数理解析研究所講究録
第 1826 巻 2013 年 36-50

Jacquet-Langlands-Shimizu correspondence for theta lifts to $GSp(2)$ and its inner forms

Hiro-aki Narita*and Takeo Okazaki
with an appendix by Ralf Schmidt

Abstract

As was first pointed out by Ibukiyama [I], the spinor L-functions of automorphic forms on the indefinite symplectic group $GSp(1,1)$ or the definite symplectic group $GSp^*(2)$ over $\mathbb{Q}$ right invariant by a (global) maximal compact subgroup are conjectured to be those of paramodular forms of some specified level on the symplectic group $GSp(2)$, which can be viewed as a generalization of the Jacquet-Langlands-Shimizu correspondence to the case of $GSp(2)$ and its inner forms $GSp(1,1)$ and $GSp^*(2)$.

This short note surveys our results presented at the RIMS-conference held during January 16-21 in 2012. They provide evidence of this conjecture by theta lifts from $GL(2)\times B^{\times}$ to the inner forms and theta lifts from $GL(2)\times GL(2)$ to $GSp(2)$ (considered by [O]), where B denotes a definite quaternion algebra over $\mathbb{Q}$. Our explicit functorial correspondence given by these theta lifts are proved to be compatible with a non-archimedean local Jacquet-Langlands correspondence for $GSp(2)$ (or $GSp(4)$) and its inner forms, which is considered in the appendix by Ralf Schmidt.

1 Basic facts

1.1 Algebraic groups.

Let B be a definite quaternion algebra over $\mathbb{Q}$ with the discriminant d_B, and let $B \ni x \mapsto \bar{x} \in B$ be the main involution of B. By n and tr we denote the reduced norm and the reduced trace of B respectively.

Let $G_{\rm nc} = GSp(1,1)$ and $G^1_{\rm nc} = Sp(1,1)$ be the $\mathbb{Q}$-algebraic groups defined by

$$G_{\rm nc}(\mathbb{Q}) := \{g \in M_2(B) \mid {}^t\bar{g}Q_{\rm nc}g = \nu(g)Q_{\rm nc},\ \nu(g) \in \mathbb{Q}^{\times}\},\ G^1_{\rm nc}(\mathbb{Q}) := \{g \in G_{\rm nc}(\mathbb{Q}) \mid \nu(g) = 1\},$$

where $Q_{\rm nc} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Furthermore let $G_{\rm c} = GSp^*(2)$ and $G^1_{\rm c} = Sp^*(2)$ be the $\mathbb{Q}$-algebraic groups defined by

$$G_{\rm c}(\mathbb{Q}) := \{g \in M_2(B) \mid {}^t\bar{g}Q_{\rm c}g = \mu(g)Q_{\rm c},\ \mu(g) \in \mathbb{Q}^{\times}\},\ G^1_{\rm c}(\mathbb{Q}) := \{g \in G_{\rm c}(\mathbb{Q}) \mid \mu(g) = 1\},$$

*Partially supported by Grand-in-Aid for Young Scientists (B) 21740025, JSPS.

where $Q_c := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

On the other hand, let $G' = GSp(2)$ be the $\mathbb{Q}$-algebraic group defined by

$$G'(\mathbb{Q}) := \left\{ g \in GL_4(\mathbb{Q}) \,\middle|\, {}^t g \begin{pmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{pmatrix} g = \lambda(g) \begin{pmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{pmatrix}, \ \lambda(g) \in \mathbb{Q}^\times \right\}.$$

We should note that G_{nc} and G_{c} are inner $\mathbb{Q}$-forms of G'. By $Z_{\mathcal{G}}$ we denote the center of $\mathcal{G} = G_{\mathrm{nc}}$, G_{c} or G_{s}.

In what follows, we often put $G = G_{\mathrm{c}}$ or G_{nc}.

1.2 Maximal compact subgroups.

Let $Q = Q_{\mathrm{nc}}$ or Q_{c}. We first introduce maximal compact subgroups at the archimedean place. We put $G^1_\infty := \{g \in M_2(\mathbb{H}) \mid {}^t\bar{g}Qg = Q\}$, where $\mathbb{H} := B \otimes_{\mathbb{Q}} \mathbb{R}$ is the Hamilton quaternion algebra. Then G^1_∞ is the maximal compact subgroup itself when $Q = Q_{\mathrm{c}}$, and

$$K^0_\infty := \{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \in M_2(\mathbb{H}) \mid a \pm b \in \mathbb{H}^1 \}$$

forms a maximal compact subgroup of G^1_∞ when $Q = Q_{\mathrm{nc}}$, where $\mathbb{H}^1 := \{u \in \mathbb{H} \mid n(u) = 1\}$. The map $K^0_\infty \ni \begin{pmatrix} a & b \\ b & a \end{pmatrix} \mapsto (a+b, a-b) \in \mathbb{H}^1 \times \mathbb{H}^1$ gives rise to an isomorphism $K^0_\infty \simeq \mathbb{H}^1 \times \mathbb{H}^1$.

We next put $G'^1_\infty := \left\{ g \in GL_4(\mathbb{R}) \,\middle|\, {}^t g \begin{pmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{pmatrix} g = \begin{pmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{pmatrix} \right\}$. Then

$$K'^0_\infty := \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \,\middle|\, A + \sqrt{-1}B \in U(2) \right\}$$

is a maximal compact subgroup of G'^1_∞, where $U(2) := \{X \in M_2(\mathbb{C}) \mid {}^t\bar{X}X = 1_2\}$ denotes the unitary group of degree two. The map $K'^0_\infty \ni \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \mapsto A + \sqrt{-1}B \in U(2)$ induces an isomorphism $K'^0_\infty \simeq U(2)$.

Let us introduce maximal compact subgroups at non-archimedean places. We first deal with the case of $G = GSp(1,1)$ or $GSp^*(2)$. We remark that $GSp(1,1)$ and $GSp^*(2)$ are isomorphic to each other over $\mathbb{Q}_p$. We can thus identify $GSp(1,1)(\mathbb{Q}_p)$ with $GSp^*(2)(\mathbb{Q}_p)$.

We let D be a divisor of d_B and fix a maximal order $\mathfrak{O}$ of B. For $p|d_B$ let $\mathfrak{P}_p$ be the maximal ideal of the p-adic completion $\mathfrak{O}_p$ of $\mathfrak{O}$ and let

$$L_p := \begin{cases} {}^t(\mathfrak{O}_p \oplus \mathfrak{O}_p) & (p \nmid d_B \text{ or } p|D), \\ {}^t(\mathfrak{O}_p \oplus \mathfrak{P}_p^{-1}) & (p|\frac{d_B}{D}). \end{cases}$$

Then $K_p := \{k \in G_p \mid kL_p = L_p\}$ is a maximal compact subgroup of G_p for each finite prime p when $G = GSp(1,1)$ or $GSp^*(2)$. Every maximal compact subgroup of G_p is conjugate to some K_p by G_p.

