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Asymptotic stability for a geophysical system

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Abstract. This paper considers the asymptotic stability for a geophysical fluid system. We state that there exists a weak solution of the system satisfying the asymptotic stability. It is also stated that there exists a unique global-in-time strong solution, which satisfies the asymptotic stability, of the system in the case when the initial datum is sufficiently small. Especially, this paper studies the asymptotic stability for the linearized system of our system. Furthermore, this paper gives one derivation of our geophysical fluid system.

1 Introduction and Main Results

Large-scale fluids such as the atmosphere and ocean are called geophysical fluids. The motion of geophysical flows is formulated as a system of the Navier-Stokes-Boussinesq equations with the Coriolis and stratification effects. We consider the following geophysical fluid system in the whole space:

$$\begin{cases} \partial_t u - \nu \Delta u + (u, \nabla) u + \Omega d \times u + \nabla \mathfrak{p} = \mathcal{G} \theta e_3, & t > 0,\ x \in \mathbb{R}^3, \\ \partial_t \theta - \kappa \Delta \theta + (u, \nabla)\theta = -N^2 u^3, & t > 0,\ x \in \mathbb{R}^3, \\ \nabla \cdot u = 0, & t > 0,\ x \in \mathbb{R}^3, \\ \lim_{|x| \to \infty} u = 0,\ \lim_{|x| \to \infty} \theta = 0, & t > 0, \\ u|_{t=0} = u_0,\ \theta|_{t=0} = \theta_0, & x \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

where the unknown functions $u = u(t,x) = (u^1, u^2, u^3)$, $\theta = \theta(t,x)$, and $\mathfrak{p} = \mathfrak{p}(t,x)$ are the fluid velocity, the thermal disturbances (temperature), and the pressure of the fluid, respectively, while $\nu > 0$, $\kappa > 0$, and $\mathcal{G} > 0$ are the viscosity, the thermal diffusivity, and the gravity, respectively. Parameters $\Omega \in \mathbb{R}$ and $N > 0$ are the rotation rate (Coriolis-parameter) and the Brunt-Väisälä frequency (stratification-parameter), respectively. We use the convention: $\Delta := \partial_1^2 + \partial_2^2 + \partial_3^2$, $\nabla := (\partial_1, \partial_2, \partial_3)$, $e_3 := (0,0,1)$, $\mathbb{S}^2 := \{(d_1,d_2,d_3) \in \mathbb{R}^3; |d| = 1\}$, and we denote the exterior product by $\times$. Here $d = (d_1,d_2,d_3) \in \mathbb{S}^2$ is the unit vector in the direction of the rotating axis, the term $\Omega d \times u$ the Coriolis force, the term $\mathcal{G}\theta e_3$ the buoyancy (flotation or heat convection), and the term $N^2 u^3$ the temperature-stratification.

This paper studies the asymptotic stability for the system (1.1). We first rewrite the system (1.1). Let us set $w = w(t,x) = (w^1, w^2, w^3, w^4) := (u^1, u^2, u^3, \sqrt{\mathcal{G}}\theta/N)$. We easily check that $(w, \mathfrak{p})$ satisfies the following system:

$$\begin{cases} \partial_t w + \mathcal{A}w + \mathcal{S}w + \widetilde{\nabla}\mathfrak{p} = -(w, \widetilde{\nabla})w, & t > 0,\ x \in \mathbb{R}^3, \\ \lim_{|x|\to\infty} w = 0, & t > 0, \\ \widetilde{\nabla} \cdot w = 0, & t > 0,\ x \in \mathbb{R}^3, \\ w|_{t=0} = w_0, & x \in \mathbb{R}^3. \end{cases} \tag{1.2}$$

Here $w_0 = (w_0^1, w_0^2, w_0^3, w_0^4) = (u_0^1, u_0^2, u_0^3, \sqrt{\mathcal{G}}\theta_0/N)$, $\widetilde{\nabla} := (\partial_1, \partial_2, \partial_3, 0)$,

$$\mathcal{A} := \begin{pmatrix} -\nu\Delta & 0 & 0 & 0 \\ 0 & -\nu\Delta & 0 & 0 \\ 0 & 0 & -\nu\Delta & 0 \\ 0 & 0 & 0 & -\kappa\Delta \end{pmatrix},$$

$$\mathcal{S} := \begin{pmatrix} 0 & -\Omega d_3 & \Omega d_2 & 0 \\ \Omega d_3 & 0 & -\Omega d_1 & 0 \\ -\Omega d_2 & \Omega d_1 & 0 & -\sqrt{\mathcal{G}}N \\ 0 & 0 & \sqrt{\mathcal{G}}N & 0 \end{pmatrix}.$$

From now on we consider (1.2) instead of (1.1). Before stating main results, we introduce function spaces and notation. Let us define real-valued function spaces as follows:

$$\begin{aligned} &C_0^\infty := C_0^\infty(\mathbb{R}^3) := \{f \in C^\infty(\mathbb{R}^3);\ f \text{ has a compact support in } \mathbb{R}^3\}, \\ &C_{0,\sigma}^\infty := C_{0,\sigma}^\infty(\mathbb{R}^3) := \{f = (f^1, f^2, f^3) \in [C_0^\infty(\mathbb{R}^3)]^3;\ \nabla \cdot f = 0\}, \\ &L_\sigma^p := L_\sigma^p(\mathbb{R}^3) := \overline{C_{0,\sigma}^\infty}^{\|\cdot\|_{L^p}},\ H_{0,\sigma}^1 := H_{0,\sigma}^1(\mathbb{R}^3) := \overline{C_{0,\sigma}^\infty}^{\|\cdot\|_{W^{1,2}}}, \\ &G_p := G_p(\mathbb{R}^3) := \{f \in [L^p(\mathbb{R}^3)]^3;\ f = \nabla g,\ g \in L_{loc}^p(\mathbb{R}^3)\} \end{aligned}$$

for $1 < p < \infty$ and $m \in \mathbb{N}$. Here $\|\cdot\|_{L^p}$ is the usual norm in the Lebesgue space L^p, and $\|\cdot\|_{W^{m,p}}$ the usual norm of the Sobolev space $W^{m,p}$. In this paper, we use the following convention:

$$\|f\|_{L^\infty} := \operatorname{ess.}\sup_{x\in\mathbb{R}^3}\{|f(x)|\},\quad \|f\|_{H^1} := \|f\|_{W^{1,2}},\ \text{and}\ H^1 := W^{1,2}(\mathbb{R}^3).$$

Moreover, by $\langle\cdot,\cdot\rangle$ we denote L^2-inner product.

