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KELVIN-HELMHOLTZ INSTABILITIES AND INTERFACIAL WAVES

DAVID LANNES

1. INTRODUCTION

1.1. **General setting.** We are interested here in the motion of the interface between two incompressible fluids of different densities $\rho^{+}>\rho^{-}$, with vorticity concentrated at the interface, and at rest at infinity (the limit case $\rho^{-}=0$ is also known as the water waves problem). We refer to [6] for a recent review of the stability issue of such interfaces. In the case of water waves $(\rho^{-}=0)$, the situation is now quite well understood since [33, 34]: the well posedness of the water waves equations requires that the *Rayleigh-Taylor criterion* is satisfied,

$$\text{(Rayleigh-Taylor)} \qquad -\partial_{z}P\,|_{\text{surface}}>0,$$

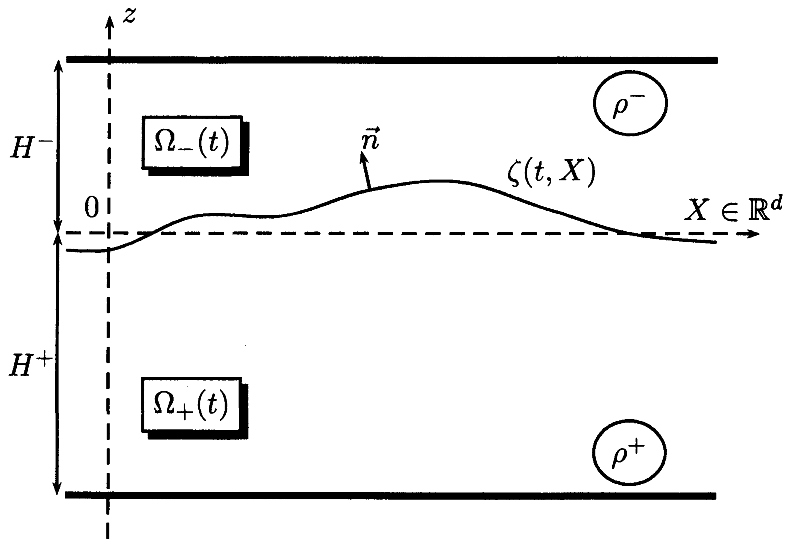
where z is the vertical coordinate and P the pressure. It is instructive to remark that the linearized version of this criterion (around the rest state) is simply $\rho^{+}g>0$, where g is the (vertical) acceleration of gravity – in other words, water must stands *below* the interface.

For the two fluids problem $(\rho^{-}>0)$ the situation is more complex. It is known, at least for $1D$ interfaces, that, outside the analytic framework of [31, 30], the evolution equations are ill-posed in absence of surface tension [15, 17, 18]. The reason of this ill-posedness is that the nonlinearity creates locally a discontinuity of the tangential velocity at the interface that induces Kelvin-Helmholtz instabilities. Taking into account the surface tension restores the local well-posedness of the equations [4, 5, 28, 9, 29, 24, 10, 25]. However, the existence time of the solution provided by these results is very small when the surface tension is small. The fact that the role of gravity (or gravity itself) is not considered in these references suggests that these general results can be improved in the "stable" configuration where the heavier fluid is placed below the lighter one.

The goal of this paper is to present one of the results of [21], namely, the derivation of a two fluids generalization of the afore mentioned Rayleigh-Taylor criterion governing the stability of two fluids interfaces:

$$[\![-\partial_{z}P^{\pm}\,|_{z=\zeta}]\!]>\frac{1}{4}\frac{(\rho^{+}\rho^{-})^{2}}{\sigma(\rho^{+}+\rho^{-})^{2}}\mathfrak{c}(\zeta)\,|\omega|_{\infty}^{4}, \tag{1}$$

where ζ is the interface parametrization, $\omega=[\![V^{\pm}\,|_{z=\zeta}]\!]$ is the jump of the horizontal velocity at the interface, and $\mathfrak{c}(\zeta)$ is a constant that depends on the geometry of the problem (two layers of finite depth in this paper) and that can be estimated quite precisely. Since the proof of [21] is quite lengthy and technical, we chose to give in this short presentation a more qualitative approach. For the sake of simplicity, we do not deal here with one of the main difficulties of [21], that is, handling the



shallow water asymptotics; this aspect of the problem is just shortly addressed in the last section.

1.2. **Notations.** - We denote by $H^s(\mathbb{R}^d)$ ($s \in \mathbb{R}$) the standard Sobolev spaces,

$$H^s(\mathbb{R}^d) = \{f \in \mathcal{S}'(\mathbb{R}^d), |f|_{H^s} < \infty\}, \quad \text{with} \quad |f|_{H^s}^2 = \int_{\mathbb{R}^d} (1+|\xi|^2)^s |\widehat{f}(\xi)|^2 d\xi.$$

and by $\dot{H}^s(\mathbb{R}^d)$ the Beppo-Levi spaces

$$\dot{H}^s(\mathbb{R}^d) = \{f \in L^2_{loc}(\mathbb{R}^d), \nabla f \in H^{s-1}(\mathbb{R}^d)^d\},$$

equipped with the (semi) norm $|f|_{\dot{H}^s} = |\nabla f|_{H^s}$ (we refer to [14, 21, 22] for some properties of these spaces).

- We use the classical notation $f(D)$ for Fourier multipliers,

$$f(D)u = \mathcal{F}^{-1}\big(f(\xi)\widehat{u}(\xi)\big).$$

2. Several formulations of the equations

We consider throughout this article the motion of the interface between two fluids of density $\rho^+ > \rho^-$, and denote by Ω_t^+ and Ω_t^- the volume they occupy at time t. Choosing the origin of the vertical axis to correspond with the interface between the two fluids at rest, we assume that Ω_t^+ (resp. Ω_t^-) is bounded below (resp. above) by an horizontal wall located at $z = -H^+$ (resp. $z = H^-$). We also denote by Γ_t the interface between both fluids, and assume that it can be parametrized as the graph of a function $\zeta(t,\cdot)$, i.e. $\Gamma_t = \{(X,z), z = \zeta(t,X)\}$; we denote by $\Gamma^\pm$ the upper and lower boundaries $\Gamma^\pm = \{z = \mp H^\pm\}$.

