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p-adic Siegel Eisenstein series of degree n

竹森 翔 (Takemori Sho)*
Department of Mathematics, Kyoto University.

1 Introduction

In this paper, we define an Siegel Eisenstein series $G_{k,\chi}^{(n)}$ of degree n and introduce a formula for its Fourier expansion. The definition of $G_{k,\chi}^{(n)}$ is different from the ordinary Siegel Eisenstein series $E_{k,\chi}^{(n)}$. But if χ satisfies a certain condition, $G_{k,\chi}^{(n)}$ coincides with $E_{k,\chi}^{(n)}$. We also introduce the theorem that states the existence of $\mathfrak{p}$-adic family of Siegel modular forms that interpolates $G_{k,\chi}^{(n)}$.

2 Statement of the main results

Let F be a totally real field with $[F:\mathbf{Q}]=m$. If K is a number field and v is a finite place of K, then we denote by $\mathcal{O}_K$ and by $\mathcal{O}_v$ the integer ring of K and that of K_v respectively. For an ideal $\mathfrak{n}$ of F, we denote the group of fractional ideals of F relatively prime to $\mathfrak{n}$ by $I_{\mathfrak{n}}$. Let χ be a narrow class character modulo $\mathfrak{n}$, that is, a character $\chi: I_{\mathfrak{n}}\rightarrow \mathbf{C}^{\times}$ trivial on any principal ideal (a) generated by a totally positive element a such that $a\equiv 1 \bmod \mathfrak{n}$. Let $\mathbb{A}_F$ be the adele ring of F and $\mathbb{A}_F^{\times}$ the idele group of F. Denote the character of finite order of $\mathbb{A}_F^{\times}/F^{\times}$ corresponding to χ by $\widetilde{\chi}$.

For an infinite place v of F, let r_v be an element of $\mathbf{Z}/2\mathbf{Z}$ satisfying the following condition.

$$\chi((a))=\prod_{v|\infty}\operatorname{sgn}(\iota_v(a))^{r_v} \text{ for } a\equiv 1 \bmod \mathfrak{n}.$$

Here v runs over the set of m real places of F and ι_v is the real embedding corresponding to v. We define a character $\operatorname{sgn}_{\chi}$ of $F^{\times}$ by

$$\operatorname{sgn}_{\chi}(a)=\prod_{v|\infty}\operatorname{sgn}^{r_v}(\iota_v(a)).$$

We define a character $\chi_f:(\mathcal{O}_F/\mathfrak{n})^{\times}\rightarrow \mathbf{C}^{\times}$ by

$$\chi_f(a)=\operatorname{sgn}_{\chi}(a)\chi((a)).$$

*takemori@math.kyoto-u.ac.jp

Let n be a positive integer. For $0 \leq i \leq n$, we denote by $w_{n,i}$ the matrix given as follows.

$$w_{n,i} = \left(\begin{array}{cc|cc} 0_i & 0 & -1_i & 0 \\ 0 & 1_{n-i} & 0 & 0_{n-i} \\ \hline 1_i & 0 & 0_i & 0 \\ 0 & 0_{n-i} & 0 & 1_{n-i} \end{array}\right).$$

We put $w_n = w_{n,n}$. We define the symplectic group of degree n by

$$\mathrm{Sp}_n(R) = \left\{g \in \mathrm{GL}_{2g}(R) \mid {}^t g w_n g = w_n\right\},$$

where R is a commutative ring. For $g \in \mathrm{Sp}_n$, we denote $g = \begin{pmatrix} a_g & b_g \\ c_g & d_g \end{pmatrix}$ with $a_g, b_g, c_g, d_g \in M_n$. Define the Siegel parabolic subgroup P_n by

$$P_n(R) = \left\{g \in \mathrm{Sp}_n(R) \mid c_g = 0\right\}.$$

We define a congruence subgroup $\Gamma_0^{(n)}(\mathfrak{n})$ by

$$\Gamma_0^{(n)}(\mathfrak{n}) = \left\{g \in \mathrm{Sp}_n(\mathcal{O}_F) \mid c_g \equiv 0 \bmod \mathfrak{n}\right\}.$$

