数理解析研究所講究録
第 1829 巻 2013 年 9-18

A revision of the conical algorithm with ω-bisection and its convergence

(ω-bisection による新しい錐分割アルゴリズムとその収束性について)

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Abstract

In this paper, we sketch out an alternative proof of the convergence of the conical algorithm with ω-subdivision for concave minimization. We show that the algorithm can converge under a more general subdivision rule including ω-bisection, and propose ω-bisection as a third pure subdivision rule alongside bisection and ω-subdivision.

Key words: Global optimization, concave minimization, conical algorithm, ω-subdivision, ω-bisection.

1 Introduction

The concave minimization is a typical multiextremal optimization problem, and known to be NP-hard from the viewpoint of computational complexity [8]. To find a globally optimal solution, Tuy made use of valid cuts and proposed in 1964 a first systematic solution method, the *conical algorithm*, which turned out later to have no guarantee of convergence. Bali [1] and Zwart [12] modified the algorithm independently and introduced the same device, i.e., *ω-subdivision*, in the early 1970s. Since then, the question whether the algorithm always converges under the ω-subdivision rule had been open for nearly three decades, until Jaumard-Meyer [4, 5] and Locatelli [6] proved it affirmatively. However, almost ten years earlier than those, Tuy showed in [10] that the algorithm converges if sequences of nested cones generated in the algorithm satisfy a certain nonsingularity condition. Unfortunately, it is still an open question, and neither Jaumard-Meyer nor Locatteli used this condition in their proofs of convergence.

In this paper, using another condition similar to Tuy's nonsingularity, we sketch out an alternative proof of convergence for the conical algorithm with ω-subdivision. We also show that the algorithm can converge under an even more general subdivision rule including ω-subdivision, and propose *ω-bisection* as a third pure subdivision rule alongside bisection and ω-subdivision.

*This author was partially supported by a Grant-in-Aid for Challenging Exploratory Research (22651057) from the Japan Society of the Promotion of Sciences.
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2 D.c. feasibility and the conical algorithm

Let $f : S(\subset \mathbb{R}^n) \to \mathbb{R}$ be a concave function and denote its upper level set for a real number α by

$$C(\alpha) = \{\mathbf{x} \in S \mid f(\mathbf{x}) \geq \alpha\}.$$

Also let $D \subset \mathbb{R}^n$ be a polyhedron defined as

$$D = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} \leq \mathbf{b}\},$$

where $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$ $(n < m)$. We assume that D has nonempty interior and is included in the interior of S, and hence f is continuous on D. Both $C(\alpha)$ and D are convex sets, but their difference $D \setminus C(\alpha)$ is not convex in general. The problem we consider is to search for a point in this *d.c. set* (difference of two convex sets) within a prescribed tolerance $\varepsilon \geq 0$, i.e.,

$$(\mathrm{DC}):\ \text{find a point } \mathbf{x} \in D \setminus C(\alpha) \text{ if there is one, or else prove that } D \subset C(\alpha - \varepsilon),$$

which is called the *d.c. feasibility problem.* For the sake of simplicity, we assume that both $C(\alpha)$ and D are bounded sets. As a consequence, $C(\gamma)$ is also bounded for any number γ other than α, since all nonempty level sets of a concave function have the same recession cone (see e.g. Theorem 8.7 in [7]).

Associated with (DC) is the concave minimization problem

$$\left|\begin{array}{ll} \text{minimize} & f(\mathbf{x}) \\ \text{subject to} & \mathbf{x} \in D. \end{array}\right. \tag{1}$$

It is known, e.g., [10, 11], that a globally ε-optimal solution $\mathbf{x}^*$ of (1) can be computed according to the following two-phase scheme:

Let $\mathbf{z}^1 \in D$ be an initial feasible solution of (1). Also let $t \leftarrow 1$.

Phase 1 (local phase). Starting from $\mathbf{z}^t$, search the vertices of D for a local minimizer of f. Then a vertex $\mathbf{x}^t$ is obtained such that $f(\mathbf{x}^t) \leq f(\mathbf{z}^t)$ and $f(\mathbf{x}^t) \leq f(\mathbf{x})$ for every vertex $\mathbf{x}$ adjacent to $\mathbf{x}^t$.