Finally, we denote by $\mathbf{U}^\pm$ the velocity field in $\Omega_t^\pm$; the horizontal component of $\mathbf{U}^\pm$ is written $V^\pm$ and its vertical one $w^\pm$. The pressure is denoted by $P^\pm$.

For the sake of clarity, it is also convenient to introduce some notation to express the difference and average of these quantity across the interface.

Notation 2.1. If A^+ and A^- are two quantities (real numbers, functions, etc.), the notations $[\![A^\pm]\!]$ and $\langle A^\pm \rangle$ stand for

$$[\![A^\pm]\!] = A^+ - A^- \quad \text{and} \quad \langle A^\pm \rangle = \frac{A^+ + A^-}{2}.$$

2.1. **The free interface Euler equations.** We assume that both fluids are inviscid, incompressible, and that the fluid motion is irrotational in the interior of these two domains. The corresponding equations are the so called *free interface Euler equations*:

- Equations in the fluid layers. In both fluid layers, the velocity field $\mathbf{U}^{\pm}$ and the pressure $P^{\pm}$ satisfy the equations

$$\operatorname{div} \mathbf{U}^{\pm}(t,\cdot) = 0, \qquad \operatorname{curl} \mathbf{U}^{\pm}(t,\cdot) = 0, \qquad \text{in } \Omega_t^{\pm} \quad (t \geq 0), \tag{2}$$

which express the incompressibility and irrotationality assumptions, and

$$\rho^{\pm}\big(\partial_t \mathbf{U}^{\pm} + (\mathbf{U}^{\pm}\cdot\nabla_{X,z})\mathbf{U}^{\pm}\big) = -\nabla_{X,z}P^{\pm} - \rho^{\pm} g \mathbf{e}_z \qquad \text{in } \Omega_t^{\pm} \quad (t \geq 0), \tag{3}$$

which expresses the conservation of momentum (Euler equation).

- Boundary conditions at the rigid bottom and lid. Impermeability of these two boundaries is classically rendered by

$$w^{\pm}(t,\cdot)_{|\Gamma^{\pm}} = 0, \qquad (t \geq 0). \tag{4}$$

- Boundary conditions at the moving interface. The fact that the interface is a bounding surface (the fluid particles do not cross it) yields the equations

$$\partial_t \zeta - \sqrt{1+|\nabla\zeta|^2}\,\mathbf{U}_n^{\pm} = 0, \qquad (t \geq 0), \tag{5}$$

where $\mathbf{U}_n^{\pm} := \mathbf{U}^{\pm}{}_{|\Gamma_t} \cdot \mathbf{n}$, $\mathbf{n}$ being the upward unit normal vector to the interface Γ_t. A direct consequence of (5) is that *there is no jump of the normal component of the velocity at the interface.* Finally, the continuity of the stress tensor at the interface gives in our particular case

$$[\![P^{\pm}(t,\cdot)_{|\Gamma_t}]\!] = \sigma k(\zeta), \qquad (t \geq 0), \tag{6}$$

where σ is the surface tension coefficient and $k(\zeta)$ denotes the mean curvature of the interface,

$$k(\zeta) = -\nabla \cdot \Big(\frac{\nabla\zeta}{\sqrt{1+|\nabla\zeta|^2}}\Big).$$

2.2. **The free interface Bernoulli equations.** Taking advantage of the irrotationality assumption made on both fluids, it is possible to reduce the number of unknowns by working with a (scalar) velocity potential $\Phi^{\pm}$ instead of the (vectorial) velocity field $\mathbf{U}^{\pm}$; this velocity potential is classically defined as

$$\mathbf{U}^{\pm}(t,\cdot) = \nabla_{X,z}\Phi^{\pm}(t,\cdot) \qquad \text{in } \Omega_t^{\pm} \quad (t \geq 0).$$

The free surface Euler equations (2)-(6) can then be written in terms of this velocity potential, and become the *free interface Bernoulli equations*:

- Equations in the fluid layers. In both fluid layers, the velocity potentials $\Phi^{\pm}$ and the pressure $P^{\pm}$ satisfy the equations

$$\Delta_{X,z}\Phi^{\pm}(t,\cdot) = 0, \qquad \text{in } \Omega_t^{\pm} \quad (t \geq 0), \tag{7}$$

and the Bernoulli equation,

$$\rho^{\pm}\big(\partial_t\Phi^{\pm} + \frac{1}{2}|\nabla_{X,z}\Phi^{\pm}|^2\big) = -P^{\pm} - \rho^{\pm} g z \qquad \text{in } \Omega_t^{\pm} \quad (t \geq 0). \tag{8}$$

- Boundary conditions at the rigid bottom and lid. Written in terms of $\Phi^{\pm}$, (4) becomes

$$\partial_z\Phi^{\pm}(t,\cdot)_{|\Gamma^{\pm}} = 0, \qquad (t \geq 0). \tag{9}$$

- Boundary conditions at the moving interface. The kinematic boundary condition (5) can be written as

$$\partial_t\zeta - \sqrt{1+|\nabla\zeta|^2}\partial_n\Phi^{\pm}_{|\Gamma_t} = 0, \qquad (t\geq 0), \tag{10}$$

where ∂_n always stands for the *upwards* normal derivative at the interface. Finally, (6) is left unchanged,

$$[\![P^{\pm}(t,\cdot)_{|\Gamma_t}]\!] = \sigma\kappa(\zeta), \qquad (t\geq 0). \tag{11}$$