Let us next deal with the case of $GSp(2)$. When p does not divide d_B, we put $K'_p := GSp(2)(\mathbb{Z}_p)$. When $p|d_B$ we put

$$K'_p := \begin{cases} \begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p & p^{-1}\mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & p\mathbb{Z}_p & \mathbb{Z}_p & p\mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix} \cap GSp(2)(\mathbb{Q}_p) & (p|\frac{d_B}{D}), \\ \begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p & p^{-2}\mathbb{Z}_p & \mathbb{Z}_p \\ p^2\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\ p^2\mathbb{Z}_p & p^2\mathbb{Z}_p & \mathbb{Z}_p & p^2\mathbb{Z}_p \\ p^2\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix} \cap GSp(2)(\mathbb{Q}_p) & (p|D). \end{cases}$$

We call this open compact subgroup of $GSp(2)(\mathbb{Q}_p)$ a paramodular subgroup of $GSp(2)(\mathbb{Q}_p)$ of level p or p^2, which is maximal when the level is p. We remark that $K_p \simeq K'_p$ for $p \nmid d_B$.

We note that we can identify $G_{\mathrm{nc}}(\mathbb{A}_f)$ with $G_{\mathrm{c}}(\mathbb{A}_f)$ since G_{nc} is isomorphic to G_{c} over $\mathbb{Q}_p$. Every maximal compact subgroup of $G(\mathbb{A}_f) = G_{\mathrm{nc}}(\mathbb{A}_f) = G_{\mathrm{c}}(\mathbb{A}_f)$ is $G(\mathbb{A}_f)$-conjugate to $K_f(D) := \prod_{p<\infty} K_p$ with $D|d_B$. In addition, we put $K'_f(D) := \prod_{p<\infty} K'_p$, which is an open compact subgroup of $G'(\mathbb{A}_f)$.

2 Theta lifts to GSp(1, 1), $GSp^*(2)$ and $GSp(2)$.

Let H and H' be $\mathbb{Q}$-algebraic groups defined by

$$H(\mathbb{Q}) = GL_2(\mathbb{Q}),\ H'(\mathbb{Q}) := B^\times$$

respectively. For a positive integer κ we let $S_\kappa(D)$ be the space of elliptic cusp forms of weight κ with level D (cf. [M-N-2, Section 3.1]). For a non-negative integer κ' we let $\mathcal{A}_{\kappa'}$ be the space of automorphic forms of weight $\sigma_{\kappa'}$ with respect to $\prod_{p<\infty} \mathfrak{O}_p^\times$ (cf. [M-N-2, Section 3.2]), where $\mathfrak{O}_p^\times$ denotes the unit group of $\mathfrak{O}_p$.

For Hecke eigenforms $(f, f') \in S_{\kappa_1}(D) \times \mathcal{A}_{\kappa_2}$ let $\pi(f)$ be the automorphic representation of $GL_2(\mathbb{A})$ generated by f and $\mathrm{JL}(\pi(f'))$ be the Jacquet-Langlands lift of the automorphic representation $\pi(f')$ generated by f'. The Hecke equivariant isomorphism between $\mathcal{A}_{\kappa_2}$ and the space of new forms in $S_{\kappa_2+2}(d_B)$ (Eichler [E-1], [E-2], Shimizu [Sh]) sends a Hecke eigenform f' to a primitive form $\mathrm{JL}(f')$. The automorphic representation $\mathrm{JL}(\pi(f'))$ is nothing but that generated by $\mathrm{JL}(f')$.

2.1 Theta lift to G

For every finite prime $p < \infty$ let $\mathbb{V}_p$ be the space of locally constant compactly supported functions on $B_p^2 \times \mathbb{Q}_p^\times$. Let $\mathcal{S}(\mathbb{H}^2)$ stand for the space of Schwartz functions on $\mathbb{H}^2$. When $G = G_{\mathrm{nc}}$ (respectively $G = G_{\mathrm{c}}$) we then introduce the space $\mathbb{V}_\infty$ of smooth functions φ on $\mathbb{H}^2 \times \mathbb{R}^\times$ such that, for each fixed $t \in \mathbb{R}^\times$, $\mathbb{H}^2 \ni X \mapsto \varphi(X,t)$ belongs to $\mathcal{S}(\mathbb{H}^2) \otimes \mathrm{End}(V_{\frac{\kappa_1+\kappa_2}{2}} \boxtimes V_{\frac{\kappa_2-\kappa_1}{2}})$ for $(\kappa_1,\kappa_2) \in (2\mathbb{Z}_{\geq 0})^{\oplus 2}$ with $\kappa_1 \leq \kappa_2$ (respectively $\mathcal{S}(\mathbb{H}^2) \otimes \mathrm{End}(\mathcal{H}_{\kappa_1-4})$ for $\kappa_1 \in 2\mathbb{Z}_{\geq 0}$ with $\kappa_1 \geq 4$), where $\mathcal{H}_{\kappa_1-4}$ denotes the space of homogeneous harmonic polynomials of degree $\kappa_1 - 4$ on $\mathbb{H}^2$. We let $\varphi_{0,p} \in \mathbb{V}_p$ be the characteristic function of $L_p \times \mathbb{Z}_p^\times$.

Let $G = G_{\mathrm{nc}}$. For $(\kappa_1,\kappa_2) \in (2\mathbb{Z}_{\geq 0})^2$ with $\kappa_1 \leq \kappa_2$ we define $\varphi_{0,\infty}^{\mathrm{nc}} = \varphi_{0,\infty}^{\mathrm{nc},\,(\kappa_1,\kappa_2)} \in \mathbb{V}_\infty$ by

$$\varphi_{0,\infty}^{\mathrm{nc}}(X,t) := \begin{cases} t^{\frac{\kappa_2+3}{2}} \sigma_{\frac{\kappa_1+\kappa_2}{2}}(X_1+X_2) \boxtimes \sigma_{\frac{\kappa_2-\kappa_1}{2}}(X_1-X_2) \exp(-2\pi t {}^t\bar{X}X) & (t>0), \\ 0 & (t<0). \end{cases}$$

Let $G = G_{\mathrm{c}}$. For $\kappa_1 \in 2\mathbb{Z}_{\geq 0}$ with $\kappa_1 \geq 4$, following [Lo, Definition 6.1], we define $\varphi_{0,\infty}^{\mathrm{c}} = \varphi_{0,\infty}^{\mathrm{c},\,(\kappa_1,\kappa_2)} \in \mathbb{V}_\infty$ by

$$\varphi_{0,\infty}^{\mathrm{c}}(X,t) := \begin{cases} t^{\frac{\kappa_1-1}{2}} \exp(-2\pi t {}^t\bar{X}X) C(X) & (t>0), \\ 0 & (t<0), \end{cases}$$

where C is the $\mathrm{Hom}(\mathcal{H}_{\kappa_1-4}, \mathcal{H}_{\kappa_1-4}^*) \simeq \mathrm{End}(\mathcal{H}_{\kappa_1-4})$-valued function on $\mathbb{H}^2$ defined by

$$C(X)(h) := h(X) \quad (h \in \mathcal{H}_{\kappa_1-4}),$$

where $\mathcal{H}_{\kappa_1-4}^*$ denotes the dual space of $\mathcal{H}_{\kappa_1-4}$.