Phase 2 (global phase). Solve (DC) for $\alpha = f(\mathbf{x}^t)$. If $D \subset C(\alpha - \varepsilon)$, then $\mathbf{x}^* \leftarrow \mathbf{x}^t$ and terminate: $\mathbf{x}^*$ is a globally ε-optimal solution of (1). Otherwise, a feasible solution $\mathbf{z} \in D$ is obtained such that $f(\mathbf{z}) < f(\mathbf{x}^t)$. Let $\mathbf{z}^{t+1} \leftarrow \mathbf{z}$, $t \leftarrow t+1$, and go to Phase 1.

Alternating between these two phases generates a sequence of vertices $\{\mathbf{x}^t \mid t = 1, 2, \dots\}$ of D such that $f(\mathbf{x}^{t+1}) < f(\mathbf{x}^t)$. Since the number of vertices of a polyhedron is finite, it terminates after finitely many repetitions if (DC) can be solved in finite time. Our goal is therefore to solve (DC) in finite time, using the conical algorithm.

OUTLINE OF THE CONICAL ALGORITHM

Let $\gamma = \alpha - \varepsilon$, and $\mathbf{v}$ be a vertex of D such that $f(\mathbf{v}) > \gamma$. In the above two-phase scheme, such a vertex can be easily found in the process of searching for $\mathbf{x}^t$ because $\mathbf{x}^t$ is a local minimizer and $f(\mathbf{x}^t) > \gamma$ when $\varepsilon > 0$. By perturbing $\mathbf{b}$ slightly if necessary, we may assume that $\mathbf{v}$ is a nondegenerate vertex of D. The system defining D is then partitioned into

$$\mathbf{B}\mathbf{v} = \mathbf{b}_B, \quad \mathbf{N}\mathbf{v} < \mathbf{b}_N,$$

where $\mathbf{B} \in \mathbb{R}^{n\times n}$, $\mathbf{N} \in \mathbb{R}^{(m-n)\times n}$ are submatrices of $\mathbf{A}$, and $\mathbf{b}_B \in \mathbb{R}^n$, $\mathbf{b}_N \in \mathbb{R}^{(m-n)}$ are the corresponding portions of $\mathbf{b}$. Let

$$\Lambda = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{B}\mathbf{x} \leq \mathbf{b}_B\}, \quad M = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{N}\mathbf{x} \leq \mathbf{b}_N\}.$$

Then we have

$$D = M \cap \Lambda.$$

Since the vertex $\mathbf{v}$ is nondegenerate, it is an interior point of M. It should also be noted that Λ is a polyhedral cone with vertex $\mathbf{v}$ and has exactly n edges. Let $\mathbf{d}_1, \ldots, \mathbf{d}_n$ be directions of the edges of Λ. These vectors are obtained immediately from a general solution of the linear system $\mathbf{A}\mathbf{x} + \mathbf{w} = \mathbf{b}$, where $\mathbf{w} \in \mathbb{R}^m$ is the vector of slack variables.

To simplify the explanation, let us translate the origin $\mathbf{0}$ to $\mathbf{v}$, and again denote by $\mathbf{N}\mathbf{x} \leq \mathbf{b}_N$ the resulting system that defines M. We may assume in the sequel that $\mathbf{b}_N > \mathbf{0}$ because $\mathbf{0}$ has moved to the interior of M. For $j = 1, \ldots, n$, let $\mathbf{q}_j$ denote the *γ-extension* of $\mathbf{d}_j$, i.e.,

$$\mathbf{q}_j = \mathrm{ext}(\mathbf{d}_j) \equiv \theta_j \mathbf{d}_j,$$

where

$$\theta_j = \sup\{\theta \mid f(\theta \mathbf{d}_j) \geq \gamma\}.$$

Then we have

$$\Lambda = \mathrm{con}(\mathbf{Q}) \equiv \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} = \sum_{j=1}^{n} \lambda_j \mathbf{q}_j,\ \boldsymbol{\lambda} \geq \mathbf{0}\},$$

where

$$\mathbf{Q} = [\mathbf{q}_1, \ldots, \mathbf{q}_n] \in \mathbb{R}^{n\times n}.$$