Next we define a weak solution of (1.2).

Definition 1.1 (Weak Solution). *Let $w_0 \in L^2_\sigma(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$. We say that a vector-valued function $(w, \mathfrak{p})(= (w^1, w^2, w^3, w^4, \mathfrak{p}))$ is a* weak solution *of* (1.2) *with the initial datum w_0, if for all $T > 0$ and for all $s, t, \varepsilon \geq 0$ such that $0 \leq s < \varepsilon < t < T$ the following properties hold:*

(i) *(function class)*

$$w \in L^\infty(0, T; L^2_\sigma(\mathbb{R}^3) \times L^2(\mathbb{R}^3)) \cap L^2(0, T; H^1_{0,\sigma}(\mathbb{R}^3) \times H^1(\mathbb{R}^3)),$$
$$\widetilde{\nabla}\mathfrak{p} \in [L^2(\varepsilon, T; [L^2(\mathbb{R}^3)]^3 \times \{0\}) + L^{5/4}(\varepsilon, T; [L^{5/4}(\mathbb{R}^3)]^3 \times \{0\})],$$

(ii) *(weak form* I*)*

$$\int_s^t \langle w, \Phi'\rangle d\tau - \nu \int_s^t \langle \nabla\overline{w}, \nabla\overline{\Phi}\rangle d\tau - \kappa \int_s^t \langle \nabla w^4, \nabla\Phi^4\rangle d\tau$$
$$- \int_s^t \langle Sw, \Phi\rangle d\tau - \int_s^t \langle (w, \widetilde{\nabla})w, \Phi\rangle d\tau = \langle w(t), \Phi(t)\rangle - \langle w(s), \Phi(s)\rangle$$

holds for all $\Phi = (\Phi^1, \Phi^2, \Phi^3, \Phi^4) \in C^1([s,t]; H^1_{0,\sigma}(\mathbb{R}^3) \times H^1(\mathbb{R}^3))$, where $\Phi' = \partial\Phi/\partial\tau$, $\overline{w} = (w^1, w^2, w^3)$, $\overline{\Phi} = (\Phi^1, \Phi^2, \Phi^3)$, and $w(0) = w_0$,

(iii) *(weak form* II*) the vector-valued function $(w, \mathfrak{p})$ satisfies the following identity:*

$$\int_\varepsilon^t \langle Sw, \widetilde{\nabla}\Psi\rangle d\tau + \int_\varepsilon^t \langle (w, \widetilde{\nabla})w, \widetilde{\nabla}\Psi\rangle d\tau + \int_\varepsilon^t \langle \widetilde{\nabla}\mathfrak{p}, \widetilde{\nabla}\Psi\rangle d\tau = 0$$

for all $\Psi \in C([\varepsilon, t]; W^{2,2}(\mathbb{R}^3))$, where $\widetilde{\nabla}\Psi = (\partial_1\Psi, \partial_2\Psi, \partial_3\Psi, 0)$.

(iv) *(strong energy inequality)*

$$\|w(t)\|^2_{L^2} + 2\nu \int_s^t \|\nabla\overline{w}(\tau)\|^2_{L^2} d\tau + 2\kappa \int_s^t \|\nabla w^4(\tau)\|^2_{L^2} d\tau \leq \|w(s)\|^2_{L^2}$$

holds for a.e. $s \geq 0$, including $s = 0$, and all $t > s$, where $w(0) = w_0$.

Let us now state main results.

Theorem 1.2 (Weak solution). *Let $w_0 \in L^2_\sigma(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$. Then there exists at least one weak solution of* (1.2) *with the initial datum w_0, satisfying*

$$\lim_{t\to\infty} \|w(t)\|_{L^2} = 0. \tag{1.3}$$

Moreover, the weak solution is smooth with respect to time when time is sufficiently large.

Theorem 1.3 (Strong solution)**.** *Let $w_0 \in H^1_{0,\sigma}(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$. Then there exists $\delta_0 > 0$ independent of w_0 such that if*

$$\|w_0\|_{H^1} < \delta_0$$

then there exists a unique global-in-time strong solution

$$w \in C([0,\infty); H^1_{0,\sigma}(\mathbb{R}^3) \times H^1) \cap C((0,\infty); [W^{2,2}]^4) \cap C^1((0,\infty); L^2_\sigma \times L^2),$$
$$\widetilde{\nabla}\mathfrak{p} \in C((0,\infty); [L^2(\mathbb{R}^3)]^3 \times \{0\})$$

of (1.2) *with the initial datum w_0, which satisfies* (1.3)*. Here $\mathfrak{p}$ is a pressure associated with w.*

Applying similar arguments in [6], we prove Theorems 1.2 and 1.3. Koba ([6]) studied the asymptotic stability of Ekman boundary layers in rotating stratified fluids. Under some assumptions on an energy inequality, he constructed a weak solution of his Ekman system satisfying the asymptotic stability and showed the existence of a unique global-in-time strong solution, which satisfies the asymptotic stability, of the system in the case when the initial datum is sufficiently small. The approach of [6] is based on the methods from ([5], [8], [9], [4]) and improves them. Kato and Fujita ([5]) constructed a unique strong solution of the Navier-Stokes system by using fractional power of the Stokes operator. In [8], Masuda proved that if a weak solution of the Navier-Stokes system satisfies the strong energy inequality then the weak solution is asymptotically stable. Miyakawa and Sohr ([9]) constructed a weak solution to the Navier-Stokes system satisfying the strong energy inequality. Moreover, they showed that the weak solution is smooth with respect to time when time is sufficiently large. Hess-Hieber-Mahalov-Saal in [4] showed the existence of a weak solution, which satisfies the asymptotic stability, of their Ekman perturbed system by using maximal L^p-regularity.