Following [M-N-1, Section 3] we introduce a metaplectic representation $r = \otimes'_{v\leq\infty} r_v$ of $G(\mathbb{A}) \times H(\mathbb{A}) \times H'(\mathbb{A})$ on the restricted tensor product $\mathbb{V} = \otimes'_{v\leq\infty}\mathbb{V}_v$ with respect to $\{\varphi_{0,p}\}_{p<\infty}$. It is associated with the standard additive character ψ of $\mathbb{A}$. For $G = G_{\mathrm{nc}}$ (respectively $G = G_{\mathrm{c}}$) we define the $\mathrm{End}(V_{\frac{\kappa_1+\kappa_2}{2}} \boxtimes V_{\frac{\kappa_2-\kappa_1}{2}})$-valued theta function (respectively $\mathrm{End}(\mathcal{H}_{\kappa_1-4})$-valued theta function) $\theta_{\kappa_1,\kappa_2}(g,h,h')$ by

$$\sum_{(X,t)\in B^2\times\mathbb{Q}^\times} r(g,h,h')\varphi_0(X,t),$$

where $\varphi_0 := \prod_{v\leq\infty} \varphi_{0,v}$ with

$$\varphi_{0,\infty} := \begin{cases} \varphi_{0,\infty}^{\mathrm{nc}} & (G = G_{\mathrm{nc}}), \\ \varphi_{0,\infty}^{\mathrm{c}} & (G = G_{\mathrm{c}}). \end{cases}$$

When $G = G_{\mathrm{nc}}$ (respectively $G = G_{\mathrm{c}}$), for $(\kappa_1,\kappa_2) \in (2\mathbb{Z}_{>0})^2$ with $\kappa_1 \leq \kappa_2$ (respectively with $\kappa_1 \geq \kappa_2$ and $\kappa_1 \geq 4$), we consider the theta lift

$$S_{\kappa_1}(D) \times \mathcal{A}_{\kappa_2} \ni (f,f') \mapsto \mathcal{L}(f,f')(g)$$

with

$$\mathcal{L}(f,f')(g) := \int_{\mathbb{R}_+^2 (H\times H')(\mathbb{Q})\backslash(H\times H')(\mathbb{A})} \overline{f(h)}\theta_{\kappa_1,\kappa_2}(g,h,h')f'(h')dhdh'.$$

By an argument similar to the proof of [M-N-1, Theorem 4.1], we verify that this is convergent on any compact subset of $G(\mathbb{A})$ when $G = G_{\mathrm{nc}}$. On the other hand, when $G = G_{\mathrm{c}}$, the theta function $\theta_{\kappa_1,\kappa_2}(g,h,h')f'(h')$ with a fixed (g,h') can be viewed as an elliptic modular form of weight κ_1 and level D (cf. [He, Section 6]). The convergence of the integral is thus reduced to that of the Petersson inner product of an elliptic modular form and an elliptic cusp form.

Theorem 2.1. *Let* $(\kappa_1, \kappa_2) \in (2\mathbb{Z}_{\geq 0})^{\oplus 2}$.
(1) The theta lift $\mathcal{L}(f,f')$ defines an automorphic forms, more precisely, it is left-$G(\mathbb{Q})$-invariant, right $K_f(D)$-invariant and right K_∞^0-equivariant (respectively G_∞^1-equivariant) with respect to the irreducible representation with highest weight $(\frac{\kappa_2-\kappa_1}{2}, \frac{\kappa_1+\kappa_2}{2})$ (respectively $(\frac{\kappa_1+\kappa_2}{2}-1, \frac{\kappa_2-\kappa_1}{2}-1)$) when $G = G_{\mathrm{nc}}$ (respectively $G = G_{\mathrm{c}}$). Furthermore $\mathcal{L}(f,f')$ has the trivial central character.
(2) Suppose that (f,f') are Hecke eigenforms. Then $\mathcal{L}(f,f')$ is also a Hecke eigenform. Furthermore, for each $p|D$, let ϵ_p (respectively ϵ'_p) be the eigenvalue for the involutive action of $\begin{pmatrix} 0 & 1 \\ -p & 0 \end{pmatrix}$ (resp. a prime element $\varpi_{B,p} \in B_p$) on f (resp. f'). Then $\mathcal{L}(f,f') \equiv 0$ unless $\epsilon_p = \epsilon'_p$.
(3) Assume furthermore that $1 < \kappa_1 < \kappa_2 + 2$ when $G = G_{\mathrm{nc}}$ (respectively $1 < \kappa_2 + 2 < \kappa_1$ when $G = G_{\mathrm{c}}$). Then $\mathcal{L}(f,f')$ is a cusp form on $G_{\mathrm{nc}}(\mathbb{A})$ generating, at the archimedean place, the discrete series representation with Harish-Chandra parameter $\lambda = (\frac{\kappa_2-\kappa_1}{2} + 1, \frac{\kappa_1+\kappa_2}{2})$ (respectively automorphic forms on $G_{\mathrm{c}}(\mathbb{A})$ generating, at the archimedean place, the discrete series representation with Harish-Chandra parameter $\lambda = (\frac{\kappa_2+\kappa_1}{2}, \frac{\kappa_1-\kappa_2}{2} - 1)$) as a $(\mathfrak{g}, K_\infty^0)$-module, where $\mathfrak{g}$ denotes the Lie algebra of G_∞^1.

Outline of proof:
(1) The assertion is essentially due to [M-N-1, Section 4].
(2) This follows from [M-N-1, Theorem 5.1] and [M-N-1, Remark 5.2 (ii)].
(3) The fact that $\mathcal{L}(f,f')$ is cuspidal when $G = G_{\mathrm{nc}}$ is shown in a manner similar to [M-N-2, Section 13.4]. To determine the representation type of $\mathcal{L}(f,f')$ at the Archimedean place, we use the result by Li-Paul-Tan-Zhu [L-P-T-Z, Theorem 5.1] on the archimedean theta correspondence, in which $\mathcal{L}(f,f')$ is involved. When $G = G_{\mathrm{c}}$ this assertion then follows immediately. In view of the archimedean theta correspondence and the discrete decomposability of the cuspidal spectrum (cf. [G-G-P]), we thus see that, when $G = G_{\mathrm{nc}}$, the archimedean component of the $G^1(\mathbb{A})$-module generated by $\mathcal{L}(f,f')(g_f*)$ with any fixed $g_f \in G(\mathbb{A}_f)$ is isomorphic to the discrete series representation in the statement as a $(\mathfrak{g}, K_\infty^0)$-module.

2.2 Theta lift to G'

We next consider the theta lift from $S_{\kappa_1}(D_1) \times S_{\kappa_2}(D_2)$ to automorphic forms on $G'(\mathbb{A})$. As in [O], we formulate the lift using the meteplectic representation r' of $G' \times H^1$ considered by Harris-Kudla [Ha-K] and Roberts [R], where H^1 denotes the $\mathbb{Q}$-algebraic group defined by

$$\{(h_1, h_2) \in GL_2 \times GL_2 \mid \det(h_1) = \det(h_2)\}.$$

Now let us introduce a quadratic space $(M_2(\mathbb{Q}), \det)$ and note that the action of $H^1(\mathbb{Q})$ on $M_2(\mathbb{Q})$ defined by

$$h \cdot X = h_1^{-1} X h_2 \quad (X \in M_2(\mathbb{Q}),\ h = (h_1, h_2) \in H^1(\mathbb{Q}))$$

induces a well-known isomorphism

$$H_1(\mathbb{Q})/\{(z,z) \mid z \in \mathbb{Q}^\times\} \simeq GSO(2,2)(\mathbb{Q}).$$