Note that $\mathbf{q}_j$'s are linearly independent and $\mathbf{Q}$ is invertible. Therefore, $\mathbf{q}_j$'s determine a unique hyperplane, which is the boundary of

$$G = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{e}\mathbf{Q}^{-1}\mathbf{x} \leq 1\},$$

where $\mathbf{e} \in \mathbb{R}^n$ is the all-ones row vector. Obviously, $G \cap \Lambda$ is a simplex with $n+1$ vertices $\mathbf{q}_j$'s and $\mathbf{0}$, all belonging to $C(\gamma)$. From the convexity of $C(\gamma)$ we see that

$$G \cap \Lambda \subset C(\gamma).$$

Accordingly, if $M \cap \Lambda$ is a subset of G, we can conclude that (DC) is solved because

$$D = M \cap \Lambda \subset G \cap \Lambda \subset C(\gamma) = C(\alpha - \varepsilon).$$

The process of checking whether $M \cap \Lambda \subset G$ or not is usually called *bounding*. We also refer to G as the *γ-valid cut*[1] for the cone Λ.

[1] In some literature, the term "γ-valid cut" refers to the closure of the complement of G.

If $M \cap \Lambda$ is not a subset of G, then either a point $\mathbf{x} \in D \setminus C(\alpha)$ is found or Λ needs to be divided into subcones for further examinations. In the latter case, an appropriate direction $\mathbf{u}$ is first selected from $\Lambda \setminus \{\mathbf{0}\}$. There exists a vector $\boldsymbol{\lambda}' \geq \mathbf{0}$ such that $\mathbf{u} = \sum_{j=1}^{n} \lambda'_j \mathbf{q}$. Let $J = \{j \mid \lambda'_j > 0\}$. Then Λ *is subdivided along* $\mathbf{u}$ into $|J|$ subcones:

$$\Lambda^j = \mathrm{con}(\mathbf{Q}^j), \quad j \in J,$$

where $\mathbf{Q}^j$ is referred to as a *child* of $\mathbf{Q}$ and defined as

$$\mathbf{Q}^j = [\mathbf{q}_1, \ldots, \mathbf{q}_{j-1}, \mathrm{ext}(\mathbf{u}), \mathbf{q}_{j+1}, \ldots, \mathbf{q}_n].$$

It is easy to see that

$$\mathrm{int}(\Lambda^i) \cap \mathrm{int}(\Lambda^j) = \emptyset \;\text{ if }\; i \neq j; \quad \Lambda = \bigcup_{j \in J} \Lambda^j.$$

In other words, the cones Λ^j's constitute a partition of Λ. This process of dividing Λ is called *branching*. After branching, the bounding process is again applied to each subcone Λ^j.

3 Pseudo-nonsingularity and Convergence of the algorithm

Suppose the conical algorithm is infinite and generates a sequence of nested cones:

$$\Lambda = \Lambda_1 \supset \cdots \supset \Lambda_k \supset \Lambda_{k+1} \supset \cdots, \tag{2}$$

where Λ_{k+1} is a cone obtained by subdividing Λ_k along a direction $\mathbf{u}^k$. For each k, the cone Λ_k is spanned by an invertible matrix $\mathbf{Q}_k$, i.e.,

$$\Lambda_k = \mathrm{con}(\mathbf{Q}_k) \equiv \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} = \sum_{j=1}^{n} \lambda_j \mathbf{q}_j^k,\ \boldsymbol{\lambda} \geq \mathbf{0}\},$$

where $\mathbf{q}_j^k$ is the jth column of $\mathbf{Q}_k$ and lies on the boundary of $C(\gamma)$. Let us denote the γ-valid cut for Λ_k by

$$G_k = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{e}\mathbf{Q}_k^{-1}\mathbf{x} \leq 1\}.$$