We define the Siegel upper half space of degree n by

$$\mathfrak{H}_n = \left\{z \in \mathrm{Sym}_n(\mathbf{C}) \mid z = x + iy,\ x,\ y \in \mathrm{Sym}_n(\mathbf{R}),\ y > 0\right\}.$$

Let k be a positive integer and assume $\chi_f(-1) = (-1)^k$. Define a Siegel Eisenstein series of degree n, character χ, weight k by

$$E_{k,\chi}^{(n)}(z) = \sum_{g \in P_n(\mathcal{O}_F) \cap \Gamma_0^{(n)}(\mathfrak{n}) \backslash \Gamma_0^{(n)}(\mathfrak{n})} \chi_f^{-1}(\det d_g) \det(c_g z + d_g)^{-k}.$$

Here $z = (z_v)_{v|\infty} \in \prod_{v|\infty} \mathfrak{H}_n$ and $\det(c_g z + d_g)^{-k}$ is defined by

$$\det(c_g z + d_g)^{-k} = \prod_{v|\infty} \det(\iota_v(c_g) z_v + \iota_v(d_g))^{-k}.$$

In the rest of this paper, we assume that $\mathfrak{n}$ is relatively prime to 2 for simplicity.

Put

$$\mathcal{P} = \left\{\mathfrak{p} : \text{a prime of } F \mid \mathfrak{p} \mid \mathfrak{n} \text{ and } \widetilde{\chi}_{\mathfrak{p}}^2 \text{ is unramified}\right\}.$$

Let $g \in \mathrm{Sp}_n(\mathcal{O}_F)$ and assume $c_g \in \mathfrak{p}^{\mathrm{ord}_{\mathfrak{p}}(\mathfrak{n})}$ if $\mathfrak{p} \notin \mathcal{P}$ and $\mathrm{rank}_{\mathcal{O}_F/\mathfrak{p}}(c_g \bmod \mathfrak{p}) = i_{\mathfrak{p}}$ with $0 \leq i_{\mathfrak{p}} \leq n$ if $\mathfrak{p} \in \mathcal{P}$. For $\mathfrak{p} \in \mathcal{P}$, the assumption for g implies $g \bmod \mathfrak{p} \in P_n(\mathcal{O}_F/\mathfrak{p}) w_{i_{\mathfrak{p}}} P_n(\mathcal{O}_F/\mathfrak{p})$. Therefore if $\mathfrak{p} \in \mathcal{P}$, there exist elements $x_{\mathfrak{p}},\ y_{\mathfrak{p}} \in \mathrm{GL}(\mathcal{O}_F/\mathfrak{p})$ satisfying

$$g \bmod \mathfrak{p} = \begin{pmatrix} x_{\mathfrak{p}} & * \\ 0 & {}^t x_{\mathfrak{p}}^{-1} \end{pmatrix} w_{i_{\mathfrak{p}}} \begin{pmatrix} y_{\mathfrak{p}} & * \\ 0 & {}^t y_{\mathfrak{p}}^{-1} \end{pmatrix}.$$

We put

$$\chi(\{i_{\mathfrak{p}}\}_{\mathfrak{p}\in\mathcal{P}}\,;g) = \prod_{\substack{\mathfrak{p}|\mathfrak{n} \\ \mathfrak{p}\notin\mathcal{P}}} (\chi_f)_{\mathfrak{p}} (\det d_g) \prod_{\mathfrak{p}\in\mathcal{P}} (\chi_f)_{\mathfrak{p}} (\det x_{\mathfrak{p}} \det y_{\mathfrak{p}}).$$

Here $(\chi_f)_{\mathfrak{p}}$ is the $\mathfrak{p}$-component of χ_f.

We define an auxiliary Siegel Eisenstein series $E'_{k,\chi}(\{i_{\mathfrak{p}}\}_{\mathfrak{p}\in\mathcal{P}}\,;z)$ as follows.