2.3. **Reduction to the interface.** Inspired by the standard Zakharov-Craig-Sulem hamiltonian formulation of the water waves equations [32, 13], we introduce the trace of $\Phi^{\pm}$ at the interface,

$$\psi^{\pm}(t,\cdot) = \Phi^{\pm}(t,\cdot)_{|\Gamma_t} \qquad (t\geq 0).$$

The knowledge of $\psi^{\pm}$ and ζ (which determines the shape of the fluid domains $\Omega^{\pm}$) allows one to recover the velocity potentials $\Phi^{\pm}$ in the interior of the fluid domains through the resolution of the boundary value problem

$$\begin{cases} \Delta_{X,z}\Phi^{\pm} = 0 & \text{in } \Omega^{\pm}_t, \\ \Phi^{\pm}{}_{|\Gamma_t} = \psi^{\pm}, & \partial_z\Phi^{\pm}{}_{|\Gamma^{\pm}} = 0 \end{cases} \tag{12}$$

(we recall that $\partial_n\Phi^{\pm}_{|\Gamma_t}$ stands for the *upward* normal partial derivative of $\Phi^{\pm}$ at the interface). Under appropriate regularity assumptions, it is therefore possible to see the normal derivative of $\Phi^{\pm}$ at the interface as an operator acting (nonlinearly) on ζ and (linearly) on $\psi^{\pm}$; we denote by $\mathcal{G}^{\pm}[\zeta]$ this operator, which is called *the Dirichlet-Neumann operator* corresponding to the two fluid layer $\Omega^{\pm}$:

$$\mathcal{G}^{\pm}[\zeta]\psi^{\pm} = \sqrt{1+|\nabla\zeta|^2}\partial_n\Phi^{\pm}{}_{|\Gamma_t},$$

where ∂_n stands for the upwards normal derivative at the interface.

By taking the trace of (8) at the interface, it is therefore possible to reduce the free surface Bernoulli equations (7)-(11) as a set of equations on ζ and $\psi^{\pm}$,

$$\partial_t\zeta - \mathcal{G}^{\pm}[\zeta]\psi^{\pm} = 0, \tag{13}$$

$$\rho^{\pm}\Big(\partial_t\psi^{\pm} + g\zeta + \frac{1}{2}|\nabla\psi^{\pm}|^2 - \frac{(\mathcal{G}^{\pm}[\zeta]\psi^{\pm} + \nabla\zeta\cdot\nabla\psi^{\pm})^2}{2(1+|\nabla\zeta|^2)}\Big) = -P^{\pm}_{|\Gamma_t}, \tag{14}$$

$$[\![P^{\pm}(t,\cdot)_{|\Gamma_t}]\!] = \sigma\kappa(\zeta). \tag{15}$$

In the case of the water waves equations corresponding to $\rho^- = 0$ and $P^- = 0$, the equations (13)-(15) correspond to the hamiltonian Zakharov-Craig-Sulem formulation[1]. In this formulation the pressure has disappeared by the choice of evaluating the Bernoulli equation on the free surface. In presence of an upper layer of nonzero density, the situation is more complex since we have to deal with two evolution equations on ψ^+ and ψ^- instead of only one evolution equation on ψ^+ in the water waves case. In order to eliminate the pressure from the equations, one must use (15), and therefore consider the difference $(14)^+ - (14)^-$. It is therefore

[1]This is a set of two evolution equations on ζ and ψ^+,

$$\partial_t\zeta - \mathcal{G}^{+}[\zeta]\psi^{+} = 0,$$

$$\partial_t\psi^{+} + g\zeta + \frac{1}{2}|\nabla\psi^{+}|^2 - \frac{(\mathcal{G}^{+}[\zeta]\psi^{+} + \nabla\zeta\cdot\nabla\psi^{+})^2}{2(1+|\nabla\zeta|^2)} = -\sigma\kappa(\zeta),$$

natural to try to rewrite the two-fluid equations as a set of two equations on the surface elevation ζ and of the quantity ψ defined as

$$\psi := \underline{\rho}^+\psi^+ - \underline{\rho}^-\psi^-,$$

where $\underline{\rho}^\pm$ stands for the relative density, $\underline{\rho}^\pm = \frac{\rho^\pm}{\rho^+ + \rho^-}$ (in particular, $\underline{\rho}^+ + \underline{\rho}^- = 1$ and $\underline{\rho}^+ - \underline{\rho}^-$ is the so called Atwood number). We obtain therefore

$$\begin{cases} \partial_t \zeta - \mathcal{G}^\pm[\zeta]\psi^\pm = 0, \\ \partial_t \psi + g'\zeta + \frac{1}{2}[\![\underline{\rho}^\pm |\nabla \psi^\pm|^2]\!] \\ \quad - \frac{1}{2}\frac{[\![\underline{\rho}^\pm(\mathcal{G}^\pm[\zeta]\psi^\pm + \nabla\zeta\cdot\nabla\psi^\pm)^2]\!]}{1+|\nabla\zeta|^2} = -\frac{\sigma}{\rho^+ + \rho^-}\kappa(\zeta), \end{cases} \tag{16}$$

where g' stands for the reduced gravity,

$$g' = (\underline{\rho}^+ - \underline{\rho}^-)g.$$

In the water waves case $\rho^- = 0$ (and therefore $\psi = \psi^+$), this coincides with the classical Zakharov-Craig-Sulem formulation. In the two fluids case $\rho^- > 0$, a new difficulty occurs, namely, one has to express ψ^+ and ψ^- in terms of ζ and ψ. This can be done by solving the system

$$\begin{cases} \underline{\rho}^+\psi^+ - \underline{\rho}^-\psi^- &= \psi, \\ \mathcal{G}^+[\zeta]\psi^+ - \mathcal{G}^+[\zeta]\psi^- &= 0; \end{cases} \tag{17}$$

the first equation is the definition of ψ, while the second traduces the continuity of the normal component of the velocity as it crosses the interface (this relation is obtained by considering the difference $(13)^+ - (13)^-$). One obtains formally the following solution to (17),

$$\psi^+ = J[\zeta]^{-1}\psi, \qquad \psi^- = (\mathcal{G}^-[\zeta])^{-1}\circ\mathcal{G}^+[\zeta]\psi^+, \tag{18}$$

with

$$J[\zeta] = \underline{\rho}^+ - \underline{\rho}^-(\mathcal{G}^-[\zeta])^{-1}\mathcal{G}^+[\zeta]. \tag{19}$$