As seen in the previous section, we have $M \cap \Lambda_k \subset C(\gamma)$ if $M \cap \Lambda_k \subset G_k$. This can be checked by solving an auxiliary problem

$$\left|\begin{array}{ll} \text{maximize} & \mathbf{e}\mathbf{Q}_k^{-1}\mathbf{x} \\ \text{subject to} & \mathbf{x} \in M \cap \Lambda_k. \end{array}\right. \tag{3}$$

Let $\boldsymbol{\omega}^k$ be an optimal solution of (3) and ζ^k the optimal value, i.e., $\zeta^k = \mathbf{e}\mathbf{Q}_k^{-1}\boldsymbol{\omega}^k$. If $f(\boldsymbol{\omega}^k) < \alpha$, then $\boldsymbol{\omega}^k$ is obviously a solution to (DC), and the conical algorithm terminates. Since the sequence (2) is infinite, that is not the case and we assume that

$$f(\boldsymbol{\omega}^k) \geq \alpha \geq \gamma, \quad k = 1, 2, \ldots. \tag{4}$$

Similarly, if $\zeta^k \leq 1$, then $M \cap \Lambda_k \subset G_k$, and we can conclude that Λ_k contains no solution to (DC). In that case, Λ_k is discarded from further consideration. However, we assume here that

$$\zeta^k > 1, \quad k = 1, 2, \ldots. \tag{5}$$

The conical algorithm is known to be convergent if the direction $\mathbf{u}^k$ of subdividing Λ_k coincides with $\boldsymbol{\omega}^k$ for every k. This subdivision rule is called ω-subdivision, and the convergence result was established independently by Jaumard-Meyar in 98 [?] and by Locatelli in 99 [6]. Almost ten years earlier than those, Tuy showed that the algorithm with ω-subdivision converges if the sequence (2) is *nonsingular*[2] [10] (see also [3]), i.e., there exists a subsequence $\{k_r \mid r = 1, 2, \ldots\}$ and a constant L such that

$$\|\mathbf{e}\mathbf{Q}_{k_r}^{-1}\| \leq L, \quad r = 1, 2, \ldots. \tag{6}$$

Unfortunately, it remains an open question whether (6) holds or not. In the rest of this section, we introduce another problem equivalent to (3) and show that the coefficients of its objective function satisfies a condition similar to (6). For this reason, we say that the sequence (2) is *pseudo-nonsingular*, from which we will derive the convergence result under a more general condition than ω-subdivision.

Linear program equivalent to (3)

The auxiliary problem (3) is a linear program of the form

$$(\mathrm{P}_k) \left| \begin{array}{ll} \text{maximize} & \mathbf{e}\mathbf{Q}_k^{-1}\mathbf{x} \\ \text{subject to} & \mathbf{N}\mathbf{x} \leq \mathbf{b}_N, \quad \mathbf{Q}_k^{-1}\mathbf{x} \geq \mathbf{0}. \end{array} \right.$$

Since the inversion of $\mathbf{Q}_k$ is not always numerically so stable, (P_k) is usually solved in the following form

$$\left| \begin{array}{ll} \text{maximize} & \mathbf{e}\boldsymbol{\lambda} \\ \text{subject to} & \mathbf{N}\mathbf{Q}_k\boldsymbol{\lambda} \leq \mathbf{b}_N, \quad \boldsymbol{\lambda} \geq \mathbf{0}. \end{array} \right. \tag{7}$$

Even if $\mathbf{Q}_k$ fails to be invertible, (7) can be defined and has an optimal solution $\boldsymbol{\lambda}^k$. The optimal solution of (P_k) is then given by $\boldsymbol{\omega}^k = \mathbf{Q}_k\boldsymbol{\lambda}^k$. The dual problem of (7) is as follows

$$\left| \begin{array}{ll} \text{minimize} & \boldsymbol{\mu}\mathbf{b}_N \\ \text{subject to} & \boldsymbol{\mu}\mathbf{N}\mathbf{Q}_k \geq \mathbf{e}, \quad \boldsymbol{\mu} \geq \mathbf{0}. \end{array} \right. \tag{8}$$