We assume that r' is associated with the additive character $\psi(\frac{1}{2}*)$ on $\mathbb{A}$. To construct the theta lift we now recall the choice of the Schwartz function on $M_2(\mathbb{A})^{\oplus 2}$ in [O]. At a finite place $v = p < \infty$, we let $\varphi'_{0,p}$ be the Schwartz function on $M_2(\mathbb{Q}_p)^{\oplus 2}$ given by the characteristic function of

$$\left\{ \left(\begin{pmatrix} a_{x_1} & b_{x_1} \\ c_{x_1} & d_{x_1} \end{pmatrix}, \begin{pmatrix} a_{x_2} & b_{x_2} \\ c_{x_2} & d_{x_2} \end{pmatrix} \right) \;\middle|\; \begin{matrix} a_{x_1} \in D_2\mathbb{Z}_p,\ b_{x_1} \in \mathbb{Z}_p,\ c_{x_1} \in D_1D_2\mathbb{Z}_p,\ d_{x_1} \in D_1\mathbb{Z}_p, \\ a_{x_2},\ b_{x_2},\ c_{x_2},\ d_{x_2} \in \mathbb{Z}_p, \end{matrix} \right\}$$

For the choice of the Schwartz function at the archimedean place we need two functions P_1 and P_2 on $M_2(\mathbb{R})$ defined as follows:

$$P_1(X) := \mathrm{tr}(X \begin{pmatrix} -\sqrt{-1} & -1 \\ -1 & \sqrt{-1} \end{pmatrix}),\ P_2(X) := \mathrm{tr}(X \begin{pmatrix} -\sqrt{-1} & 1 \\ -1 & -\sqrt{-1} \end{pmatrix}) \quad (X \in M_2(\mathbb{R}))$$

Let $\mathbb{C}[s_1, s_2]$ denote the polynomial ring of two variables s_1 and s_2 over $\mathbb{C}$. As our choice of the test function at $v = \infty$ we take the $\mathbb{C}[s_1, s_2]$-valued Schwartz function $\varphi_{\infty,0}$ on $M_2(\mathbb{R})^{\oplus 2}$ as follows:

$$\varphi'_{\infty,0}(X_1, X_2) :=$$

$$\exp(-\pi\, \mathrm{tr}\,({}^tX_1X_1 + {}^tX_2X_2))P_1(s_1X_1 + s_2X_2)^{\frac{\kappa_1+\kappa_2}{2}} \times \begin{cases} P_2(s_2X_1 - s_1X_2)^{\frac{\kappa_1-\kappa_2}{2}} & (\kappa_1 \geq \kappa_2) \\ \bar{P}_2(s_2X_1 - s_1X_2)^{\frac{\kappa_2-\kappa_1}{2}} & (\kappa_1 \leq \kappa_2) \end{cases}$$

Put $\varphi'_0 := \underset{v \leq \infty}{\otimes} \varphi'_{v,0}$ and define the theta series $\theta'_{\kappa_1,\kappa_2}(g,h)$ on $G'(\mathbb{A}) \times H^1(\mathbb{A})$ as

$$\sum_{(X_1,X_2) \in M_2(\mathbb{Q})^{\oplus 2}} r'(g,h)\varphi'_0(X_1, X_2).$$

We view $f_1 \boxtimes f_2 := f_1 f_2$ as an automorphic form on $H^1(\mathbb{A})$ or $(H \times H)(\mathbb{A})$ for $(f_1, f_2) \in S_{\kappa_1}(D_1) \times S_{\kappa_2}(D_2)$. We embed $\mathbb{A}^\times$ into $H^1(\mathbb{A})$ by

$$\mathbb{A}^\times \ni a \mapsto (a \cdot 1_2, a \cdot 1_2) \in H^1(\mathbb{A}).$$

For $(\kappa_1, \kappa_2) \in (2\mathbb{Z}_{\geq 0})^{\oplus 2}$ we then define the theta lifting from $S_{\kappa_1}(D_1) \times S_{\kappa_2}(D_2)$ to $G'(\mathbb{A})$ by

$$S_{\kappa_1}(D_1) \times S_{\kappa_2}(D_2) \ni (f_1, f_2) \to \mathcal{L}'(f_1, f_2)(g),$$

where $\Lambda' = (\frac{\kappa_1+\kappa_2}{2}, -\frac{|\kappa_1-\kappa_2|}{2})$ and

$$\mathcal{L}'(f_1, f_2)(g) := \int_{\mathbb{A}^\times H^1(\mathbb{Q})\backslash H^1(\mathbb{A})} \theta'_{\kappa_1,\kappa_2}(g, hh') f_1 \boxtimes f_2(hh') dh$$

with an invariant measure dh on $\mathbb{A}^\times H^1(\mathbb{Q})\backslash H^1(\mathbb{A})$. Here for each $g \in G'(\mathbb{A})$, we take $h' = (h_1', h_2') \in (H \times H)(\mathbb{A})$ so that $\nu'(g) = \det(h_1')\det(h_2')^{-1}$. We note that this theta lift does not depend on the choice of h'.

We now quote the following theorem (cf. [O])):

Theorem 2.2. *For two non-zero primitive cusp forms $(f_1, f_2) \in S_{\kappa_1}(D_1) \times S_{\kappa_2}(D_2)$, $\mathcal{L}'(f_1, f_2)$ is a non-zero generic cusp form on $G'(\mathbb{A}) = GSp(2)(\mathbb{A})$ with the trivial central character satisfying the following properties:*

1. *$\mathcal{L}'(f_1, f_2)$ is a paramodular form of level D_1D_2, namely, at a prime $p|N := D_1D_2$, it is right invariant by a paramodular group*

$$K'_{p^{\mathrm{ord}_p N}} := \begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p & N^{-1}\mathbb{Z}_p & \mathbb{Z}_p \\ N\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\ N\mathbb{Z}_p & N\mathbb{Z}_p & \mathbb{Z}_p & N\mathbb{Z}_p \\ N\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix} \cap GSp(2)_{\mathbb{Q}_p},$$

2. *When $\kappa_1 \neq \kappa_2$, $\mathcal{L}'(f_1, f_2)$ lies, at the archimedean place, in the minimal K'^0_∞-type $\tau_{\Lambda'}$ of the large discrete series representations $\pi_{\lambda'}$ with*

$$\lambda' = (\frac{\kappa_1 + \kappa_2}{2} - 1, -\frac{|\kappa_1 - \kappa_2|}{2}), \quad \Lambda' = (\frac{\kappa_1 + \kappa_2}{2}, -\frac{|\kappa_1 - \kappa_2|}{2}).$$

3 The Jacquet-Langlands-Shimizu correspondence for the theta lifts

3.1 Automorphic L-functions

We now define the spinor L-function for $\mathcal{L}(f, f')$, modifying the definition of [M-N-3, Section 2.6] at the archimedean place.