Let us now explain about construction of weak solutions of (1.2) and construction of strong solutions of (1.2). Using the Yosida approximation, maximal L^p-regularity, real interpolation theory, and an energy inequality of the system (1.2), one can construct a weak solution of the system. See ([9], [4], [6]). Applying semigroup theory on Hilbert spaces and an energy inequality of (1.2), we can show the existence of a strong solution of (1.2). See ([5], [6], [7]). Koba ([7]) constructed strong solutions of the spatial inhomogeneous Boussinesq system in various domains.

In the rest of this paper, we consider the asymptotic stability for the linearized system of (1.2) and discuss derivation of (1.1). In Section 2, we study the stability for a linear system satisfying an energy inequality. In Section 3, we derive our system (1.1) from the incompressible Navier-Stokes system by using physical and mathematical assumptions.

Finally, we state some references for geophysical fluids and the Boussinesq approximation. Greenspan ([3]), Pedlosky ([10]), and Benoit ([1]) are textbooks for geophysical fluids and rotating fluids. Fife ([2]) studied the Benard problem and the Boussinesq approximation from a mathematical point of view.

2 Linear Stability

In this section, we investigate the asymptotic stability for the following system:

$$\begin{cases} w_t + \mathcal{A}w + \mathcal{S}w + \widetilde{\nabla}\mathfrak{p} = 0, & t > 0,\ x \in \mathbb{R}^3, \\ \lim_{|x|\to\infty} w = 0, & t > 0, \\ \widetilde{\nabla} \cdot w = 0, & t > 0,\ x \in \mathbb{R}^3, \\ w|_{t=0} = w_0, & x \in \mathbb{R}^3. \end{cases} \tag{2.1}$$

Here $w = w(t,x) = (w^1, w^2, w^3, w^4)$, $\mathfrak{p} = \mathfrak{p}(t,x)$, $\widetilde{\nabla} = (\partial_1, \partial_2, \partial_3, 0)$,

$$\mathcal{A} := \begin{pmatrix} -\nu\Delta & 0 & 0 & 0 \\ 0 & -\nu\Delta & 0 & 0 \\ 0 & 0 & -\nu\Delta & 0 \\ 0 & 0 & 0 & -\kappa\Delta \end{pmatrix},$$

$$\mathcal{S} := \begin{pmatrix} 0 & \mathcal{S}_1 & \mathcal{S}_2 & \mathcal{S}_3 \\ -\mathcal{S}_1 & 0 & \mathcal{S}_4 & \mathcal{S}_5 \\ -\mathcal{S}_2 & -\mathcal{S}_4 & 0 & \mathcal{S}_6 \\ -\mathcal{S}_3 & -\mathcal{S}_5 & -\mathcal{S}_6 & 0 \end{pmatrix},$$

$\nu, \kappa > 0$, and $\mathcal{S}_\ell \in L^\infty(\mathbb{R}^3)$ $(\ell = 1, 2, \ldots, 6)$. Note that $L^\infty(\mathbb{R}^3)$ is a real-valued function space in this paper. It is easy to check that the system (2.1) contains the linearized system of (1.2). This section proves the proposition.

Proposition 2.1. *Let $w_0 \in L^2_\sigma(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$. Then there exists a unique global-in-time strong solution*

$$w \in C([0,\infty); L^2_\sigma(\mathbb{R}^3) \times L^2(\mathbb{R}^3)) \cap C((0,\infty); [W^{2,2}]^4) \cap C^1((0,\infty); L^2_\sigma \times L^2),$$
$$\widetilde{\nabla}\mathfrak{p} \in C((0,\infty); G_2(\mathbb{R}^3) \times \{0\})$$

of (2.1) *with the initial datum w_0, where $\mathfrak{p}$ is a pressure associated with w. Assume in addition that $\mathcal{S}_1$,$\mathcal{S}_2$,$\mathcal{S}_3$,$\mathcal{S}_4$,$\mathcal{S}_5$,$\mathcal{S}_6$ do not depend on x_1. Then*

$$\lim_{t\to\infty} \|w(t)\|_{L^2(\mathbb{R}^3)} = 0.$$

To prove Proposition 2.1, we introduce the three tools: the Fourier transformation $\mathcal{F}$, the extended Helmholtz projection $\widetilde{P}$, and the tangential operator $\widetilde{\partial}_1$.

Definition 2.2 (Fourier transformation). *Let f and g be in the class of rapidly decreasing $\mathbb{K}$-valued functions $\mathscr{S}(\mathbb{R}^3;\mathbb{K})(\mathbb{K} = \mathbb{R}, \mathbb{C})$. The* Fourier transform *$\mathcal{F}[f]$ and the* inverse Fourier transform *$\mathcal{F}^{-1}[g]$ are defined by*

$$\mathcal{F}[f(x)](\xi) := \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \mathrm{e}^{-i(x_1\xi_1+x_2\xi_2+x_3\xi_3)} f(x)dx,$$
$$\mathcal{F}^{-1}[g(\xi)](x) := \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \mathrm{e}^{i(x_1\xi_1+x_2\xi_2+x_3\xi_3)} g(\xi)d\xi,$$

where $i = \sqrt{-1}$, $x = (x_1, x_2, x_3), \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$.

Definition 2.3 (Helmholtz projection). *Let $\mathcal{P}$ be the operator defined by*

$$\mathcal{P}f(x) := \mathcal{F}^{-1}\left[\begin{pmatrix} 1-\frac{\xi_1^2}{|\xi|^2} & -\frac{\xi_1\xi_2}{|\xi|^2} & -\frac{\xi_1\xi_3}{|\xi|^2} \\ -\frac{\xi_2\xi_1}{|\xi|^2} & 1-\frac{\xi_2^2}{|\xi|^2} & -\frac{\xi_2\xi_3}{|\xi|^2} \\ -\frac{\xi_3\xi_1}{|\xi|^2} & -\frac{\xi_3\xi_2}{|\xi|^2} & 1-\frac{\xi_3^2}{|\xi|^2} \end{pmatrix} \mathcal{F}[f](\xi)\right](x)$$

for $f \in \mathscr{S}(\mathbb{R}^3;\mathbb{K})(\mathbb{K} = \mathbb{R}, \mathbb{C})$. We call $\mathcal{P}$ the Helmholtz projection.