This problem also has an optimal solution $\boldsymbol{\mu}^k$, and by the assumption (5) we have

$$\mathbf{e}\boldsymbol{\lambda}^k = \boldsymbol{\mu}^k\mathbf{b}_N = \zeta^k > 1.$$

For the dual solution $\boldsymbol{\mu}^k$, let us define another linear program

$$(\mathrm{P}'_k) \left| \begin{array}{ll} \text{maximize} & \boldsymbol{\mu}^k\mathbf{N}\mathbf{x} \\ \text{subject to} & \mathbf{N}\mathbf{x} \leq \mathbf{b}_N, \quad \mathbf{Q}_k^{-1}\mathbf{x} \geq \mathbf{0}, \end{array} \right.$$

[2] Tuy used the term "nondegenerate", following [2], instead of "nonsingular". However, since it is easily confused with nondegeneracy of polyhedra, we use "nonsingular" in view of its relation to the invertibility of $\mathbf{Q}_k$.

which is equivalent to (P_k) in the following sense.

Lemma 3.1. *An optimal solution of (P'_k) is* $\boldsymbol{\omega}^k = \mathbf{Q}_k\boldsymbol{\lambda}^k$, *and the optimal value is equal to* ζ^k. *Conversely, if* $\mathbf{x}'$ *is an optimal solution of (P'_k), then* $\mathbf{x}'$ *is an optimal solution of (P_k).*

Let us investigate the relationship between (P_k) and (P'_k) in more detail. Let

$$\Lambda_k^+ = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} = \sum_{j\in J_k} \mathbf{q}_j^k \lambda_j,\ \boldsymbol{\lambda} \geq \mathbf{0}\}, \quad J_k = \{j \mid \lambda_j^k > 0\}.$$

Apparently, Λ_k^+ is the minimal face of Λ_k containing the optimal solution $\boldsymbol{\omega}^k$ of (P_k) and (P'_k).

Lemma 3.2. *It holds that*

$$\boldsymbol{\mu}^k\mathbf{N}\mathbf{x} \geq \mathbf{e}\mathbf{Q}_k^{-1}\mathbf{x}, \quad \forall \mathbf{x} \in \Lambda_k. \tag{9}$$

In particular,

$$\boldsymbol{\mu}^k\mathbf{N}\mathbf{x} = \mathbf{e}\mathbf{Q}_k^{-1}\mathbf{x} \ \ \textit{if}\ \ \mathbf{x} \in \Lambda_k^+. \tag{10}$$

Let

$$H_k = \{\mathbf{x} \in \mathbb{R}^n \mid \boldsymbol{\mu}^k\mathbf{N}\mathbf{x} \leq 1\}.$$

Immediately from Lemma 3.2, we see the relationship between this halfspace H_k and the γ-valid cut G_k:

$$G_k \cap \Lambda_k^+ = H_k \cap \Lambda_k^+ \subset H_k \cap \Lambda_k \subset G_k \cap \Lambda_k \subset C(\gamma). \tag{11}$$

PSEUDO-NONSINGULARITY OF (2)

Let us give here the formal definition of pseudo-nonsingularity.

Definition 3.1. The sequence of nested cones $\{\Lambda_k \mid k = 1, 2, \ldots\}$ is said to be *pseudo-nonsingular* if there exists a constant L such that

$$\|\boldsymbol{\mu}^k\mathbf{N}\| \leq L, \quad k = 1, 2, \ldots. \tag{12}$$

The conical algorithm is also said to be *pseudo-nonsingular* if every sequence of nested cones that it generates is pseudo-nonsingular.

Note that this definition requires the norm in (12) to be bounded from above for every k, unlike the original nonsingularity (6).

To show the pseudo-nonsingularity of the sequence (2), we only have to show the existence of a constant L satisfying (12). For this purpose, we need to introduce a further lemma.