In [M-N-1, Section 5.1] we introduced three Hecke operator $\mathcal{T}_p^i$ with $0 \le i \le 2$ for $p \nmid d_B$. Let Λ_p^i be the Hecke eigenvalue of $\mathcal{T}_p^i$ for F with $0 \le i \le 2$. For $p \nmid d_B$ we put

$$Q_{F,p}(t) := 1 - p^{-\frac{3}{2}}\Lambda_p^1 t + p^{-2}(\Lambda_p^2 + p^2 + 1)t^2 - p^{-\frac{3}{2}}\Lambda_p^1 t^3 + t^4$$

For this we note that $Q_{F,p}(p^{-s})^{-1}$ coincides with the local spinor L-function for an unramified principal series of the group of $GSp(2)_{\mathbb{Q}_p}$. On the other hand, in [M-N-1, Section 5.2], we introduced two Hecke operators $\mathcal{T}_p^i$ with $0 \le i \le 1$ for $p|d_B$. Let Λ'^i_p be the Hecke eigenvalue of $\mathcal{T}_p^i$ for F with $0 \le i \le 1$. For $p|d_B$ we put

$$Q_{F,p}(t) := \begin{cases} (1 - p^{-\frac{3}{2}}(\Lambda'^1_p - (p-1)\Lambda'^0_p)t + t^2)(1 - \Lambda'^0_p p^{-\frac{1}{2}}t) & (p|\frac{d_B}{D}), \\ (1 + \Lambda'^0_p p^{-\frac{1}{2}}t)(1 - \Lambda'^0_p p^{-\frac{1}{2}}t) & (p|D). \end{cases}$$

The first one is due to Sugano [Su, (3.4)]. The first factor of the second one comes from the numerator of the formal Hecke series.

We define the spinor L-function $L(F, \mathrm{spin}, s)$ of a Hecke eigenform F on $G(\mathbb{A})$ with the following properties:

- F is right $K_f(D)$-invariant and right K^0_∞-equivariant with respect to the irreducible representation of highest weight $(\frac{\kappa_2-\kappa_1}{2}, \frac{\kappa_1+\kappa_2}{2})$,
- F generates, as a $(\mathfrak{g}, K^0_\infty)$-module, the discrete series representation with Harish Chandra parameter $(\frac{\kappa_2-\kappa_1}{2}+1, \frac{\kappa_1+\kappa_2}{2})$ with $(\kappa_1, \kappa_2) \in 2\mathbb{Z}^{\oplus 2}$ such that $1 < \kappa_1 < \kappa_2+2$, where recall that $\mathfrak{g}$ denotes the Lie algebra of G^1_∞ (cf. Theorem 2.1 (3)).

The definition is as follows:

$$L(F, \mathrm{spin}, s) := \prod_{v \leq \infty} L_v(F, \mathrm{spin}, s),$$

where

$$L_v(F, \mathrm{spin}, s) := \begin{cases} Q_{F,p}(p^{-s})^{-1} & (v = p < \infty), \\ \Gamma_{\mathbb{C}}(s + \frac{\kappa_1-1}{2})\Gamma_{\mathbb{C}}(s + \frac{\kappa_2+1}{2}) & (v = \infty). \end{cases}$$

By virtue of Theorem 2.1 (3) we can use this definition for $F = \mathcal{L}(f, f')$ when (f, f') are Hecke eigenforms.

We generalize [M-N-3, Proposition 2.9] to have the following:

Proposition 3.1. *The spinor L-function for $\mathcal{L}(f, f')$ decomposes into*

$$L(\mathcal{L}(f, f'), \mathrm{spin}, s) = L(\pi(f), s)L(\mathrm{JL}(\pi(f')), s),$$

where $L(\pi(f), s)$ (resp. $L(\mathrm{JL}(\pi(f')), s)$) denotes the standard L-function of $\pi(f)$ (resp. $\mathrm{JL}(\pi(f'))$).

Of course, we thus see that $L(\mathcal{L}(f, f'), \mathrm{spin}, s)$ has the meromorphic continuation and satisfies the functional equation between s and $1-s$ since so do $L(\pi(f), s)$ and $L(\mathrm{JL}(\pi(f')), s)$.

We now recall that there is Novodvorsky's zeta integral of the spinor L-function for a generic cusp form on $G'(\mathbb{A})$ (cf. [No]). By means of the zeta integral, the theorem as follows (cf. [O]) describes the spinor L-function for a generic form $\mathcal{L}'(f_1, f_2)$.

Theorem 3.2. *Let the notations be as in Theorem 2.2. Then the global spinor L-function of $\mathcal{L}'(f_1, f_2)$ decomposes into*

$$L(\pi(f_1), s)L(\pi(f_2), s).$$

As an immediate consequence of Proposition 3.1 and this theorem we obtain the following:

Corollary 3.3. *Let $f \in S_{\kappa_1}(D)$ be a primitive form and $f' \in \mathcal{A}_{\kappa_2}$ be a Hecke eigenform. Then we have*

$$L(\mathcal{L}(f, f'), \mathrm{spin}, s) = L(\mathcal{L}'(f, \mathrm{JL}(f')), \mathrm{spin}, s).$$

3.2 Automorphic representations generated by the theta lifts

We study locally and globally the representation $\pi(\mathcal{L}(f,f'))$ of $G(\mathbb{A}) = GSp(1,1)(\mathbb{A})$ or $GSp^*(2)(\mathbb{A})$ generated by $\mathcal{L}(f,f')$ (respectively the representation $\pi(\mathcal{L}(f,\mathrm{JL}(f')))$ of $G'(\mathbb{A}) = GSp(2)(\mathbb{A})$ generated by $\mathcal{L}'(f,\mathrm{JL}(f'))$).

(1) The case of G:

We first discuss the case of G. We note that the Lie algebra of the group G_∞/Z_{G_∞} is isomorphic to the Lie algebra $\mathfrak{g}$ of G^1_∞. The group G_∞/Z_{G_∞} is isomorphic to G^1_∞ when $G = G_{\mathrm{c}}$ but it is neither connected or isomorphic to G^1_∞ when $G = G_{\mathrm{nc}}$. For $G = G_{\mathrm{nc}}$ let K_∞ be a maximal compact subgroup of G_∞/Z_{G_∞}. We can regard K^0_∞ as the connected component of the identity for K_∞. Take $\sigma := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in G(\mathbb{R})$. We can then identify K_∞ with $K^0_\infty \cup K^0_\infty\sigma$. For $(\kappa_1,\kappa_2) \in (2\mathbb{Z}_{\geq 0})^{\oplus 2}$ with $1 < \kappa_1 + 2 < \kappa_2$ let $\pi^{(\kappa_1,\kappa_2)}_\infty$ be the discrete series representation of G^1_∞ with Harish Chandra parameter $(\frac{\kappa_2-\kappa_1}{2}+1, \frac{\kappa_1+\kappa_2}{2})$. Then we introduce another representation $\pi^{(\kappa_1,\kappa_2)}_{\infty,\sigma}$ of G^1_∞ defined by

$$\pi^{(\kappa_1,\kappa_2)}_{\infty,\sigma}(g) = \pi^{(\kappa_1,\kappa_2)}_\infty(\sigma g \sigma^{-1}) \quad \forall g \in G^1_\infty.$$

This is equivalent to the discrete series representation with Harish Chandra parameter $(\frac{\kappa_1+\kappa_2}{2}, \frac{\kappa_2-\kappa_1}{2}+1)$, which is not isomorphic to $\pi^{(\kappa_1,\kappa_2)}_\infty$. There is an irreducible $(\mathfrak{g}, K_\infty)$-module $V^{(\kappa_1,\kappa_2)}_\infty$ which is equivalent to $\pi^{(\kappa_1,\kappa_2)}_\infty \oplus \pi^{(\kappa_1,\kappa_2)}_{\infty,\sigma}$ as $(\mathfrak{g}, K^0_\infty)$-modules.