$$E'_{k,\chi}(\{i_{\mathfrak{p}}\}_{\mathfrak{p}\in\mathcal{P}}\,;z) = \sum_{g} \chi(\{i_{\mathfrak{p}}\}_{\mathfrak{p}\in\mathcal{P}}\,;g)^{-1} \det(c_g z + d_g),$$

where g runs over the set $P_n(F) \cap \mathrm{Sp}_n(\mathcal{O}_F)\backslash\mathrm{Sp}_n(\mathcal{O}_F)$ satisfying the property $c_g \in \mathfrak{p}^{\mathrm{ord}_{\mathfrak{p}}(\mathfrak{n})}$ if $\mathfrak{p} \notin \mathcal{P}$ and $\mathrm{rank}_{\mathcal{O}_F/\mathfrak{p}}(c_g \bmod \mathfrak{p}) = i_{\mathfrak{p}}$ if $\mathfrak{p} \in \mathcal{P}$. By the definition, we have $E'_{k,\chi}(\{i_{\mathfrak{p}}\}_{\mathfrak{p}\in\mathcal{P}}\,;z) = E^{(n)}_{k,\chi}$ if $i_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in \mathcal{P}$.

Let $\mathfrak{p}$ be a prime of F and assume $\mathfrak{p} \in \mathcal{P}$ and $(\mathfrak{p}, 2) = 1$. For $0 \le i \le n$ and $s \in \mathbf{C}$, we put $M_{in}(s, \widetilde{\chi}_{\mathfrak{p}}) = 0$ if i is odd and put

$$M_{in}(s, \widetilde{\chi}_{\mathfrak{p}}) = \widetilde{\chi}_{\mathfrak{p}}(-1)\mathrm{N}\mathfrak{p}^{-i/2} \prod_{a=0}^{i/2} (1 - \mathrm{N}\mathfrak{p}^{-1-2a})(1 - \widetilde{\chi}^2_{\mathfrak{p}}(\mathfrak{p})\mathrm{N}\mathfrak{p}^{-2s+2a+n-i-2}),$$

if i is even. We set

$$m_i(k, \chi) = M_{in}(-k + \frac{n+1}{2}, \widetilde{\chi}_{\mathfrak{p}}).$$

By definition, the right hand side does not depend on n.

We define an Eisenstein $G^{(n)}_{k,\chi}$ as a linear combination of $E'_{k,\chi}(\{i_{\mathfrak{p}}\}_{\mathfrak{p}\in\mathcal{P}}\,;z)$.

Definition 2.1. If $\mathcal{P} \neq \emptyset$, we define

$$G^{(n)}_{k,\chi}(z) = \sum_{\{i_{\mathfrak{p}}\}_{\mathfrak{p}\in\mathcal{P}}} \left(\prod_{\mathfrak{p}\in\mathcal{P}} m_{i_{\mathfrak{p}}}(k, \chi) \right) E'_{k,\chi}(\{i_{\mathfrak{p}}\}_{\mathfrak{p}\in\mathcal{P}}\,;z).$$

Here $\{i_{\mathfrak{p}}\}_{\mathfrak{p}\in\mathcal{P}}$ runs over all the non-empty subsets of $\prod_{\mathfrak{p}\in\mathcal{P}} \{0, \dots, n\}$. If $\mathcal{P} = \emptyset$, we define

$$G^{(n)}_{k,\chi} = E^{(n)}_{k,\chi}.$$

Remark 2.1. We can define $G^{(n)}_{k,\chi}$ more naturally by using the intertwining operator. But to shorten the statement, we define $G^{(n)}_{k,\chi}$ in this way. (See subsection 3.1)

The first main theorem of this paper is the result for Fourier coefficients for $G^{(n)}_{k,\chi}$. We prepare some notation.

Let $B \in \mathrm{Sym}^*_n(\mathcal{O}_F)$ be a half integral matrix of size n. Put $r = \mathrm{rank}\, B$. There exists a matrix $A \in \mathrm{GL}_n(F)$ such that

$${}^tABA = \begin{pmatrix} B' & 0 \\ 0 & 0 \end{pmatrix},$$

with $B' \in \mathrm{Sym}_r(F)$. Then $\det B' \in F^\times/F^{\times 2}$ does not depend on the choice of A. If r is even we denote by χ_B the narrow class character of F associated with the extension $F(\sqrt{(-1)^{r/2}\det B'})/F$ by the global class field theory.