Definition 2.4 (Extended Helmholtz projection). *Let $\mathcal{P}$ be the Helmholtz projection. Set*

$$\widetilde{P} := \begin{pmatrix} \mathcal{P} & \\ & 1 \end{pmatrix}.$$

We call $\widetilde{P}$ an extended Helmholtz projection.

It is clear that $\widetilde{P} : [L^p(\mathbb{R}^3)]^4 \to L^p_\sigma(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$ $(1 < p < \infty)$.

Definition 2.5 (Tangential operator). *We define the* tangential operator $\widetilde{\partial_1}$ *in* $L^2_\sigma(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ *as follows:*

$$\widetilde{\partial_1} := \begin{pmatrix} \partial_1 & 0 & 0 & 0 \\ 0 & \partial_1 & 0 & 0 \\ 0 & 0 & \partial_1 & 0 \\ 0 & 0 & 0 & \partial_1 \end{pmatrix},$$

$$D(\widetilde{\partial_1}) := [L^2_\sigma(\mathbb{R}^3) \times L^2(\mathbb{R}^3)] \cap [H^1(\mathbb{R}; L^2(\mathbb{R}^2))]^4,$$

where

$$H^1(\mathbb{R}; L^2(\mathbb{R}^2)) := \{f \in L^2(\mathbb{R}^3);\ \|\partial_1 f\|_{L^2(\mathbb{R}^3)} < \infty\}.$$

From [6, Sec. 3.3], we deduce the following lemma.

Lemma 2.6. *The operator* $\widetilde{\partial_1}$ *has the following properties:*

(i) $\widetilde{\partial_1} : D(\widetilde{\partial_1}) \to L^2_\sigma(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$.

(ii) $\widetilde{\partial_1}$ *is a closed operator.*

(iii) The range $R(\widetilde{\partial_1})$ *is a dense subset of* $L^2_\sigma(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$.

Multiplying (2.1) by the extended Helmholtz projection $\widetilde{P}$, we obtain the following abstract system:

$$\begin{cases} w_t + Lw = 0, \ t > 0, \\ w|_{t=0} = w_0. \end{cases}$$

Here the linear operator L is defined by

$$\begin{cases} Lw := \widetilde{P}(\mathcal{A} + \mathcal{S})w, \\ D(L) := [[L^2_\sigma(\mathbb{R}^3) \times L^2(\mathbb{R}^3)] \cap [W^{2,2}(\mathbb{R}^3)]^4] \oplus i[[L^2_\sigma \times L^2] \cap [W^{2,2}]^4]. \end{cases}$$

Moreover, we define the two linear operator A and L^* as follows:

$$\begin{cases} Aw := \widetilde{P}\mathcal{A}w, \\ D(A) := [[L^2_\sigma(\mathbb{R}^3) \times L^2(\mathbb{R}^3)] \cap [W^{2,2}(\mathbb{R}^3)]^4] \oplus i[[L^2_\sigma \times L^2] \cap [W^{2,2}]^4], \end{cases}$$

$$\begin{cases} L^*w := \widetilde{P}(\mathcal{A} - \mathcal{S})w, \\ D(L^*) := [[L^2_\sigma(\mathbb{R}^3) \times L^2(\mathbb{R}^3)] \cap [W^{2,2}(\mathbb{R}^3)]^4] \oplus i[[L^2_\sigma \times L^2] \cap [W^{2,2}]^4]. \end{cases}$$

We easily check that L^* is its adjoint operator of the operator L. See [7] for details. Next we deduce useful properties of A and L.

Lemma 2.7. *The operator A has the following properties:*

(i) $-A$ *generates an analytic semigroup on* $[L^2_\sigma(\mathbb{R}^3) \times L^2(\mathbb{R}^3)] \oplus i[L^2_\sigma(\mathbb{R}^3) \times L^2(\mathbb{R}^3)]$.

(ii) For all $f \in D(\widetilde{\partial}_1)$ *and* $t > 0$

$$\|\widetilde{\partial}_1 \mathrm{e}^{-tA} f - \mathrm{e}^{-tA} \widetilde{\partial}_1 f\|_{L^2} = 0.$$

(iii) For all $f \in [H^1(\mathbb{R}^3)]^4$ *and each* $j = 1, 2, 3$

$$\|\widetilde{P} \partial_j f - \partial_j \widetilde{P} f\|_{L^2} = 0.$$

(iv) There is $C > 0$ *depending only on* (ν, κ) *such that for all* $f \in L^2_\sigma \times L^2$ *and* $t > 0$

$$\|\nabla \mathrm{e}^{-tA} f\|_{L^2} \leq \frac{C}{t^{1/2}} \|f\|_{L^2}.$$

Proof of Lemma 2.7. Using the Fourier-transformation, the definition of the extended Helmholtz projection, and the formula:

$$\mathrm{e}^{-tA} f = \mathcal{F}^{-1}[\mathrm{diag}\{\mathrm{e}^{-\nu t|\xi|^2}, \mathrm{e}^{-\nu t|\xi|^2}, \mathrm{e}^{-\nu t|\xi|^2}, \mathrm{e}^{-\kappa t|\xi|^2}\} \mathcal{F}[f](\xi)](x),$$

we prove Lemma 2.7. □

Lemma 2.8. *The two operators L and L^* have the following properties:*

(i) Each operator $-L$ *and* $-L^*$ *generates an analytic semigroup on* $[L^2_\sigma(\mathbb{R}^3) \times L^2(\mathbb{R}^3)] \oplus i[L^2_\sigma(\mathbb{R}^3) \times L^2(\mathbb{R}^3)]$.