Let us state here a few facts about the Dirichlet-Neumann operators that show that (18) and (19) do make sense; in what follows, t_0 is any real number such that $t_0 > d/2$, and $\zeta \in H^{t_0+1}(\mathbb{R}^d)$ does not touch the bottoms (i.e. $\inf_{\mathbb{R}^d}(H^\pm \pm \zeta) > 0$). We also recall that the Beppo-Levi spaces have been defined in §1.2.

(1) The operator $G^\pm[\zeta]$ is a continuous operator mapping $\dot{H}^{s+1/2}(\mathbb{R}^d)$ into $H^{s-1/2}(\mathbb{R}^d)$ for all $0 \le s \le t_0 + 1/2$ (see Theorem 3.15 of [22]).
(2) This mapping is not onto and the inverse $(\mathcal{G}^\pm[\zeta])^{-1}$ is not well defined as a mapping $H^{s-1/2}(\mathbb{R}^d) \to \dot{H}^{s+1/2}(\mathbb{R}^d)$.
(3) However, the range of $\mathcal{G}^+[\zeta]$ is included in the range of $\mathcal{G}^-[\zeta]$, and the operator $(\mathcal{G}^-[\zeta])^{-1}\mathcal{G}^+[\zeta] : \dot{H}^{s+1/2}(\mathbb{R}^d) \to \dot{H}^{s+1/2}(\mathbb{R}^d)$ is well defined and bounded (see [21], Proposition 1).
(4) From the previous point $J[\zeta] : \dot{H}^{s+1/2}(\mathbb{R}^d) \to \dot{H}^{s+1/2}(\mathbb{R}^d)$ is well defined. It is also invertible (Neumann series) for small values of $\underline{\rho}^-$. It remains invertible in the general case using Fredholm's index theory (Lemma 2 of [21]).

It follows from this brief analysis that the formal solution (18)-(19) is well defined, and therefore that ψ^+ and ψ^- can be expressed in terms of ζ and ψ. More precisely, one can write

$$\mathcal{G}^\pm[\zeta]\psi^\pm = \mathcal{G}[\zeta]\psi \quad \text{with} \quad \mathcal{G}[\zeta] = \mathcal{G}^-[\zeta]J[\zeta](\mathcal{G}^-[\zeta])^{-1}\mathcal{G}^+[\zeta].$$

The two-fluids generalization of the Zakharov-Craig-Sulem formulation of the water waves equations is then given by a closed set of two evolution equations on ζ and ψ,

$$\left\{\begin{aligned} &\partial_t\zeta - \mathcal{G}[\zeta]\psi = 0,\\ &\partial_t\psi + g'\zeta + \frac{1}{2}[\![\underline{\rho}^\pm|\nabla\psi^\pm|^2]\!]\\ &\qquad -\frac{1}{2}\frac{[\![\underline{\rho}^\pm(\mathcal{G}[\zeta]\psi + \nabla\zeta\cdot\nabla\psi^\pm)^2]\!]}{1+|\nabla\zeta|^2} = -\frac{\sigma}{\rho^+ + \rho^-}\kappa(\zeta)\end{aligned}\right. \tag{20}$$

(we kept the notation $\psi^\pm$ here, but they must be understood as functions of ζ and ψ through (18)).

Remark 2.2. Benjamin and Bridges [7] (see also [19, 12]) showed that the two fluids equations are Hamiltonian with ζ and ψ as the two canonical variables and the total energy as the Hamiltonian.

3. A STABILITY CRITERION

We know from the previous section that (16) can be viewed as a set of two evolution equations on ζ and ψ. We show here that they are well posed under suitable assumptions on the initial data, and provided that the stability criterion (1) is satisfied.

3.1. The linearized equations around the rest state $(\underline{\zeta},\underline{\psi}) = (0,0)$. It is instructive to consider the linearization of (20) around the rest state, namely,

$$\left\{\begin{aligned} &\partial_t\zeta - \mathcal{G}[0]\psi = 0,\\ &\partial_t\psi + \big(g' - \frac{\sigma}{\rho^+ + \rho^-}\Delta\big)\zeta = 0.\end{aligned}\right. \tag{21}$$

By simple Fourier analysis, one can explicitly compute $\mathcal{G}^\pm[0]$ and therefore $\mathcal{G}[0]$,

$$\begin{aligned} \mathcal{G}^\pm[0] &= |D|\tanh(H^\pm|D|),\\ \mathcal{G}[0] &= |D|\frac{\tanh(H^+|D|)\tanh(H^-|D|)}{\underline{\rho}^+\tanh(H^-|D|) + \underline{\rho}^-\tanh(H^+|D|)}.\end{aligned} \tag{22}$$

It follows that the equations (21) are well posed in Sobolev spaces, even in absence of surface tension ($\sigma = 0$). The linearization around the rest state therefore misses one of the most important features of the two fluids problem: *the Kelvin-Helmholtz instabilities.*