For a prime $\mathfrak{p}$ of F such that $\mathfrak{p} \nmid \mathfrak{n}$, there exists a matrix $U \in \mathrm{GL}_n(\mathcal{O}_\mathfrak{p})$ that satisfies

$$ {}^tUBU = \begin{pmatrix} B'_\mathfrak{p} & 0 \\ 0 & 0 \end{pmatrix}, $$

with $B'_\mathfrak{p} \in \mathrm{Sym}^*_r(\mathcal{O}_\mathfrak{p})$. The matrix $B'_\mathfrak{p}$ is unique up to unimodular equivalence. Therefore $\Phi^{(r)}_\mathfrak{p}(B'_\mathfrak{p};T)$ does not depend on the choice of U, where $\Phi^{(r)}_\mathfrak{p}(B'_\mathfrak{p};T)$ is the polynomial obtained by the Siegel series. (In the notation of [4] 13.6. Theorem, we have $\Phi^{(r)}_\mathfrak{p}(B'_\mathfrak{p};T) = f_{B'_\mathfrak{p}}(T)$.) Thus we put $\Phi^{(r)}_\mathfrak{p}(B,T) = \Phi^{(r)}_\mathfrak{p}(B'_\mathfrak{p},T)$.

Theorem 2.1. *Let $0 \le B \in \mathrm{Sym}^*_n(\mathcal{O}_F)$ be a half integral positive semi-definite matrix of size n and $k > n+1$ an integer. Let χ be a primitive narrow class character of F of conductor $\mathfrak{n}$. Put $r = \mathrm{rank}\, B$. Then the following assertions hold.*

If r is even, then $a(B, G^{(n)}_{k,\chi})$ is given by

$$ 2^{rm/2}\prod_{\mathfrak{p}\nmid\mathfrak{n}} \Phi^{(r)}_\mathfrak{p}(B;\chi(\mathfrak{p})\mathrm{N}\mathfrak{p}^{k-r-1}) $$
$$ \times L(1-k,\chi)^{-1} L^{(\mathfrak{n})}(1+r/2-k,\chi_h\chi)\prod_{i=1}^{r/2} L^{(\mathfrak{n})}(1+2i-2k,\chi^2)^{-1}. $$

If r is odd, then $a(B, G^{(n)}_{k,\chi})$ is given by

$$ 2^{(r+1)m/2}\prod_{\mathfrak{p}\nmid\mathfrak{n}} \Phi^{(r)}_\mathfrak{p}(B;\chi(\mathfrak{p})\mathrm{N}\mathfrak{p}^{k-r-1}) $$
$$ \times L(1-k,\chi)^{-1} \prod_{i=1}^{(r-1)/2} L^{(\mathfrak{n})}(1+2i-k,\chi^2)^{-1}. $$

For a Hecke L-function $L(s,\chi)$ and an ideal $\mathfrak{n}$, we denote $L^{(\mathfrak{n})}(s,\chi) = \prod_{\mathfrak{p}\nmid\mathfrak{n}}(1-\chi(\mathfrak{p})\mathrm{N}\mathfrak{p}^{-s})^{-1}$, where the index $\mathfrak{p}$ runs over the set of primes of F relatively prime to the conductor of χ and the ideal $\mathfrak{n}$.

From this theorem and the definition of $G^{(n)}_{k,\chi}$, we have a formula for the Fourier coefficients of $E^{(n)}_{k,\chi}$ if $\mathcal{P} = \emptyset$.

Remark 2.2. When $F = \mathbf{Q}$, Katsurada [1] proved the explicit formula for $\Phi^{(r)}_\mathfrak{p}(B,T)$, thus in this case we have the explicit formula for Fourier coefficients of $G^{(n)}_{k,\chi}$.

For a prime $\mathfrak{p}$ of F, we put

$$ P_+(\mathfrak{p}^\alpha) = \left\{ a\mathcal{O}_F \mid a\text{:positive definite and } a \equiv \mathrm{mod}\,\mathfrak{p}^\alpha \right\}. $$

We define the narrow ray class group of F of conductor $\mathfrak{p}^\alpha$ by $\mathrm{Cl}_F(\mathfrak{p}^\alpha) = I_\mathfrak{p}/P_+(\mathfrak{p}^\alpha)$. We consider the projective limit

$$\mathrm{Cl}_F(\mathfrak{p}^\infty) = \varprojlim \mathrm{Cl}_F(\mathfrak{p}^\alpha).$$

We put $G = \mathrm{Cl}_F(\mathfrak{p}^\infty)$.