Proposition 3.4. *Suppose that f and f' are Hecke eigenforms and that $1 < \kappa_1 + 2 < \kappa_2$ for $G = G_{\mathrm{nc}}$ (respectively $1 < \kappa_2 + 2 < \kappa_1$ for $G = G_{\mathrm{c}}$). Then the representation $\pi(\mathcal{L}(f,f'))$ of $G(\mathbb{A}_\mathbb{Q})$ is irreducible.*

The point of proof is to use [N-P-S, Theorem 3.1]. We then reduce the global irreducibility to the local irreducibility at the archimedean place. When $G = G_{\mathrm{nc}}$, we can verify that the archimedean component of $\pi(\mathcal{L}(f,f'))$ is isomorphic to $V^{(\kappa_1,\kappa_2)}_\infty$.

We can therefore decompose $\pi(\mathcal{L}(f,f'))$ into the restricted tensor product $\prod'_{v\leq\infty} \pi_v$ and are able to determine each local component π_v. To state our result on it we need several notation.

For a primitive cusp form $f \in S_{\kappa_1}(D)$ let $\pi(f)$ be an irreducible cuspidal representation of $GL_2(\mathbb{A})$, which admits a decomposition into the restricted tensor product $\pi(f) = \prod'_{v\leq\infty} \pi(f)_v$. Then, for $v = p \nmid D$, $\pi(f)_p$ is an unramified principal series representation of $GL_2(\mathbb{Q}_p)$. Let $\chi_{f,p}$ denote the unramified character of $\mathbb{Q}_p^\times$ which induces $\pi(f)_p$.

For a Hecke eigenform $f' \in \mathcal{A}_{\kappa_2}$ let $\pi(f')$ be the irreducible automorphic representation of $H'(\mathbb{A})$ generated by f', and let $\pi(f') = \prod'_{v\leq\infty} \pi(f')_v$ be the decomposition into the restricted tensor product of local representations. When $p \nmid d_B$, $\pi(f')_p$ is an unramified principal series representation of $B_p^\times \simeq GL_2(\mathbb{Q}_p)$. We let $\chi_{f',p}$ be the unramified character of $\mathbb{Q}_p^\times$ inducing $\pi(f')_p$. When $p|d_B$, $\pi(f')_p$ is a character of $B_p^\times$ of order at most two. Thus we have

$$\pi(f')_p = \delta_p \circ n$$

with a character δ_p of $\mathbb{Q}_p^\times$ of order at most two, where recall that the notation n stands for the reduced norm of B (cf. Section 1.1). In view of Theorem 2.1 (2), $\delta_p(p) = \epsilon'_p = \epsilon_p$ is necessary for $p|D$ in order that $\mathcal{L}(f, f') \not\equiv 0$.

Following the notation of the appendix, let ν be the p-adic absolute value of $\mathbb{Q}_p$ and let ξ be the non-trivial unramified character of $\mathbb{Q}_p^\times$ of order two for $p|d_B$. We further note that, in the appendix, the notation $\chi 1_B \rtimes \sigma$ is used for the induced representation of $GSp(1,1)(\mathbb{Q}_p)$ defined by two quasi-character χ and σ of $\mathbb{Q}_p^\times$ when $p|d_B$. On the other hand, with three unramified quasi-characters χ_1, χ_2 and σ of $\mathbb{Q}_p^\times$, $\chi_1 \times \chi_2 \rtimes \sigma$ denotes the unramified principal series representation of $GSp(2)(\mathbb{Q}_p)$, which is referred to as "type I" on the table of the appendix.

Proposition 3.5. *Let the notation be as above.*
(1) Let $v = p \nmid d_B$. Then π_p is an unramified principal series representation of $GSp(1,1)(\mathbb{Q}_p) \simeq GSp^*(2)(\mathbb{Q}_p) \simeq GSp(2)(\mathbb{Q}_p)$ given by $(\chi_{f',p} \cdot \chi_{f,p}^{-1}) \times (\chi_{f',p}^{-1} \cdot \chi_{f,p}^{-1}) \rtimes \chi_{f,p}$.
(2) Let $v = p|d_B$. When $v = p|\frac{d_B}{D}$, π_p is isomorphic to the irreducible representation of $GSp(1,1)(\mathbb{Q}_p) \simeq GSp^*(2)(\mathbb{Q}_p)$ of type II_a with $\sigma = \chi_{f,p}$ and $\chi = \chi_{f,p}^{-1} \cdot \delta_p$. When $v = p|D$ and δ_p is trivial (respectively non-trivial), π_p is isomorphic to the irreducible representation of $GSp(1,1)(\mathbb{Q}_p) \simeq GSp^*(2)(\mathbb{Q}_p)$ of type V_a with $\sigma = \xi$ (respectively $\sigma = 1$), where, for the representations of $GSp(1,1)(\mathbb{Q}_p) \simeq GSp^*(2)(\mathbb{Q}_p)$ of type II_a and V_a, see the appendix.
(3) When $v = \infty$ and $G = G_{\rm nc}$, π_∞ is isomorphic to $V_\infty^{(\kappa_1,\kappa_2)}$. When $v = \infty$ and $G = G_{\rm c}$, π_∞ is isomorphic to the irreducible representation with Harish-Chandra parameter $(\frac{\kappa_1+\kappa_2}{2}, \frac{\kappa_1-\kappa_2}{2} - 1)$ modulo center.

The archimedean component of $\pi(\mathcal{L}(f, f'))$ is already determined in the proof of Proposition 3.4. It thus suffices to consider the non-archimedean components. For every finite prime p, π_p is a spherical representation of $G_p = GSp(1,1)(\mathbb{Q}_p)$ or $GSp(2)(\mathbb{Q}_p)$ (cf. [C]). As we see in [C], π_p is uniquely determined by the Hecke eigenvalues. We calculate Hecke eigenvalues of $\mathcal{L}(f, f')$ explicitly in terms of eigenvalues for (f, f') to obtain the assertion.

(2) The case of G':

We next deal with the automorphic representation $\pi(\mathcal{L}'(f, \mathrm{JL}(f')))$ of $GSp(2)(\mathbb{A})$ generated by $\mathcal{L}'(f, \mathrm{JL}(f'))$. According to [R, Theorem 8.3], $\pi'(f, \mathrm{JL}(f'))$ is an irreducible cuspidal representation. It thereby admits a decomposition into the restricted tensor product $\pi(\mathcal{L}'(f, \mathrm{JL}(f'))) = \prod'_{v \le \infty} \pi'_v$. For each finite prime $v = p$, π_p is involved in the local theta correspondence for $GSO(2,2)(\mathbb{Q}_p)$ and $GSp(2)(\mathbb{Q}_p)$, which is explicitly described in Gan-Takeda [G-T-2]. To describe each π_p we use the notation of the appendix. To describe the archimedean component π'_∞, we need to introduce, for two even integers (κ_1, κ_2) with $1 < \kappa_1 + 2 < \kappa_2$, the irreducible admissible representation $V'^{(\kappa_1,\kappa_2)}_\infty$ of $GSp(2)(\mathbb{R})$ whose restriction to $Sp(2)(\mathbb{R})$ is the direct sum of the two large discrete series representation of $Sp(2)(\mathbb{R})$ with Harish Chandra parameters $(\frac{\kappa_1+\kappa_2}{2}, -\frac{\kappa_2-\kappa_1}{2} - 1)$ and $(\frac{\kappa_2-\kappa_1}{2} + 1, -\frac{\kappa_1+\kappa_2}{2})$.