In order to prepare the ground for the nonlinear analysis of the equations, let us remark that the equations can be made "symmetric" if we multiply them by S_0 with

$$S_0 = \begin{pmatrix} g' - \frac{\sigma}{\rho^+ + \rho^-}\Delta & 0 \\ 0 & \mathcal{G}[0]\end{pmatrix};$$

denoting $U=(\zeta,\psi)$, a natural energy for the system is therefore

$$
\begin{aligned}
E_0(U) &= (U, S_0 U)\\
&= g'|\zeta|^2_{H^1_\sigma} + \frac{\sigma}{\rho^+ + \rho^-}|\nabla\zeta|_2^2 + |\psi|^2_{\dot H^{1/2}_*},
\end{aligned}
\tag{23}
$$

with the norms $|\cdot|_{H^1_\sigma}$ and $|\cdot|_{\dot H^{1/2}_*}$.

$$
|\zeta|^2_{H^1_\sigma} = |\zeta|_2^2 + \frac{\sigma}{g'(\rho^+ + \rho^-)}|\nabla\zeta|_2^2, \qquad |\psi|_{\dot H^{1/2}_*} = |\mathfrak{P}\psi|_2 \qquad (\text{and } \mathfrak{P} = \sqrt{\mathcal{G}[0]});
$$

two important features of these norms are the following,

(1) The H^1 and H^1_σ norms are equivalent, but with equivalence constants depending on σ; it is important to distinguish these norms if we want an existence time that has a sharp dependence on σ.
(2) The $\dot H^{1/2}$ and $\dot H^{1/2}_*$ are equivalent but with equivalence constant that depend on H^+ and H^-; if one wants to be able to handle shallow water asymptotics, it is important to distinguish them.

3.2. **The linearized equations around a constant shear.** In order to understand simply the occurrence of Kelvin-Helmholtz instabilities, let us consider the linearized equations around a constant shear. More precisely, we consider the linearization of the two fluids equations (13)-(15) around the constant flow

$$
\underline{\mathbf{U}}^\pm = \begin{pmatrix} c^\pm \\ 0 \end{pmatrix}, \qquad \underline{\zeta} = 0,
$$

which yields the following system

$$
\begin{cases}
\partial_t\zeta + c^\pm\cdot\nabla\zeta - \underline{w}^\pm = 0,\\
\partial_t\psi + g'\zeta + [\![\underline{\rho}^\pm c^\pm\cdot\nabla\psi^\pm]\!] = \dfrac{\sigma}{\rho^+ + \rho^-}\Delta\zeta,
\end{cases}
$$

with $\psi = \underline{\rho}^+\psi^+ - \underline{\rho}^-\psi^-$ and $\underline{w}^\pm = \mathcal{G}^\pm[0]\psi^\pm$. The quantities $\psi^\pm$ (and therefore $\underline{w}^\pm$ since $\underline{w}^\pm = \pm|D|\tanh(H^\pm|D|)\psi^\pm$) can be expressed in terms of ζ and ψ by adapting the computations of §2.3. This leads to

$$
-\underline{w}^\pm = \frac{\underline{\rho}^-\tanh(H^+|D|)}{\underline{\rho}^-\tanh(H^+|D|) + \underline{\rho}^+\tanh(H^-|D|)}[\![c^\pm]\!]\cdot\nabla\zeta + \mathcal{G}[0]\psi,
$$

where $\mathcal{G}[0]$ is as in (22). For the term $[\![\underline{\rho}^\pm c^\pm\cdot\nabla\psi^\pm]\!]$ that appears in the second equation, we write

$$
[\![\underline{\rho}^\pm c^\pm\cdot\nabla\psi^\pm]\!] = \langle c^\pm\rangle\cdot\nabla\psi + [\![c^\pm]\!]\cdot\nabla\langle\underline{\rho}^\pm\psi^\pm\rangle.
$$

The dependence on ζ is specific to the two fluids system (i.e. it disappears if $\rho^- = 0$), and is responsible for the Kelvin-Helmholtz instabilities through the operator $e(D)$ in the resulting set of equations in ζ, ψ,

$$
\begin{cases}
\partial_t\zeta + T(D)\zeta - \mathcal{G}[0]\psi = 0,\\
\partial_t\psi + T(D)\psi + (g' - \underline{\rho}^+\underline{\rho}^-[\![c^\pm]\!]\cdot e(D)([\![c^\pm]\!]\bullet) - \dfrac{\sigma}{\rho^+ - \rho^-}\Delta)\zeta = 0,
\end{cases}
\tag{24}
$$

with

$$\begin{aligned} T(D) &= \frac{c^+\underline{\rho}^+\tanh(H^+|D|)+c^-\underline{\rho}^-\tanh(H^+|D|)}{\underline{\rho}^-\tanh(H^+|D|)+\underline{\rho}^+\tanh(H^-|D|)}\cdot\nabla,\\ e(D) &= \frac{1}{\underline{\rho}^-\tanh(H^+|D|)+\underline{\rho}^+\tanh(H^-|D|)}\frac{DD^T}{|D|}. \end{aligned}$$

The diagonal terms in (24) are governed by the operator $T(D)$, which can be seen as a nonlocal transport term; consequently, they do no play any role in the stability analysis. Since $\mathcal{G}[0]$ is a positive operator, the linear stability of (24) depends therefore on the sign of the *stability operator*

$$\mathrm{Ins}(D) = g' - \underline{\rho}^+\underline{\rho}^-[\![c^\pm]\!]\cdot e(D)([\![c^\pm]\!]\bullet) - \frac{\sigma}{\rho^+-\rho^-}\Delta,$$

in the sense that stability for all frequencies is obtained if $\mathrm{Ins}(\xi) > 0$ for all $\xi \in \mathbb{R}^d$. Conversely, Kelvin-Helmholtz instability is the mechanism that amplifies the frequencies for which $\mathrm{Ins}(\xi) < 0$. Therefore,

$$\text{All frequency are stable} \iff \forall\xi\in\mathbb{R}^d,\quad \mathrm{Ins}(\xi)>0.$$