Let Ω be the completion of $\overline{F}_\mathfrak{p}$ and A the integer ring of Ω. We fix the embedding of $\overline{\mathbf{Q}}$ to $\mathbf{C}$ and Ω. We denote $\mathrm{Meas}(G, A)$ by the bounded $\mathfrak{p}$-adic measure on G with values in A. Let p be the residual characteristic of $F_\mathfrak{p}$. Since $I_\mathfrak{p}$ can be considered as a dense subgroup of G and the norm map $N : I_\mathfrak{p} \to \mathbf{Z}_p^\times$ is continuous, we can extend N to G. We denote the extended character by the same letter. Let ω be the Teichmüller character of $\mathbf{Z}_p^\times$ and put $\omega_F = \omega \circ N$.

Theorem 2.2. *Let $\mathfrak{p}$ be a prime of F such that $(\mathfrak{p}, 2) = 1$, p a residual characteristic of $F_\mathfrak{p}$ and χ a narrow ray class character of conductor $\mathfrak{p}^\nu$. Denote $\mathcal{O}_F[\chi]$ by the ring generated by $\mathrm{Im}(\chi)$ over $\mathcal{O}_F$. Then there exists a formal Fourier expansion $\mathbf{G}^{(n)}(\chi; T)$*

$$\mathbf{G}^{(n)}(\chi; T) = \sum_{0 \le B \in \mathrm{Sym}_n^{(*)}(\mathcal{O}_F)} \mathbf{a}(B; T)\mathbf{e}(Bz),$$

where $\mathbf{a}(B; T)$ is an element of the quotient ring of the formal power series ring $\mathrm{Frac}\,\mathcal{O}_F[\chi][\![T]\!]$, and satisfies the following condition. If $k > n+1$ and $\chi \cdot \omega_F^{-k}$ is not the trivial character modulo $\mathfrak{p}$

$$\mathbf{G}^{(n)}(\chi; u^k - 1) = G_{k, \chi \cdot \omega_F^{-k}},$$

where u is a fixed generator of $1 + \mathbf{Z}_p$. Moreover, there exists a nonzero formal power series $\mathbf{b}(T)$ and a $\mathfrak{p}$-adic measure $\mu_B \in \mathrm{Meas}(G, A)$ for each B that satisfy

$$\mathbf{b}(u^s - 1)\mathbf{a}(B; u^s - 1) = \int_G \chi(x)N(x)^{-1}\langle N(x)\rangle^s d\mu_B, \quad \text{for } s \in \mathbf{Z}_p.$$

Here for $a \in \mathbf{Z}_p^\times$, we put $\langle a \rangle = a\omega^{-1}(a)$.

Remark 2.3. In the interpolation property, we assume the character is not trivial character mod $\mathfrak{p}$. H. Kawamura [2] proved the existence p-adic family of Siegel Eisenstein series that interpolates Eisenstein series with trivial character modulo p.

3 Sketch of the proof of the main theorem

Since we can derive theorem 2.2 by theorem 2.1 and the existence of $\mathfrak{p}$-adic Heck L-functions for totally real fields, we only prove theorem 2.1.

3.1 Definition of an Eisenstein series $\widetilde{G}^{(n)}_{k,\chi}$

In this subsection, we give a more natural definition of $G^{(n)}_{k,\chi}$.

For a place v of F, we denote the space for the normalized induction by

$$\mathrm{Ind}_{P_n}^{\mathrm{Sp}_n}(\widetilde{\chi}_{\mathfrak{p}}|\cdot|_{\mathfrak{p}}^{s}).$$

We define the intertwining operator

$$M_{w_n}^{(s)}:\mathrm{Ind}_{P_n}^{\mathrm{Sp}_n}(\widetilde{\chi}_v|\cdot|_v^{s})\to\mathrm{Ind}_{P_n}^{\mathrm{Sp}_n}(\widetilde{\chi}_v^{-1}|\cdot|_v^{-s})$$

by

$$M_{w_n}^{(s)}(f)(g)=\int_{\mathrm{Sym}_n(F_v)}f\left(w_n\begin{pmatrix}1&x\\0&1\end{pmatrix}g\right)dx,$$

for $g\in\mathrm{Sp}_n(F_v)$. Here we take a Haar measure of $\mathrm{Sym}_n(F_v)$ so that we have $\int_{\mathrm{Sym}_n(\mathcal{O}_v)}dx=1$. The integral is convergent if $\mathrm{Re}\,s$ is sufficiently large and has meromorphic continuation to the whole complex plane.