Proposition 3.6. *Let the notation be as above.*
(1) Let $v = p \nmid d_B$. Then π'_p is isomorphic to π_p, namely an unramified principal series representation of $GSp(1,1)(\mathbb{Q}_p) \simeq GSp^*(2)(\mathbb{Q}_p) \simeq GSp(2)(\mathbb{Q}_p)$ given by $(\chi_{f',p} \cdot \chi_{f,p}^{-1}) \times (\chi_{f',p}^{-1} \cdot \chi_{f,p}^{-1}) \rtimes \chi_{f,p}$.

(2) Let $v = p|d_B$. *When* $v = p|\frac{d_B}{D}$, π'_p *is isomorphic to the irreducible representation of* $GSp(2)(\mathbb{Q}_p)$ *of type* II_a *with* $\sigma = \chi_{f,p}$ *and* $\chi = \chi_{f,p}^{-1} \cdot \delta_p$. *When* $v = p|D$ *and* δ_p *is trivial (respectively non-trivial),* π'_p *is isomorphic to the irreducible representation of* $GSp(2)(\mathbb{Q}_p)$ *of type* V_a *with* $\sigma = \xi$ *(respectively* $\sigma = 1$*), where, for the representations of* $GSp(2)(\mathbb{Q}_p)$ *of type* II_a *and* V_a*, see the appendix.*
(3) When $v = \infty$, π'_∞ *is isomorphic to* $V'^{(\kappa_1,\kappa_2)}_\infty$.

Using Przebinda [Prz], the representation π'_∞ at the infinite prime $v = \infty$ is determined by the same reasoning as in the case of $GSp(1,1)(\mathbb{R})$. The representation π_p is included in the table 2 of Section 14 or Theorem 8.2 (iv), (v), (vi) of Gan-Takeda [G-T]. Then, looking also at the table of the appendix, we have the assertion on π_p.

3.3 Conjecture and conclusion

Let $\mathcal{A}_G$ and $\mathcal{A}_{G'}$ denote the equivalence classes of irreducible automorphic representations of $G(\mathbb{A})$ and $G'(\mathbb{A})$ respectively. We note that the L-group ${}^LG$ of G is the same as the L-group ${}^LG'$ of G', where ${}^LG = {}^LG'$ is the direct product of $GSp(2)(\mathbb{C})$ and the Weil group of $\mathbb{Q}$ (for the notion of L-group see [La] and [B] et al). As the choice of the L-morphism between ${}^LG$ and ${}^LG'$ we can take the idenfity map. The Langlands principle of functoriality predicts the following:

Conjecture 3.7 (Langlands). *The L-morphism induced by the identity map would gives rise to a natural transfer from* $\mathcal{A}_G$ *to* $\mathcal{A}_{G'}$ *which preserves L-functions, namely an L-function of an irreducible automorphic representation of* $G(\mathbb{A})$ *is one of some irreducible automorphic representation of* $G'(\mathbb{A})$.

Let us now introduce

$$\mathcal{A}_G(K_f(D)) := \{\pi = \prod_{v\leq\infty} \pi_v \in \mathcal{A}_G \mid \pi_p \text{ has a } K_p\text{-fixed vector for } v = p < \infty\},$$
$$\mathcal{A}_{G'}(K'_f(D)) := \{\pi' = \prod_{v\leq\infty} \pi'_v \in \mathcal{A}_{G'} \mid \pi'_p \text{ has a } K'_p\text{-fixed vector for } v = p < \infty\},$$

where see Section 1.2 for $K_f(D)$ and $K'_f(D)$.

Based on the observation by R. Schmidt including the table of irreducible admissible representations of $G(\mathbb{Q}_p) = G_{\text{nc}}(\mathbb{Q}_p) = G_{\text{c}}(\mathbb{Q}_p)$ and $G'(\mathbb{Q}_p) = G_{\text{s}}(\mathbb{Q}_p)$ in the appendix (see also [RS, Section A.8]), we can formulate the conjecture as follows:

Conjecture 3.8. *The above transfer would map* $\mathcal{A}_G(K_f(D))$ *into* $\mathcal{A}_{G'}(K'_f(D))$ *and an L-function of* $\pi \in \mathcal{A}_G(K_f(D))$ *is one of some* $\pi' \in \mathcal{A}_{G'}(K'_f(D))$.

We remark that this was first pointed out by Ibukiyama [I] for the case of $G = G_{\text{c}}$ and $D = 1$. As a consequence of Corollary 3.3, Propositions 3.5 and 3.6 we have known that our theta lifts $\mathcal{L}(f, f')$ and $\mathcal{L}'(f, \text{JL}(f'))$ provide evidence of Conjecture 3.8. We state it as follows:

Theorem 3.9. *Suppose that two even integers* (κ_1, κ_2) *satisfy* $1 < \kappa_1 + 2 < \kappa_2$ *when* $G = G_{\text{nc}}$ *(respectively* $1 < \kappa_2 + 2 < \kappa_1$ *when* $G = G_{\text{c}}$*). For any given primitive form* $f \in S_{\kappa_1}(D)$ *and Hecke eigenform* $f' \in \mathcal{A}_{\kappa_2}$*, the map*

$$\mathcal{A}_G(K_f(D)) \ni \pi(\mathcal{L}(f, f')) \mapsto \pi(\mathcal{L}'(f, \mathrm{JL}(f'))) \in \mathcal{A}_{G'}(K'_f(D))$$

preserves the coincidence of the global spinor L-functions and is compatible with the non-archimedean local Jacquet-Langlands correspondence for G *and* $G' = GSp(2)$ *(cf. Appendix). Namely, this map satisfies the expected properties of the transfer in the conjecture.*

A Appendix: The spherical representations of $\mathrm{GSp}(1,1)$ and local Langlands parameters for $\mathrm{GSp}(4)$ (by Ralf Schmidt)

Let F be a non-archimedean local field of characteristic zero. Let B be the non-split quaternion algebra over F, and let $x \mapsto \bar{x}$ be its standard involution. We consider $\mathrm{GSp}(1,1)$ and $GSp(4)$ (or $GSp(2)$) over F. Let $\mathfrak{o}_B$ be a maximal order in $B(F)$, and let $\mathfrak{p}_B$ be the unique maximal ideal of $\mathfrak{o}_B$. Let

$$K_1 = \{g \in \mathrm{GSp}(1,1)(F) \cap \begin{bmatrix} \mathfrak{o}_B & \mathfrak{o}_B \\ \mathfrak{o}_B & \mathfrak{o}_B \end{bmatrix} : \nu(g) \in \mathfrak{o}^\times\},$$

$$K_2 = \{g \in \mathrm{GSp}(1,1)(F) \cap \begin{bmatrix} \mathfrak{o}_B & \mathfrak{p}_B \\ \mathfrak{p}_B^{-1} & \mathfrak{o}_B \end{bmatrix} : \nu(g) \in \mathfrak{o}^\times\}.$$

We remark that these groups K_1 and K_2 are maximal compact subgroups of $\mathrm{GSp}(1,1)(F)$, and every maximal compact subgroup is conjugate to either K_1 or K_2.

The following table lists all irreducible, admissible representations of $GSp(1,1)(F)$ which are constituents of representations of the form $\chi 1_{B^\times} \rtimes \sigma$, where χ and σ are characters of $F^\times$. The table also lists all the irreducible, admissible representations of $\mathrm{GSp}(4, F)$ supported in the Borel subgroup, using the notations and classification scheme of [R-S]. Representations with the same L-parameter $W'_F \to \mathrm{GSp}(4, \mathbb{C})$ appear in the same row; this is nothing but the Langlands functorial transfer from $\mathrm{GSp}(1,1)$ to $\mathrm{GSp}(4)$ coming from the natural inclusion of dual groups. The actual L-parameters can be found in Table A.7 of [R-S].