(ii) If $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4, \mathcal{S}_5, \mathcal{S}_6$ *do not depend on* x_1, *then for all* $f \in D(\widetilde{\partial}_1)$ *and* $t > 0$

$$\|\widetilde{\partial}_1 \mathrm{e}^{-tL} f - \mathrm{e}^{-tL} \widetilde{\partial}_1 f\|_{L^2} = 0,$$
$$\|\widetilde{\partial}_1 \mathrm{e}^{-tL^*} f - \mathrm{e}^{-tL^*} \widetilde{\partial}_1 f\|_{L^2} = 0.$$

Proof of Lemma 2.8. Applying Lemma 2.7 and perturbation theory on semigroup, we deduce (i). By Lemma 2.7 and the assumption of (ii), and an argument similar to that in [6, Sec. 3.5], we see (ii). □

Let us now prove Proposition 2.1 by using Lemmas 2.6 and 2.8.

Proof of Proposition 2.1. Let $w_0 \in L^2_\sigma(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$. Set $w(t) := e^{-tL} w_0 (t \geq 0)$. Since e^{-tL} is an analytic semigroup on $L^2_\sigma \times L^2$, it follows from semigroup theory that

$$w \in C([0,\infty); L^2_\sigma(\mathbb{R}^3) \times L^2(\mathbb{R}^3)) \cap C((0,\infty); [W^{2,2}]^4) \cap C^1((0,\infty); L^2_\sigma \times L^2)$$

and w satisfies

$$\begin{cases} w_t + Lw = 0, \ t > 0, \\ w|_{t=0} = w_0. \end{cases} \tag{2.2}$$

Multiplying (2.2) by w, integrating by parts, and integrating with respect to time, we see that for all $s, t \geq 0 (0 \leq s < t)$

$$\|w(t)\|^2_{L^2} + 2\min\{\nu,\kappa\} \int_s^t \|\nabla w(\tau)\|^2_{L^2} d\tau \leq \|w(s)\|^2_{L^2} \leq \|w_0\|^2_{L^2}.$$

Here we used the fact that

$$\langle \mathcal{S}w, w\rangle = (\mathcal{S}w, w)_{L^2} = 0.$$

Fix $\varepsilon > 0$. Since $R(\widetilde{\partial_1})$ is dense in $L^2_\sigma \times L^2$ by Lemma 2.6, there exist $a \in L^2_\sigma \times L^2$ and $b \in D(\widetilde{\partial_1})$ such that

$$\begin{aligned} &\|w_0 - a_0\|_{L^2} < \varepsilon/2, \\ &a = \partial_1 b. \end{aligned}$$

Set $U(t) = e^{-tL} a$, $V(t) = e^{-tL}(w_0 - a)$, and $W(t) = e^{-tL} b$. As before, we see that for all $s, t \geq 0 (0 \leq s < t)$

$$\|U(t)\|^2_{L^2} + 2\min\{\nu,\kappa\} \int_s^t \|\nabla U(\tau)\|^2_{L^2} d\tau \leq \|U(s)\|^2_{L^2} \leq \|a\|^2_{L^2}, \tag{2.3}$$

$$\|V(t)\|^2_{L^2} + 2\min\{\nu,\kappa\} \int_s^t \|\nabla V(\tau)\|^2_{L^2} d\tau \leq \|V(s)\|^2_{L^2} \leq \|w_0 - a\|^2_{L^2}, \tag{2.4}$$

$$\|W(t)\|^2_{L^2} + 2\min\{\nu,\kappa\} \int_s^t \|\nabla W(\tau)\|^2_{L^2} d\tau \leq \|W(s)\|^2_{L^2} \leq \|b\|^2_{L^2}. \tag{2.5}$$

By (2.4), we check that

$$\begin{aligned} \|e^{-tL} w_0\|_{L^2} \leq& \|w_0 - a\|_{L^2} + \|e^{-tL} a\|_{L^2} \\ \leq& \frac{\varepsilon}{2} + \|e^{-tL} a\|_{L^2}. \end{aligned} \tag{2.6}$$

From (2.3), we find that for $t > s$

$$\|e^{-tL}a\|_{L^2} \leq \|e^{-sL}a\|_{L^2}.$$

Integrating both its sides of the above inequalities with respect to s and using the Hölder inequality, we see that

$$\begin{aligned}\|e^{-tL}a\|_{L^2} \leq & \frac{1}{t}\int_0^t \|e^{-sL}a\|_{L^2}ds \\ \leq & \frac{1}{t^{1/2}}\left(\int_0^t \|e^{-sL}a\|_{L^2}^2 ds\right)^{1/2}. \end{aligned} \tag{2.7}$$

Since $a = \partial_1 b$ and $e^{-sL}\partial_1 b = \partial_1 e^{-sL}b$, we use (2.5) to check that

$$\begin{aligned}\int_0^t \|e^{-sL}a\|_{L^2}^2 ds = & \int_0^t \|\partial_1 e^{-sL}b\|_{L^2}^2 ds \\ \leq & \int_0^t \|\nabla e^{-sL}b\|_{L^2}^2 ds \\ \leq & \frac{\|b\|_{L^2}^2}{2\min\{\nu, \kappa\}}. \end{aligned} \tag{2.8}$$

Combining (2.6), (2.7), and (2.8), we obtain

$$\|e^{-tL}w_0\|_{L^2} \leq \frac{\varepsilon}{2} + \sqrt{\frac{1}{2\min\{\kappa, \nu\}}}t^{-1/2}\|b\|_{L^2}.$$

Hence we see that there is $T_0 > 0$ such that for all $t > T_0$

$$\|e^{-tL}w_0\|_{L^2} < \varepsilon.$$

Since ε is arbitrary, we conclude that

$$\lim_{t\to\infty} \|w(t)\|_{L^2} = 0.$$

Set

$$\widetilde{\nabla}\mathfrak{p} := -w_t - \mathcal{A}w - \mathcal{S}w.$$

It is easy check that $(w, \widetilde{\nabla}\mathfrak{p})$ is a strong solution of (2.1) with the initial datum w_0. Since (2.2) is a linear system, we see the uniqueness of the strong solution. Therefore Proposition 2.1 is proved. $\square$

3 Derivation of a Geophysical Fluid System

This section gives one derivation of a geophysical fluid system. We derive our geophysical fluid system from the incompressible Navier-Stokes system by using mathematical and physical assumptions. The argument on the derivation of our system (1.1) may not be rigorous from a physical point of view, but the argument here makes a lot of sense to the reader. We first prepare one tool. By direct calculations, we obtain the following lemma.