Lemma 3.3. *The optimal value* ζ^k *of (P_k) and (P'_k) is nonincreasing in k, i.e.,*

$$\zeta^k \geq \zeta^{k+1} > 1, \quad k = 1, 2, \ldots.$$

Theorem 3.4. *The sequence of nested cones (2) is pseudo-nonsingular.*

Since (2) is an arbitrary sequence of nested cones, the conical algorithm is also pseudo-nonsingular. In the next section, we will use the pseudo-nonsingularity and prove the convergence of the conical algorithm under a certain class of subdivision rules, including $\boldsymbol{\omega}$-subdivision.

CONVERGENCE OF THE ALGORITHM WITH ω-SUBDIVISION

Let $\mathbf{y}^k$ denote the intersection of the ray from $\mathbf{0}$ through $\mathbf{u}^k$ with the boundary of G_k. One of the main results in this paper is the following, which guarantees that G_k approximates $C(\gamma)$ asymptotically on Λ_k.

Theorem 3.5. *Let $\{\Lambda_k \mid k = 1,2,\dots\}$ be a sequence of nested cones such that Λ_{k+1} is obtained by subdividing Λ_k along $\mathbf{u}^k \in \Lambda_k^+$. Then,*

$$\liminf_{k \to +\infty} \|\mathrm{ext}(\mathbf{u}^k) - \mathbf{y}^k\| = 0. \tag{13}$$

To prove Theorem 3.5 rigorously, we need two more lemmas, but omit them because of space limitations. The convergence result with the usual ω-subdivision can be thought of as a corollary of Theorem 3.5.

Corollary 3.6. *Let $\{\Lambda_k \mid k = 1,2,\dots\}$ be a sequence of nested cones such that Λ_{k+1} is obtained by subdividing Λ_k along $\mathbf{u}^k = \boldsymbol{\omega}^k$. Then $\{\mathbf{y}^k \mid k = 1,2,\dots\}$ has an accumulation point $\mathbf{y}^0 \in D$ such that $f(\mathbf{y}^0) = \gamma$.*

Unfortunately, Theorem 3.5 does not, by itself, ensure the convergence of the algorithm to a solution of the d.c. feasibility problem (DC). It merely implies the existence of a subsequence $\{k_r \mid r = 1,2,\dots\}$ such that the γ-extension of $\mathbf{u}^{k_r}$ approaches the γ-valid cut G_{k_r} asymptotically. To achieve the convergence to a solution of (DC), we need to further restrict the selection of the subdivision direction $\mathbf{u}^k$ for each k. One way is obviously ω-subdivision. In the next section, we will develop an alternative to ω-subdivision, named *ω-bisection*, which bisects Λ_k by splitting a two-dimensional face of Λ_k^+ into two pieces.

4 Conical algorithm based on ω-bisection

To develop the ω-bisection, we assume in the rest of the paper that f is strictly concave, i.e., if $\mathbf{x}, \mathbf{y} \in S$ and $\mathbf{x} \neq \mathbf{y}$, then we have

$$f[(1-\lambda)\mathbf{x} + \lambda\mathbf{y}] > (1-\lambda)f(\mathbf{x}) + \lambda f(\mathbf{y}), \quad \forall \lambda \in (0,1). \tag{14}$$

Under this assumption, we can observe the following. As before, $\mathbf{y}^k$ denotes the intersection of the ray in direction $\mathbf{u}^k$ with the boundary of G_k.

Lemma 4.1. *Let $\{\Lambda_k \mid k = 1,2,\dots\}$ be a sequence of nested cones such that Λ_{k+1} is obtained by subdividing Λ_k along $\mathbf{u}^k$ lying on a two-dimensional face of Λ_k^+. Then $\{\mathbf{q}_j^k \mid k = 1,2,\dots\}$ has an accumulation point $\mathbf{q}_j^0 \in \partial C(\gamma)$ for each $j = 1,\dots,n$. Among the $\mathbf{q}_j^0$, there exists an accumulation point $\mathbf{y}^0$ of $\{\mathbf{y}^k \mid k = 1,2,\dots\}$.*

ω-BISECTION RULE

On the basis of the above observation, let us now attempt to develop a systematic procedure for ω-bisection.

