambda|, |\Lambda| + \lambda_3; -m_\xi + |\Lambda| - \lambda_3 \right]$$

where $|\Lambda| := \lambda_1 + \lambda_2 + \lambda_3$.

**Step 6**. Finally, we find "the good" $K_\infty$-finite vector in $\tau_\mu^{\rm good}$ as

$$w^{\rm good} = \left| \begin{matrix} \mu_1^{\rm good}, \mu_2^{\rm good} \\ \lambda_1 + \lambda_2 - 2m_\xi \end{matrix} \right\rangle \otimes \mathbf{1}_{\mu_3^{\rm good}},$$

again by appealing to the recursive relations in Step 2. ■

[1]Because $\pi_\Lambda$ is large, the Blattner parameter coincides with the Harish-Chandra parameter $\Lambda$ in this case.

Here are some comments. It was Oda and Koseki who first tried to investigate the Archimedean component of Gelbart Piatetski-Shapiro integral. In [K-O] they treated GCD of whole $A_\infty$-radial part of $\mathcal{Z}_\infty(s; W, f^{(s)}_{\xi,\Phi})$ as an application of their explicit formula of Whittaker function on $SU(2,1)$. But $\mathcal{Z}_\infty(s; W, f^{(s)}_{\xi,\Phi})$ is integration on $A_\infty$ and on $K_{H\infty}$. So the quite many members in Koseki-Oda's family should be abandoned in the view point of GCD definition for local $L$-factor.

Even after considering $K_{H\infty}$-integral, the GCD of archimedean zeta integrals $\mathcal{Z}_\infty(s; W, f^{(s)}_\xi)$ for ALL the $K_\infty$-finite $W$ and normalized section $f^{(s)}_\xi$ has an odd form compared with the Langlands factor. We reported this phenomenon in the RIMS workshop 2006 [Is].

After all, the above proof shows that by taking GCD of $\mathcal{Z}_\infty(s; W, f^{(s)}_\xi)$ we can NOT gain the genuine Langlands factor.

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Yoshi-hiro Ishikawa
The Graduate School of Natural Science and Technology,
Department of Mathematics, Okayama University,
Naka 3-1-1 Tushima Okayama, 700-8530, Japan