Lemma 3.1. *Let $\Omega > 0$ and $t \in \mathbb{R}$. Let $d = (d_1, d_2, d_3) \in \mathbb{R}^3$ such that $d_1^2 + d_2^2 + d_3^2 = 1$. Set*

$$M_\Omega := \begin{pmatrix} 0 & -\Omega d_3 & \Omega d_2 \\ \Omega d_3 & 0 & -\Omega d_1 \\ -\Omega d_2 & \Omega d_1 & 0 \end{pmatrix}$$

and

$$R_\Omega(t) := \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$$

with

$$\begin{aligned}
r_{11} = r_{11}(\Omega, t) &:= 1 + (d_1^2 - 1)(1 - \cos(\Omega t)), \\
r_{12} = r_{12}(\Omega, t) &:= d_1 d_2 (1 - \cos(\Omega t)) - d_3 \sin(\Omega t), \\
r_{13} = r_{13}(\Omega, t) &:= d_1 d_3 (1 - \cos(\Omega t)) + d_2 \sin(\Omega t), \\
r_{21} = r_{21}(\Omega, t) &:= d_1 d_2 (1 - \cos(\Omega t)) + d_3 \sin(\Omega t), \\
r_{22} = r_{22}(\Omega, t) &:= 1 + (d_2^2 - 1)(1 - \cos(\Omega t)), \\
r_{23} = r_{23}(\Omega, t) &:= d_2 d_3 (1 - \cos(\Omega t)) - d_1 \sin(\Omega t), \\
r_{31} = r_{31}(\Omega, t) &:= d_1 d_3 (1 - \cos(\Omega t)) - d_2 \sin(\Omega t), \\
r_{32} = r_{32}(\Omega, t) &:= d_2 d_3 (1 - \cos(\Omega t)) + d_1 \sin(\Omega t), \\
r_{33} = r_{33}(\Omega, t) &:= 1 + (d_3^2 - 1)(1 - \cos(\Omega t)).
\end{aligned}$$

Let $W_0 := (W_0^1, W_0^2, W_0^3) \in \mathbb{R}^3$. Set $W(t) := R_\Omega(t) W_0 (t \in \mathbb{R})$. Then W satisfies

$$\begin{cases} W_t - M_\Omega W = 0, \ t \in \mathbb{R}, \\ W|_{t=0} = W_0. \end{cases}$$

Furthermore, the following equalities hold:

$$M_\Omega W = \Omega d \times W,$$
$$R_\Omega(t) = I + (1 - \cos(\Omega t))M_\Omega^2/\Omega^2 + \sin(\Omega t)M_\Omega/\Omega,$$

$$\begin{aligned}
&R_\Omega(-t) = [R_\Omega(t)]^T,\\
&R_\Omega(t)R_\Omega(-t) = R_\Omega(-t)R_\Omega(t) = I,\\
&dR_\Omega(t)/dt = M_\Omega R_\Omega(t),\\
&dR_\Omega(-t)/dt = -M_\Omega R_\Omega(-t),
\end{aligned}$$

$$\begin{aligned}
&r_{11}^2 + r_{12}^2 + r_{13}^2 = 1,\\
&r_{21}^2 + r_{22}^2 + r_{23}^2 = 1,\\
&r_{31}^2 + r_{32}^2 + r_{33}^2 = 1,\\
&r_{11}r_{21} + r_{12}r_{22} + r_{13}r_{23} = 0,\\
&r_{11}r_{31} + r_{12}r_{32} + r_{13}r_{33} = 0,\\
&r_{21}r_{31} + r_{22}r_{32} + r_{23}r_{33} = 0,
\end{aligned}$$

and

$$\begin{aligned}
&d_1(r_{23} + r_{32}) = d_2(r_{13} + r_{31}) = d_3(r_{12} + r_{21}),\\
&d_1(r_{22} - r_{33}) = d_2r_{21} - d_3r_{13} = d_2r_{12} - d_3r_{31},\\
&d_2(r_{11} - r_{33}) = d_1r_{12} - d_3r_{23} = d_1r_{21} - d_3r_{32},\\
&d_3(r_{11} - r_{22}) = d_1r_{13} - d_2r_{32} = d_1r_{31} - d_2r_{23}.
\end{aligned}$$

Now we derive our system (1.1). Here we do not consider the initial conditions and boundary conditions. The procedure for deriving our system (1.1) is as follows:

Incompressible Navier-Stokes system (INS)
$\Longrightarrow$ Navier-Stokes system with rotational effect (NSR)
$\Longrightarrow$ Navier-Stokes-Coriolis system (NSC)
$\Longrightarrow$ Navier-Stokes-Boussinesq system
with Coriolis and stratification effects (NSBCS)
$\Longrightarrow$ Geophysical fluid system (1.1).