For each pair $\{i, j\} \subset J_k$, let

$$\mathbf{y}_{ij}^k = (\lambda_i^k \mathbf{q}_i^k + \lambda_j^k \mathbf{q}_j^k)/(\lambda_i^k + \lambda_j^k). \tag{15}$$

This point $\mathbf{y}_{ij}^k$ is the intersection of the segment $[\mathbf{q}_i^k, \mathbf{q}_j^k]$ with the hyperplane spanned by $n-1$ vectors $\mathbf{q}_1^k, \ldots, \mathbf{q}_{i-1}^k, \mathbf{q}_{i+1}^k, \ldots, \mathbf{q}_{j-1}^k, \mathbf{q}_{j+1}^k, \ldots, \mathbf{q}_n^k$ and $\boldsymbol{\omega}^k$. The segment $[\mathbf{q}_i^k, \mathbf{q}_j^k]$ is split into two pieces $[\mathbf{q}_i^k, \mathbf{y}_{ij}^k]$ and $[\mathbf{y}_{ij}^k, \mathbf{q}_j^k]$, the shorter of which has a length of

$$\delta_{ij}^k = \|\mathbf{q}_i^k - \mathbf{q}_j^k\| \min\{\lambda_i^k, \lambda_j^k\}/(\lambda_i^k + \lambda_j^k). \tag{16}$$

Among the $\mathbf{y}_{ij}^k$, we select as $\mathbf{u}^k$ the one with the largest δ_{ij}^k, i.e., $\mathbf{y}_{st}^k$ with

$$\{s, t\} \in \arg\max\{\delta_{ij}^k \mid \{i, j\} \subset J_k\}, \tag{17}$$

and subdivide the cone Λ_k along the direction $\mathbf{u}^k = \mathbf{y}_{st}^k$ into two subcones:

$$\Lambda_k^j = \mathrm{con}(\mathbf{Q}_k^j), \quad j = s, t, \tag{18}$$

where

$$\mathbf{Q}_k^j = [\mathbf{q}_1^k, \ldots, \mathbf{q}_{j-1}^k, \mathrm{ext}(\mathbf{y}_{st}^k), \mathbf{q}_{j+1}^k, \ldots, \mathbf{q}_n^k]. \tag{19}$$

Either Λ_k^s or Λ_k^t is adopted as Λ_{k+1} in the sequence of nested cones $\{\Lambda_k \mid k = 1, 2, \ldots\}$.

Suppose that $\{\Lambda_k \mid k = 1, 2, \ldots\}$ is generated according to the rule given by (15)–(19). Then we have the following results.

Lemma 4.2. *There exists an index set $J_0 \subset \{1, \ldots, n\}$ such that $J_k = J_0$ for infinitely many k. Moreover,*

(i) for each pair $\{i, j\} \subset J_0$, the sequence $\{\mathbf{y}_{ij}^k \mid k = 1, 2, \ldots\}$ has an accumulation point $\mathbf{y}_{ij}^0 \in \{\mathbf{q}_i^0, \mathbf{q}_j^0\}$, and

(ii) for each $j \in J_0$, the sequence $\{\lambda_j^k \mid k = 1, 2, \ldots\}$ has an accumulation point $\lambda_j^0 \geq 0$ such that $\sum_{j \in J_0} \lambda_j^0 \geq 1$.

In particular, if $\lambda_i^0, \lambda_j^0 > 0$ for $\{i, j\} \subset J_0$, then it holds that $\mathbf{y}_{ij}^0 = \mathbf{q}_i^0 = \mathbf{q}_j^0$.