Remarking that

$$\begin{aligned} &[\![c^\pm]\!]\cdot e(\xi)([\![c^\pm]\!]\bullet) \sim \underline{\rho}^+\underline{\rho}^-|[\![c^\pm]\!]|^2|\xi| &&\text{as}\quad |\xi|\to\infty,\\ &[\![c^\pm]\!]\cdot e(\xi)([\![c^\pm]\!]\bullet) \sim \underline{\rho}^+\underline{\rho}^-|[\![c^\pm]\!]|^2H_0^{-1} &&\text{as}\quad |\xi|\to 0 \end{aligned}$$

(with $H_0 = \underline{\rho}^-H^+ + \underline{\rho}^-H^-$), one can deduce the following facts:

(1) In absence of surface tension, high frequencies are always *unstable*;
(2) In presence of surface tension, high frequencies are always *stable*;
(3) Low frequencies are stable if $|[\![c^\pm]\!]|^2 < \frac{g'H_0}{\underline{\rho}^+\underline{\rho}^-}$.

In the situation we are interested in this article, the fluid is assumed to be at rest at infinity, and the situation of §3.1 may look more relevant. However, when considering the full (nonlinear) equations, the discontinuity of the velocity must be taken into account, and the situation considered here gives a qualitative insight of the phenomena at stake in the nonlinear case. Therefore, from the three fact discussed above, we can expect the following consequences for the full two fluids equations (20),

(1) In absence of surface tension, the equations (20) are ill-posed because of the high-frequency instabilities[2];
(2) In presence of surface tension, the equations (20) are locally well posed on a very small time scale depending strongly[3] on the surface tension coefficient σ;
(3) Under an additional assumption on the low frequency behavior, one can expect a longer existence time[4].

[2]To be more precise, the equations are then elliptic in space-time.

[3]Since the components g' and $e(D)$ of $\mathrm{Ins}(D)$ are of lower order than $-\frac{\sigma}{\rho^++\rho^-}\Delta$, they can be neglected in this analysis. In particular, the sign of g' (or equivalently, the fact that the heavier fluid is below or above the lighter one) is of no importance under this approach, while this should intuitively play a strong role on the stability of the flow.

[4]In particular, the sign of g' should be relevant in this additional assumption, as it is in the third observation made above in the linear case.

The first point has been established by Iguchi-Tanaka-Tani [17] and made more precise in [18, 35]. The second point (local well posedness in presence of surface tension) has been proved for instance in [4, 5, 28, 9, 29]. The goal of this paper is to deal with the third case, and in particular to obtain an existence time that is consistent with physical observations[5].

3.3. **Quasilinearization of the equations.** The equations (20) are fully nonlinear. A typical strategy to handle such equations is to differentiate them to form a quasilinear system. For instance, the eikonal equation

$$\partial_t u + F(\nabla u) = 0$$

is fully nonlinear, but if we introduce $V = \nabla u$ and take the gradient of this equation, we obtain

$$\begin{cases} \partial_t u + F(V) = 0, \\ \partial_t V + dF(V)\cdot \nabla V = 0, \end{cases}$$

which is a quasilinear system. This strategy has been implemented in the water waves case in the Zakharov-Craig-Sulem formulation in [20, 3] (using a Nash-Moser iteration) and in [16] (see also [22]). We want to implement it here for the two fluids equations (20); compared to the standard "quasilinearization approach" described above, it is worth insisting on two aspects:

(1) The equations must be differentiated several times to obtain a closed quasilinear system;
(2) The new variables that one must introduce (the analogous of $V = \nabla u$ for the eikonal equation above) are not simply the derivatives of the original unknowns ζ and ψ but "good unknowns" that we will comment on later.

In order to motivate the introduction of the good unknowns, let us first give a differentiation formula for $\mathcal{G}[\zeta]\psi$. We denote by $\underline{U}^\pm = (\underline{V}^\pm, \underline{w}^\pm)$ the evaluation of the velocity at the interface[6] and $\underline{w} = \underline{\rho^+ w^+} - \underline{\rho^- w^-}$. The linearization formula is then a consequence of a shape derivative formula for $\mathcal{G}[\zeta]$ (Lemma 7 of [21]), which is itself a generalization of the shape derivative formula for the Dirichlet-Neumann operator proved in [20] (see also Chapter 3 of [22]); it is given by the following relation

$$\forall \alpha \in \mathbb{N}^d, \quad |\alpha| = 1, \qquad \partial^\alpha(\mathcal{G}[\zeta]\psi) = \mathcal{G}[\zeta]\psi_{(\alpha)} - \mathcal{T}[U]\partial^\alpha\zeta,$$

where $\psi_{(\alpha)} = \partial^\alpha\psi - \underline{w}\partial^\alpha\zeta$, $U = (\zeta, \psi)$, and

$$\mathcal{T}[U]f = \nabla\cdot(f\underline{V}^+) + \underline{\rho}^-\mathcal{G}[\zeta]\circ(\mathcal{G}^-[\zeta])^{-1}(\nabla\cdot(f[\![\underline{V}^\pm]\!]).$$

For all $\alpha \in \mathbb{N}^d$, $|\alpha| = 1$, one can then show after some computations that the "good unknown" $(\zeta_{(\alpha)}, \psi_{(\alpha)}) = (\partial^\alpha\zeta, \partial^\alpha\psi - \underline{w}\partial^\alpha\zeta)$ solves the system

$$\begin{cases} \partial_t\zeta_{(\alpha)} + \mathcal{T}[U]\zeta_{(\alpha)} - \mathcal{G}[\zeta]\psi_{(\alpha)} = 0, \\ \partial_t\psi_{(\alpha)} - \mathcal{T}[U]^*\psi_{(\alpha)} + \mathit{Ins}[U]\zeta = 0, \end{cases} \tag{25}$$

with the *instability operator* $\mathit{Ins}[U]$ defined as

$$\mathit{Ins}[U]f = \mathfrak{a}f - \underline{\rho}^+\underline{\rho}^-[\![\underline{V}^\pm]\!]\cdot\mathcal{E}[\zeta](f[\![V^\pm]\!]) - \frac{\sigma}{\rho^+ + \rho^-}\nabla\cdot\mathcal{K}[\nabla\zeta]\nabla f,$$

[5]The time scale obtained in the framework of the second point is typically 10^6 to 10^9 smaller than the physical one [21].