Let $\nu, r > 0$, $\Omega > 0$, and $d = (d_1, d_2, d_3) \in \mathbb{R}^3$ such that $d_1^2 + d_2^2 + d_3^2 = 1$. Fix $\varepsilon > 0$. We call ε a *scale parameter* here. Set for $t \geq 0$

$$\mathcal{R}_r := \{x = (x_1, x_2, x_3) \in \mathbb{R}^3; 1 < |x| < 1 + r\},$$
$$\mathcal{R}_r^\varepsilon(t) := \{y = (y_1, y_2, y_3) \in \mathbb{R}^3; y = R_{\Omega\varepsilon^2}(t)x \text{ for } x \in \mathcal{R}_r\}$$

with

$$R_{\Omega\varepsilon^2}(t) := \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + [1 - \cos(\Omega\varepsilon^2 t)]M^2 + \sin(\Omega\varepsilon^2 t)M.$$

Here

$$M := \begin{pmatrix} 0 & -d_3 & d_2 \\ d_3 & 0 & -d_1 \\ -d_2 & d_1 & 0 \end{pmatrix}.$$

We consider the incompressible Navier-Stokes system in a rotating spherical shell:

$$(INS) \begin{cases} v_t + (v, \nabla)v - \nu\Delta v + \nabla p = f, & t > 0,\ x \in \mathcal{R}_r^\varepsilon(t), \\ \nabla \cdot v = 0, & t > 0,\ x \in \mathcal{R}_r^\varepsilon(t). \end{cases}$$

Here $v = v(t, x) = (v^1, v^2, v^3)$ is the fluid velocity, $p = p(t, x)$ the pressure of the fluid, and $f = f(t, x) = (f^1, f^2, f^3)$ the external force. We assume that (v, p, f) are smooth functions. Set

$$V(t, x) = [R_{\Omega\varepsilon^2}(t)]^T v(t, R_{\Omega\varepsilon^2}(t)x),$$
$$P(t, x) = p(t, R_{\Omega\varepsilon^2}(t)x),$$
$$F(t, x) = [R_{\Omega\varepsilon^2}(t)]^T f(t, R_{\Omega\varepsilon^2}(t)x).$$

Using Lemma 3.1, we see that

$$(NSR) \begin{cases} V_t + (V, \nabla)V - \nu\Delta V + \nabla P + \Omega\varepsilon^2 d \times V \\ \qquad\qquad - (\Omega\varepsilon^2 d \times (x_1, x_2, x_3), \nabla)V = F,\ t > 0,\ x \in \mathcal{R}_r, \\ \nabla \cdot V = 0,\ t > 0,\ x \in \mathcal{R}_r. \end{cases}$$

Set

$$P_\infty^\varepsilon := \Omega^2\varepsilon^4 \Big[\frac{d_2^2 + d_3^2}{2}x_1^2 + \frac{d_1^2 + d_3^2}{2}x_2^2 + \frac{d_1^2 + d_2^2}{2}x_3^2 \\ - d_1 d_2 x_1 x_2 - d_1 d_3 x_1 x_3 - d_2 d_3 x_2 x_3\Big],$$

$$w(t,x) := V(t,x) - \Omega\varepsilon^2 d \times x, \;\; x = (x_1, x_2, x_3),$$
$$q := P - P^{\varepsilon}_{\infty}.$$

By direct calculations, we check that (w,q) satisfies the following system:

$$(NSC) \begin{cases} w_t + (w,\nabla)w - \nu\Delta w + \nabla q + 2\Omega\varepsilon^2 d \times w = F, & t>0, \; x \in \mathcal{R}_r, \\ \nabla \cdot w = 0, & t>0, \; x \in \mathcal{R}_r. \end{cases}$$

Next we consider fluids effected by heat and stratification effects. Fix $X_0 = (x_0, y_0, z_0) \in \mathcal{R}_r$ and $T_0 > 0$. Set for $\delta > 0$

$$B_\delta(T_0, X_0) := \{(t,x) \in \mathbb{R}_+ \times \mathbb{R}^3; |t - T_0| < \delta, \; |x - X_0| < \delta\}.$$

Now we consider (NSC) in a neighborhood of the point (T_0, X_0):

$$\begin{cases} U_t + (U,\nabla)U - \nu\Delta U + \nabla q + 2\Omega\varepsilon^2 d \times U = F & \text{in } B_\delta(T_0,X_0), \\ \nabla \cdot U = 0 & \text{in } B_\delta(T_0,X_0). \end{cases}$$

Here $B_\delta(T_0,X_0) \subset \mathbb{R}_+ \times \mathcal{R}_r$ and $U = U(t,x) = (U^1, U^2, U^3) := w|_{B_\delta(T_0,X_0)}$. Assume that the fluid in $B_\delta(T_0,X_0)$ is effected by heat and stratification effects. Applying the following Boussinesq approximation:

$$\begin{aligned} & q = Q + q_0, \\ & \partial_x q_0 \simeq 0, \\ & \partial_y q_0 \simeq 0, \\ & \partial_z q_0 \simeq -\mathcal{G}\Theta, \\ & -N^2 U^3 = d\Theta/dt - \kappa\Delta\Theta + (U,\nabla)\Theta, \\ & F \equiv 0, \end{aligned}$$

or

$$\begin{aligned} & F = \mathcal{G}\Theta, \\ & -N^2 U^3 = d\Theta/dt - \kappa\Delta\Theta + (U,\nabla)\Theta, \end{aligned}$$

we obtain

$$\begin{cases} U_t + (U,\nabla)U - \nu\Delta U + \nabla Q + 2\Omega\varepsilon^2 d \times U = \mathcal{G}\Theta e_3 & \text{in } B_\delta(T_0,X_0), \\ \Theta_t + (U,\nabla)\Theta - \kappa\Delta\Theta = -N^2 U^3 & \text{in } B_\delta(T_0,X_0), \\ \nabla \cdot U = 0 & \text{in } B_\delta(T_0,X_0). \end{cases}$$

Here Θ is temperature, $\kappa > 0$, and $\mathcal{G}, N > 0$. Set

$$\mathbb{R}_{T_0} := \{t \in \mathbb{R}; t > t_0\}$$

and

$$W_\varepsilon := \{(t,x) = (t, x_1, x_2, x_3) \in \mathbb{R}_{T_0} \times \mathbb{R}^3; (\varepsilon^2 t + T_0, \varepsilon x + X_0) \in W_\varepsilon\}.$$

It is clear that as $\varepsilon \to 0$

$$W_\varepsilon \to \mathbb{R}_{T_0} \times \mathbb{R}^3 \approx \mathbb{R}_+ \times \mathbb{R}^3.$$

Now we consider two cases in order to drive our system (1.1).