Lemma 4.3. *Let $\boldsymbol{\eta}^k$ denote the intersection of the ray from $\mathbf{0}$ through $\boldsymbol{\omega}^k$ with ∂G_k. Then $\{\boldsymbol{\eta}^k \mid k = 1, 2, \ldots\}$ has an accumulation point $\boldsymbol{\eta}^0 \in D$ such that $f(\boldsymbol{\eta}^0) = \gamma$.*

ALGORITHM DESCRIPTION

Before closing this section, let us summarize the conical algorithm for solving (DC) with ω-bisection.

algorithm conic_ω_bisect$(D, f, \alpha, \varepsilon)$

$\gamma \leftarrow \alpha - \varepsilon$;

determine a cone Λ with vertex $\mathbf{v} = \mathbf{0}$ and a polyhedron M such that $D = M \cap \Lambda$, $f(\mathbf{v}) > \gamma$, and $\mathbf{v}$ is an interior point of M;

let Λ be spanned by n vectors $\mathbf{q}_1,\ldots,\mathbf{q}_n$ with $f(\mathbf{q}_j)=\gamma$, and $\mathbf{Q}\leftarrow[\mathbf{q}_1,\ldots,\mathbf{q}_n]$;
$\mathscr{P}\leftarrow\emptyset$; $\mathscr{T}\leftarrow\{\mathbf{Q}\}$; $stop\leftarrow false$; $k\leftarrow 1$;
while $stop=false$ do
  for each $\mathbf{Q}\in\mathscr{T}$ do
    compute an optimal solution $\boldsymbol{\lambda}(\mathbf{Q})$ of the linear program $\max\{\mathbf{e}\boldsymbol{\lambda}\mid \mathbf{Q}\lambda\in M,\boldsymbol{\lambda}\geq\mathbf{0}\}$;
    $\zeta(\mathbf{Q})\leftarrow\mathbf{e}\boldsymbol{\lambda}(\mathbf{Q})$;
    if $\zeta(\mathbf{Q})>1$ then
      $\mathscr{P}\leftarrow\mathscr{P}\cup\{\mathbf{Q}\}$;
    end if
    if $f(\mathbf{Q}\boldsymbol{\lambda}(\mathbf{Q}))<\alpha$ then
      $\mathbf{z}\leftarrow\mathbf{Q}\boldsymbol{\lambda}(\mathbf{Q})$; $stop\leftarrow true$;
    end if
  end for
  if $\mathscr{P}=\emptyset$ then
    $stop\leftarrow true$;
  else
    choose $\mathbf{Q}$ with the largest $\zeta(\mathbf{Q})$ from $\mathscr{P}$, and let $\mathbf{Q}_k\leftarrow\mathbf{Q}$;
    $\boldsymbol{\lambda}^k\leftarrow\boldsymbol{\lambda}(\mathbf{Q}_k)$; $\boldsymbol{\omega}^k\leftarrow\mathbf{Q}_k\boldsymbol{\lambda}^k$;
    generate the children $\mathbf{Q}_k^s$ and $\mathbf{Q}_k^t$ of $\mathbf{Q}_k$ from $\boldsymbol{\lambda}^k$ according to (15)–(19);
    $\mathscr{P}\leftarrow\mathscr{P}\setminus\{\mathbf{Q}_k\}$; $\mathscr{T}\leftarrow\{\mathbf{Q}_k^s,\mathbf{Q}_k^t\}$; $k\leftarrow k+1$;
  end if
end while
if $\mathscr{P}\neq\emptyset$ then
  print "$\mathbf{z}$ is a point in $D\setminus C(\alpha)$.";
else
  print "D is a subset of $C(\gamma)$.";
end if
end.

Theorem 4.4. *Suppose $\varepsilon=0$. If the algorithm conic_ω_bisect terminates, then it either generates a point $\mathbf{z}\in D\setminus C(\alpha)$ or proves that $D\subset C(\alpha)$. If not, the sequence $\{\boldsymbol{\omega}^k\mid k=1,2,\ldots\}$ has an accumulation point $\boldsymbol{\omega}^0\in D$ such that $f(\boldsymbol{\omega}^0)=\alpha$.*

Corollary 4.5. *If $\varepsilon>0$, the algorithm conic_ω_bisect terminates in a finite number of iterations, and either generates a point $\mathbf{z}\in D\setminus C(\alpha)$ or proves that $D\subset C(\alpha-\varepsilon)$.*

In the forthcoming paper, we will report the numerical result of comparison between this algorithm and the usual one with ω-subdivision, along with the detailed proofs for all propositions in this paper.

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