We define a compact subgroup $C_{0,v}$ of $\mathrm{Sp}_n(F_v)$ as follows.

(i) If v is real or $\widetilde{\chi}_v$ is unramified then we define $C_{0,v}=C_v$.

(ii) If $v=\mathfrak{p}$ is a finite place and $\widetilde{\chi}_{\mathfrak{p}}$ is ramified then we define

$$C_{0,v}=\left\{g\in\mathrm{Sp}_n(\mathcal{O}_v)\;\middle|\;c_g\equiv 0\bmod\mathfrak{p}^{\nu}\right\}.$$

Here $\mathfrak{p}^{\nu}$ is the conductor of $\widetilde{\chi}_v$.

We define a character κ_v of $C_{0,v}$ as follows.

(i) If v is real then we define

$$\kappa_v\left(\begin{pmatrix}u&-v\\v&u\end{pmatrix}\right)=\det(u+iv)^{-k}.$$

(ii) If v is finite and $\widetilde{\chi}_v$ is unramified then we define $\kappa_v=1$.

(iii) If v is finite and $\widetilde{\chi}_v$ is ramified then we define

$$\kappa_v(\gamma)=\widetilde{\chi}_v(\det d_\gamma).$$

We denote by $\phi_v(s,\cdot)$ the element of $\mathrm{Ind}_{P_n}^{\mathrm{Sp}_n}(\widetilde{\chi}_v|\cdot|_v^{s})$ satisfying the following conditions.

$$\begin{aligned}&\mathrm{supp}\phi_v(s,\cdot)=P_n(F_v)C_{0,v},\\&\phi_v(s,g\gamma)=\kappa_v(\gamma)\phi_v(s,g)\quad\text{for all }\gamma\in C_{0,v},\\&\phi_v(s,1)=1.\end{aligned}$$

We also denote by $\phi'_v(-s,\cdot)$ the element of $\mathrm{Ind}_{P_n}^{\mathrm{Sp}_n}(\widetilde{\chi}_v^{-1}|\cdot|_v^{-s})$ satisfying the following conditions.

$$\begin{aligned} &\mathrm{supp}\phi'_v(-s,\cdot) = P_n(F_v)w_nC_{0,v}, \\ &\phi'_v(-s,g\gamma) = \kappa_v(\gamma)\phi'_v(-s,g) \quad \text{for all } \gamma \in C_{0,v}. \\ &\phi'_v(-s,w_n) = 1. \end{aligned} \tag{3.1}$$

For a place v of F, we define $\varphi_v(s,\cdot) \in \mathrm{Ind}_{P_n}^{\mathrm{Sp}_n}(\widetilde{\chi}_v|\cdot|_v^{s})$ as follows.

(i) If v is real or $\widetilde{\chi}_v$ is unramified then we define

$$\varphi_v(s,g) = \phi_v(s,g).$$

(ii) If v is finite and $\widetilde{\chi}_v$ is ramified then we define

$$\varphi_v(s,g) = M_{w_n}^{(-s)}(\phi'_v(-s,\cdot))(g). \tag{3.2}$$

For $g = (g_v)_v \in \mathrm{Sp}_n(\mathbb{A}_F)$, we put

$$\varphi(s,g) = \prod_v \varphi_v(s,g_v),$$

where v runs over the set of the places of F.

We define an Eisenstein series on $\mathrm{Sp}_n(\mathbb{A}_F)$ by

$$\widetilde{\mathcal{G}}_{s,\chi}^{(n)}(g) = \sum_{\gamma\in P_n(F)\backslash \mathrm{Sp}_n(F)} \varphi(s,\gamma g).$$

We define $\widetilde{G}_{k,\chi}^{(n)}$ by the function on $\prod_{v|\infty}\mathfrak{H}_n$ corresponding to $\widetilde{\mathcal{G}}_{k-(n+1)/2,\chi}^{(n)}$.

We can prove the proposition bellow by explicit calculation of the value of the intertwining operator. We omit the proof.