The columns labeled K_1 and K_2 indicate, in the case when the inducing characters are unramified, the dimension of the space of K_1 resp. K_2 invariant vectors in a representation of $G(F)$.

		GSp(1,1)	GSp(4)	K_1	K_2
I		—	$\chi_1 \times \chi_2 \rtimes \sigma$ (irreducible)		
II	a	$\chi 1_{B^\times} \rtimes \sigma$	$\chi \mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$	1	1
	b	—	$\chi 1_{\mathrm{GL}(2)} \rtimes \sigma$		
III	a	—	$\chi \rtimes \sigma \mathrm{St}_{\mathrm{GSp}(2)}$		
	b	—	$\chi \rtimes \sigma 1_{\mathrm{GSp}(2)}$		
IV	a	$\sigma \mathrm{St}_{\mathrm{GSp}(1,1)}$	$\sigma \mathrm{St}_{\mathrm{GSp}(4)}$	0	0
	b	—	$L(\nu^2, \nu^{-1}\sigma \mathrm{St}_{\mathrm{GSp}(2)})$		
	c	$\sigma 1_{\mathrm{GSp}(1,1)}$	$L(\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-3/2}\sigma)$	1	1
	d	—	$\sigma 1_{\mathrm{GSp}(4)}$		
V	a	$\delta(\nu^{1/2}\xi 1_{B^\times}, \nu^{-1/2}\sigma)$	$\delta([\xi, \nu\xi], \nu^{-1/2}\sigma)$	1	0
	b	$L(\nu^{1/2}\xi 1_{B^\times}, \nu^{-1/2}\sigma)$	$L(\nu^{1/2}\xi \mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$	0	1
	c	$L(\nu^{1/2}\xi 1_{B^\times}, \xi\nu^{-1/2}\sigma)$	$L(\nu^{1/2}\xi \mathrm{St}_{\mathrm{GL}(2)}, \xi\nu^{-1/2}\sigma)$	0	1
	d	—	$L(\nu\xi, \xi \rtimes \nu^{-1/2}\sigma)$		
VI	a	—	$\tau(S, \nu^{-1/2}\sigma)$		
	b	—	$\tau(T, \nu^{-1/2}\sigma)$		
	c	$\nu^{1/2} 1_{B^\times} \rtimes \nu^{-1/2}\sigma$	$L(\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$	1	1
	d	—	$L(\nu, 1_{F^\times} \rtimes \nu^{-1/2}\sigma)$		

The notation ν stands for the valuation of F. For the IIa type representation, χ is such that $\chi^2 \neq \nu^{\pm 1}$ and $\chi \neq \nu^{\pm 3/2}$. For the representations in group V, the character ξ is assumed to be non-trivial and quadratic.

References

[B] A. Borel, Automorphic L-functions, Proc. Sympos. Pure Math. 33 part 2 (1977) 27-61.

[C] P. Cartier, Representations of $\mathfrak{p}$-adic groups: a survey, Proc. Symp. Pure Math. **33** (1979) part 1 111–155.

[E-1] M. Eichler, Über die Darstellbarkeit von Modulformen durch Thetareihen, J. Reine Angew. Math. 195 (1955) 156-171.

[E-2] M. Eichler, Quadratische Formen und Modulfunktionen, Acta Arith. 4 (1958) 217-239.

[G-G-P] I. M. Gel'fand, M. I. Graev and I. I. Piatetski-Shapiro, Representation theory and automorphic functions, W.B. Saunders Company (1969).

[G-T] W. T. Gan and S. Takeda, Theta correspondence for $GSp(4)$, Representation theory 15 (2011) 670-718.

[Ha-K] M. Harris and S. Kudla, Arithmetic automorphic forms for the non-holomorphic discrete series of $GSp(2)$, Duke Math. J. 66 (1992) 59-121.

[He] E. Hecke, Analytische arithmetik der positiven quadratischen formen, in Math. Werke, Vandenheck and ruprecht in Göttingen (1983) 789-918.

[I] T. Ibukiyama, Paramodular forms and compact twist, Automorphic forms on $GSp(4)$, Proceedings of the 9th Autumn workshop on number theory (2006) 37-48.

[J-L] H. Jacquet and R.P.Langlands, Automorphic forms on $GL(2)$, Lecture Notes in Math. 114, Springer-Verlag (1970).

[La] R. Langlands, Problems in the theory of automorphic forms, Lecture Notes in Math. 170, Springer-Verlag (1970) 18-86.

[L-P-T-Z] J. S. Li, A. Paul, E. C. Tan and C. B. Zhu, The explicit duality correspondence of $(Sp(p,q), O^*(2n))$, J. Funct. Anal. 200 (2003) 71-100.

[Lo] R. Löschel, Thetakorrespondenz automorpher Formen, Inaugural-Dissertation zur Erlangung des Doktorgrades der Mathematisch-Naturwissenschaftlichen Fakultät der Universität zu Köln (1997).

[M-N-1] A. Murase and H. Narita, Commutation relations of Hecke operators for Arakawa lifting, Tohoku Math. J. 60 (2008) 227–251.

[M-N-2] A. Murase and H. Narita, Fourier expansion of Arakawa lifting I: An explicit formula and examples of non-vanishing lifts, Israel J. Math. 187 (2012) 317–369.

[M-N-3] A. Murase and H. Narita, Fourier expansion of Arakawa lifting II: Relation with central L-values, preprint.

[Na] H. Narita, Theta lifting from elliptic cusp forms to automorphic forms on $Sp(1,q)$, Math. Z. 259 (2008) 591-615.

[N-O] H. Narita and T.Okazaki, Jacquet-Langlands-Shimizu correspondence for theta lifts to $GSp(2)$ and its inner forms, with an appendix by Ralf Schmidt, preprint, (2012).

[N-P-S] H. Narita, A. Pitale and R. Schmidt, Irreducibility criteria for local and global representations, to appear in Proceedings of the American Mathematical Society.

[No] M. Novodvorsky, Automorphic L-functions for symplectic group $GSp(4)$, Proc. Symp. Pure Math. 33 part 2 (1979) 87-95.

[O] T. Okazaki, Paramodular forms on $GSp_2(\mathbb{A})$, preprint.

[Prz] T. Przebinda, The oscillator duality correspondence for the pair $O(2,2)$ and $Sp(2,\mathbb{R})$, Memoirs of A. M. S. vol.79, No.403 (1989).

[R] B. Roberts, Global L-packets for $GSp(2)$ and theta lifts, Documenta Math. 6 (2001) 247-314.

[R-S] B. Roberts and R. Schmidt, Local new forms for $GSp(4)$, Lecture Notes in Math. 1918, Springer-Verlag (2007).

[Sh] H. Shimizu, Theta series and automorphic forms on GL_2, J. Math. Soc. Japan 24 (1972) 638–683.

[Su] T. Sugano, On holomorphic cusp forms on quaternion unitary groups of degree 2, J. Fac. Sci. Univ. Tokyo 31 (1985) 521–568.

Hiro-aki Narita
Department of Mathematics, Faculty of Science
Kumamoto University
Kurokami, Kumamoto 860-8555, Japan
E-mail address: narita@sci.kumamoto-u.ac.jp