[6]Proceeding as in §2.3 one can show that these quantities depend only on ψ and ζ.

with

$$\mathfrak{a} = g' + [\![\underline{\rho}^{\pm}(\partial_t + \underline{V}^{\pm}\cdot\nabla)\underline{w}^{\pm}]\!]$$

(note that $\mathfrak{a}$ can be expressed in terms of the pressure, $\mathfrak{a} = [\![-\partial_z P^{\pm}_{|_{z=\zeta}}]\!]$) and

$$\mathcal{E}[\zeta] = \nabla\circ(\underline{\rho}^{+}\mathcal{G}^{-} - \underline{\rho}^{-}\mathcal{G}^{+})^{-1}\circ\nabla^T,$$

$$\mathcal{K}[\nabla\zeta] = \frac{(1+|\nabla\zeta|^2)\mathrm{Id}\ - \nabla\zeta\otimes\nabla\zeta}{(1+|\nabla\zeta|^2)^{3/2}}.$$

The system (25) has therefore the same structure as (24), with the following adaptations,

$$g' \rightsquigarrow \mathfrak{a},\quad T(D)\rightsquigarrow \mathcal{T}[U],\quad \mathcal{G}[0]\rightsquigarrow\mathcal{G}[\zeta],\quad [\![c^{\pm}]\!]\rightsquigarrow[\![\underline{V}^{\pm}]\!],\quad \mathrm{Ins}(D)\rightsquigarrow \mathcal{I}ns[U];$$

in particular, a non zero jump $[\![\underline{V}^{\pm}]\!]$ creates Kelvin-Helmholtz instabilities through the operator $\mathcal{E}[\zeta]$ in $\mathcal{I}ns[U]$ that make the equations ill posed in absence of surface tension. In presence of surface tension, and as for (24), high frequency are always stable, but an additional condition is needed for low frequencies. A careful study of the instability operator $\mathcal{I}ns[U]$ shows that a sufficient condition for all frequencies is the following

$$\mathfrak{a} = [\![-\partial_z P^{\pm}_{|_{z=\zeta}}]\!] > \frac{1}{4}\frac{(\rho^+\rho^-)^2}{\sigma(\rho^+ + \rho^-)^2}\mathfrak{c}(\zeta)|[\![\underline{V}^{\pm}]\!]|^4_{\infty}, \tag{26}$$

where $\mathfrak{c}(\zeta)$ is a constant depending on ζ, H^+ and H^-. We can then prove the following theorem.

Theorem 3.1. *Let $U^0 = (\zeta^0,\psi^0)^T$ be smooth enough and satisfy the non vanishing depth condition*

$$\exists h^{\pm}_{min} > 0,\qquad \inf_{X\in\mathbb{R}^d}(H^{\pm}\pm\zeta^0(X)) \geq h^{\pm}_{min}.$$

If moreover U^0 satisfies the stability criterion (26) then there exists $T>0$ and a unique solution to (16) with initial condition U^0. Moreover, T depends on the surface tension coefficient σ through $\mathfrak{d}(U^0)$ only, with

$$\mathfrak{d}(U^0) = [\![-\partial_z P^{\pm}_{|_{z=\zeta^0}}]\!] - \frac{1}{4}\frac{(\rho^+\rho^-)^2}{\sigma(\rho^+ + \rho^-)^2}\mathfrak{c}(\zeta^0)|[\![\underline{V}^{0\pm}]\!]|^4_{\infty}.$$

Remark 3.2. In the water waves case ($\rho^- = 0$), the stability criterion (26) coincides with the standard Rayleigh-Taylor criterion

$$-\partial_z P_{|_{z=\zeta}} > 0,$$

which is a well known condition to have a solution of the water waves equations in absence of surface tension [33, 34, 20, 23, 27, 11]. In presence of surface tension, the Rayleigh-Taylor criterion is not necessary to have local-well posedness of the equations of the water waves equations, but the existence time extremely small if the Rayleigh-Taylor criterion is not satisfied (it corresponds to the existence time one obtains when the water is above vacuum and not below). *The stability criterion* (26) *must therefore be understood as a two fluids generalization of the Rayleigh-Taylor criterion.*

Sketch of proof. We give here the main steps of the proof, and refer to [21] for full details.

Step 1. In order to "quasilinearize" the equations, one has to differentiate them several times (up to order $N = 5$ is enough in dimension $d = 1, 2$). The good unknowns $(\zeta_{(\alpha)}, \psi_{(\alpha)})$ mentioned above can be generalized for all $\alpha \in \mathbb{N}^d$, $|\alpha| \leq N$ by taking

$$\zeta_{(\alpha)} = \partial^\alpha \zeta, \qquad \psi_{(\alpha)} = \partial^\alpha \psi - \underline{w} \partial^\alpha \zeta;$$

they all satisfy a system of the form (25) up to a residual (R_α^1, R_α^2).
Step 2. We introduce the following higher order generalization of the energy $E_0(U)$ introduced in (23): for all $N \in \mathbb{N}$, we define $E_N(U)$ as

$$E_N(U) = |\nabla \psi|^2_{H^{t_0+2}} + \sum_{\alpha \in \mathbb{N}^d, |\alpha| \leq N} |\zeta_{(\alpha)}|^2_{H^1_\sigma} + |\psi_{(\alpha)}|^2_{\dot{H}^{1/2}_*} \tag{27}$$