Proposition 3.1. *Assume that* $\mathfrak{n}$ *is relatively prime to* 2. *Then we have*

$$G_{k,\chi}^{(n)} = \widetilde{G}_{k,\chi}^{(n)}.$$

3.2 Functional equation of Whittaker functions

The key ingredient for the proof of the main theorem is the following theorem by T. Ikeda (Kyoto University).

Theorem 3.1 (T. Ikeda). *Let k be a local field. Let ψ be a nontrivial additive character of k and χ be a quasi character of $k^\times$. Suppose $B \in \mathrm{Sym}_n(k)$ and* $\det B \neq 0$. *For $f \in \mathrm{Ind}_{P_n}^{\mathrm{Sp}_n}(\widetilde{\chi}_{\mathfrak{p}}|\cdot|_{\mathfrak{p}}^{s})$, we put*

$$W_B(f)(g) = \int_{\mathrm{Sym}_n(k)} f\left(w_n\begin{pmatrix}1_n & x\\ 0_n & 1_n\end{pmatrix}\right)\psi(-\mathrm{Tr}Bx)dx.$$

Then

$$W_B \circ M_{w_n} = \chi(\det B)^{-1}|\det B|^{-s}c(s,B)W_B.$$

The notation is as follows.

Let n be even. D_B is defined by $D_B = (-1)^{n/2}\det B$. χ_B is the character of $k^\times$ corresponding to $k(\sqrt{D_B})/k$. $c(s,B)$ is given as follows.

$$c(s,B) = |2|^{-ns}\frac{\alpha(D_B)}{\alpha(1)}\chi(2)^{-n}\varepsilon'(s+\frac{1}{2},\chi\chi_B,\psi)$$
$$\times\, \varepsilon'(s-\frac{n-1}{2},\chi,\psi)^{-1}\prod_{r=1}^{n/2}\varepsilon'(2s-n+2r,\chi^2,\psi)^{-1}.$$

Here $= \varepsilon(s,\omega,\psi)$ is the epsilon factor, $\varepsilon'(s,\omega,\psi) = \varepsilon(s,\omega,\psi)\frac{L(1-s,\omega^{-1})}{L(s,\omega)}$ and $\alpha(*)$ is the Weil index.

$$\frac{\alpha(1)}{\alpha(D_B)} = \varepsilon(\frac{1}{2},\chi_B,\psi).$$

Let n be odd. Then

$$c(s,B) = |2|^{-(n-1)s}\chi(2)^{-(n-1)}\zeta_B$$
$$\times\, \varepsilon'(s-\frac{n-1}{2},\chi,\psi)^{-1}\prod_{r=1}^{(n-1)/2}\varepsilon'(2s-n+2r,\chi^2,\psi)^{-1}.$$

Here

$$\zeta_B = ((-1)^{(n-1)/2},\det B)(-1,-1)^{(n^2-1)/8}h(B),$$

and $(*,*)$ is the Hilbert symbol.

3.3 Sketch of the proof

sketch of the proof of theorem 2.1. For simplicity, we assume the class number of F is one. Denote Φ by the Siegel operator. Then by the definition of $G^{(n)}_{k,\chi}$, we have $\Phi G^{(n)}_{k,\chi} = G^{(n-1)}_{k,\chi}(z)$. Thus it is enough to compute $a(B, G^{(n)}_{k,\chi})$ when $\det B \neq 0$. We can prove that $a(B, G^{(n)}_{k,\chi})$ has Euler product expression. By [3] (4.34K), (4.35K), [4] 13.6. Theorem, we know the Euler factor at infinite places and unramified places. Thus it is enough to compute the Euler factors at ramified places. Let $\mathfrak{p} \mid \mathfrak{n}$. Then the Euler factor at $\mathfrak{p}$ is given by

$$\int_{\mathrm{Sym}_n(F_\mathfrak{p})}\varphi_v\left(k-(n+1)/2, w_n\begin{pmatrix}1_n & x\\ 0_n & 1_n\end{pmatrix}\right)\mathrm{e}(-\mathrm{Tr}Bx)dx$$
$$= W_B \circ M_{w_n}(\phi'(-k+(n+1)/2,\cdot))(1_n).$$

By theorem 3.1, it is enough to compute $W_B(\phi'(-k+(n+1)/2,\cdot))(1_n)$, but it is easy to verify that this equals to 1. □

References

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