3.1 Case I

Let us consider the following system:

$$\begin{cases} U_t + (U,\nabla)U - \nu\Delta U + \nabla Q + 2\Omega\varepsilon^2 d \times U = \mathcal{G}\Theta e_3 & \text{in } W_\varepsilon, \\ \Theta_t + (U,\nabla)\Theta - \kappa\Delta\Theta = -N^2 U^3 & \text{in } W_\varepsilon, \\ \nabla \cdot U = 0 & \text{in } W_\varepsilon. \end{cases}$$

Assume that there is $N_0 \in \mathbb{R}$ such that

$$N = N_0 \varepsilon^2.$$

Set

$$\begin{aligned} \widetilde{U}^\varepsilon(t,x) &:= \varepsilon U(\varepsilon^2 t + T_0, \varepsilon x + X_0), \\ \widetilde{\Theta}^\varepsilon(t,x) &:= \varepsilon^3 \Theta(\varepsilon^2 t + T_0, \varepsilon x + X_0), \\ \widetilde{Q}^\varepsilon(t,x) &:= \varepsilon^2 Q(\varepsilon^2 t + T_0, \varepsilon x + X_0). \end{aligned}$$

Then

$$\begin{cases} \widetilde{U}^\varepsilon_t + (\widetilde{U}^\varepsilon,\nabla)\widetilde{U}^\varepsilon - \nu\Delta \widetilde{U}^\varepsilon + \nabla \widetilde{Q}^\varepsilon + 2\Omega d \times \widetilde{U}^\varepsilon = \mathcal{G}\widetilde{\Theta}^\varepsilon e_3 & \text{in } W_\varepsilon, \\ \widetilde{\Theta}^\varepsilon_t + (\widetilde{U}^\varepsilon,\nabla)\widetilde{\Theta}^\varepsilon - \kappa\Delta\widetilde{\Theta}^\varepsilon = -N_0^2 \widetilde{U}^{\varepsilon,3} & \text{in } W_\varepsilon, \\ \nabla \cdot \widetilde{U}^\varepsilon = 0 & \text{in } W_\varepsilon. \end{cases}$$

Using the following assumption: as $\varepsilon \to 0$

$$
\begin{aligned}
\widetilde{U}^{\varepsilon} &\to u,\\
\widetilde{\Theta}^{\varepsilon} &\to \theta,\\
\widetilde{Q}^{\varepsilon} &\to \mathfrak{p},
\end{aligned}
$$

we obtain

$$
\begin{cases}
u_t + (u,\nabla)u - \nu\Delta u + \nabla\mathfrak{p} + 2\Omega d \times u = \mathcal{G}\theta e_3 & \text{in } \mathbb{R}_+ \times \mathbb{R}^3,\\
\theta_t + (u,\nabla)\theta - \kappa\Delta\theta = -N_0^2 u^3 & \text{in } \mathbb{R}_+ \times \mathbb{R}^3,\\
\nabla \cdot u = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}^3.
\end{cases}
$$

Therefore we get our system (1.1).

3.2 Case II

Let us consider the following system:

$$
\begin{cases}
U_t + (U,\nabla)U - \nu\Delta U + \nabla Q + 2\Omega\varepsilon^2 d \times U = \mathcal{G}\Theta e_3 & \text{in } W_\varepsilon,\\
\Theta_t + (U,\nabla)\Theta - \kappa\Delta\Theta = -N^2 U^3 & \text{in } W_\varepsilon,\\
\nabla \cdot U = 0 & \text{in } W_\varepsilon.
\end{cases}
$$

Assume that there are $N_0, \mathcal{G}_0 \in \mathbb{R}$ such that

$$
\begin{aligned}
\mathcal{G} &= \mathcal{G}_0\varepsilon^2,\\
N &= N_0\varepsilon.
\end{aligned}
$$

Set

$$
\begin{aligned}
\widetilde{U}^{\varepsilon}(t,x) &:= \varepsilon U(\varepsilon^2 t + T_0, \varepsilon x + X_0),\\
\widetilde{\Theta}^{\varepsilon}(t,x) &:= \varepsilon \Theta(\varepsilon^2 t + T_0, \varepsilon x + X_0),\\
\widetilde{Q}^{\varepsilon}(t,x) &:= \varepsilon^2 Q(\varepsilon^2 t + T_0, \varepsilon x + X_0).
\end{aligned}
$$

Then

$$
\begin{cases}
\widetilde{U}^{\varepsilon}_t + (\widetilde{U}^{\varepsilon},\nabla)\widetilde{U}^{\varepsilon} - \nu\Delta\widetilde{U}^{\varepsilon} + \nabla\widetilde{Q}^{\varepsilon} + 2\Omega d \times \widetilde{U}^{\varepsilon} = \mathcal{G}_0\widetilde{\Theta}^{\varepsilon} e_3 & \text{in } W_\varepsilon,\\
\widetilde{\Theta}^{\varepsilon}_t + (\widetilde{U}^{\varepsilon},\nabla)\widetilde{\Theta}^{\varepsilon} - \kappa\Delta\widetilde{\Theta}^{\varepsilon} = -N_0^2 \widetilde{U}^{\varepsilon,3} & \text{in } W_\varepsilon,\\
\nabla \cdot \widetilde{U}^{\varepsilon} = 0 & \text{in } W_\varepsilon.
\end{cases}
$$

Using the following assumption: as $\varepsilon \to 0$

$$\begin{aligned} \widetilde{U}^\varepsilon &\to u, \\ \widetilde{\Theta}^\varepsilon &\to \theta, \\ \widetilde{Q}^\varepsilon &\to \mathfrak{p}, \end{aligned}$$

we obtain

$$\begin{cases} u_t + (u, \nabla)u - \nu\Delta u + \nabla \mathfrak{p} + 2\Omega d \times u = \mathcal{G}_0 \theta e_3 & \text{in } \mathbb{R}_+ \times \mathbb{R}^3, \\ \theta_t + (u, \nabla)\theta - \kappa\Delta\theta = -N_0^2 u^3 & \text{in } \mathbb{R}_+ \times \mathbb{R}^3, \\ \nabla \cdot u = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}^3. \end{cases}$$

Therefore we get our system (1.1).

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