(the term $|\nabla \psi|^2_{H^{t_0+2}}$, with $t_0 > d/2$ is due to technical reasons), and we also denote by $\mathfrak{m}^N(U)$ any constant of the form

$$\mathfrak{m}^N(U) = C(\frac{1}{h^{\pm}_{min}}, E_N(U)).$$

Step 3. We want to show that the big system formed by all the systems satisfied by the $(\zeta_{(\alpha)}, \psi_{(\alpha)})$ for all $|\alpha| \leq N$ is quasilinear; this means that the residual R_α^1 for the first equation is controlled by $\mathfrak{m}^N(U)$ in H^1_σ-norm, and that the residual R_α^2 is controlled by $\mathfrak{m}^N(U)$ in $|\psi|_{\dot{H}^{1/2}_*}$-norm (see §(23) for the definition of these norms). It turns out that this is not true and that, when $|\alpha| = N$, subprincipal terms cannot be put into the residual terms. This difficulty is related to the presence of the (second order) surface tension term, and several techniques have been proposed in the case of the water waves equations. We use here the technique proposed in [26] and which consists in using not only space derivatives to quasilinearize the system, but also time derivatives.
Step 4. On can then construct a solution by a standard iterative scheme. □

4. Shallow water asymptotics

We do not consider here the problem of shallow water asymptotics which consists in describing the behavior of the solutions when the shallowness parameter μ is very small, with μ defined as

$$\mu = \frac{H^2}{L^2} \quad \text{with} \quad H = \frac{H^+ H^-}{\rho^+ H^- + \rho^- H^+},$$

and L the typical order for the wavelength of the interfacial waves under consideration. Let us just mention that the main difficulty is to show that the existence time provided by Theorem 3.1 is uniform with respect to μ. Once this is done, one can derive asymptotic models as $\mu \to 0$ (see [8] for a systematic derivation of shallow water models for two fluids interfaces). Unfortunately, the limit $\mu \to 0$ is singular and proving that the existence time is uniform with respect to μ is not a straightforward adaptation of what we saw in the previous section. One of the reasons why this limit is singular is because standard symbolic analysis is itself singular. Indeed, the standard symbolic approximation of the Dirichlet-Neumann operator can be stated as

$$\mathcal{G}^+[\zeta]\psi = g_1(X, D) + R_0, \tag{28}$$

where R_0 is an operator of order zero and $g_1(X,D)$ is the pseudo-differential operator[7] of symbol

$$g_1(X,\xi)=\sqrt{|\xi|^2+\big(|\nabla\zeta|^2|\xi|^2-(\nabla\zeta\cdot\xi)^2\big)}.$$

The principal part of the operator $\mathcal{G}^+[\zeta]$ is therefore $g_1(X,D)$, and one can check that it does not depend on the bottom. The above decomposition could be generalized into

$$\mathcal{G}^+[\zeta]\psi=g_1(X,D)+g_0(X,D)+\dots+\dots g_{-k}(X,D)+R_{-k-1},$$

with $g_j(X,\xi)$ $(-k\leq j\leq 1)$ a symbol of order j and R_{-k-1} an operator of order $-k-1$. One would then check that none of the symbols $g_j(X,\xi)$ depends on the bottom. The reason for this is because the contribution of the bottom to the Dirichlet-Neumann operator is analytic by standard elliptic theory and therefore transparent to any homogeneous symbolic expansion (at any order). In the shallow water limit, where the role of the bottom topography is crucial, this is problematic.

In the water waves case, it is possible to bypass the use of the symbolic approximation of the Dirichlet-Neumann operator [3, 16], but it is not clear whether this approach can be generalized to the two fluids case. In [21], another approach has therefore been proposed. It consists in adding a "tail" to the principal symbol of the Dirichlet Neumann operator; this tail is of order $-\infty$ and accounts for the effects of the bottom on the Dirichlet-Neumann operator. More precisely, we use the decomposition

$$\mathcal{G}^+[\zeta]\psi=g(X,D)+R_0, \tag{29}$$

where R_0 is a zero order operator, while the symbol $g(X,\xi)$ is given by

$$g(X,\xi)=g_1(X,\xi)\tanh\Big((H^++\zeta)\int_{-H^+}^0\frac{|\xi|^2+(z+H^+)^2\big(|\nabla\zeta|^2|\xi|^2-(\nabla\zeta\cdot\xi)^2\big)}{1+(z+H^+)^2|\nabla\zeta|^2}dz\Big).$$

The difference between (28) and (29) is that an infinitely smoothing "tail" has been added to the symbolic description; this tail $t(X,\xi)$ is

$$t(X,\xi)=1-\tanh\Big((H^++\zeta)\int_{-H^+}^0\frac{|\xi|^2+(z+H^+)^2\big(|\nabla\zeta|^2|\xi|^2-(\nabla\zeta\cdot\xi)^2\big)}{1+(z+H^+)^2|\nabla\zeta|^2}dz\Big).$$

The symbolic description (29) now depends on the bottom, and using this symbolic approximation "with tail" removes the shallow water singularity, and it is possible to handle the shallow water limit.

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[7] We recall that a pseudo-differential operator $\sigma(X,D)$ of symbol $\sigma(x,\xi)$ is defined by

$$\sigma(x,D)=(2\pi)^{-d}\int_{\mathbb{R}^d}e^{ix\cdot\xi}\sigma(x,\xi)\widehat{u}(\xi)d\xi.$$

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DMA, ECOLE NORMALE SUPÉRIEURE DE PARIS ET CNRS UMR 8553,, 45 RUE D'ULM, 75005 PARIS, FRANCE, DAVID.LANNES@ENS.FR