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On asymptotic behavior of solutions to the compressible Navier-Stokes equation around a time-periodic parallel flow

Jan Březina
Graduate School of Mathematics, Kyushu University

1 Introduction

In this paper we study the stability of a time-periodic parallel flow to the compressible Navier-Stokes equation with time-periodic external force and time-periodic boundary conditions.

We consider the system of equations

$$\partial_{\tilde{t}}\tilde{\rho}+\mathrm{div}\,(\tilde{\rho}\tilde{v})=0, \tag{1.1}$$

$$\tilde{\rho}(\partial_{\tilde{t}}\tilde{v}+\tilde{v}\cdot\nabla\tilde{v})-\mu\Delta\tilde{v}-(\mu+\mu')\nabla\mathrm{div}\,\tilde{v}+\nabla\tilde{P}(\tilde{\rho})=\tilde{\rho}\tilde{\boldsymbol{g}}, \tag{1.2}$$

in an n dimensional infinite layer $\Omega_\ell=\mathbb{R}^{n-1}\times(0,\ell)$:

$$\Omega_\ell=\{\tilde{x}={}^T(\tilde{x}',\tilde{x}_n)\,;\tilde{x}'={}^T(\tilde{x}_1,\ldots,\tilde{x}_{n-1})\in\mathbb{R}^{n-1},\ 0<\tilde{x}_n<\ell\}.$$

Here, $n\geq 2$; $\tilde{\rho}=\tilde{\rho}(\tilde{x},\tilde{t})$ and $\tilde{v}={}^T(\tilde{v}^1(\tilde{x},\tilde{t}),\ldots,\tilde{v}^n(\tilde{x},\tilde{t}))$ denote the unknown density and velocity at time $\tilde{t}\geq 0$ and position $\tilde{x}\in\Omega_\ell$, respectively; $\tilde{P}$ is the pressure, smooth function of $\tilde{\rho}$, where for given $\rho_*>0$ we assume

$$\tilde{P}'(\rho_*)>0;$$

μ and μ' are the viscosity coefficients that are assumed to be constants satisfying $\mu>0$, $\frac{2}{n}\mu+\mu'\geq 0$; div, ∇ and Δ denote the usual divergence, gradient and Laplacian with respect to $\tilde{x}$.

In (1.2) we assume $\tilde{\boldsymbol{g}}$ to have the form

$$\tilde{\boldsymbol{g}}={}^T(\tilde{g}^1(\tilde{x}_n,\tilde{t}),0,\ldots,0,\tilde{g}^n(\tilde{x}_n)),$$

with $\tilde{g}^1$ being $\tilde{T}$-periodic function in time, where $\tilde{T}>0$. Here and in what follows ^{T} denotes transposition.

The system (1.1)–(1.2) is considered under boundary condition

$$\tilde{v}|_{\tilde{x}_n=0}=\tilde{V}^1(t)\boldsymbol{e}_1,\ \ \tilde{v}|_{\tilde{x}_n=\ell}=0, \tag{1.3}$$

and initial condition

$$(\tilde{\rho},\tilde{v})|_{\tilde{t}=0}=(\tilde{\rho}_0,\tilde{v}_0), \tag{1.4}$$

where $\tilde{V}^1$ is a $\tilde{T}$-periodic function of time. Here, $\boldsymbol{e}_1={}^T(1,0,\ldots,0)\in\mathbb{R}^n$.

Under suitable conditions on $\tilde{\boldsymbol{g}}$ and $\tilde{V}^1$, problem (1.1)–(1.3) has a smooth time-periodic solution $\bar{u}_p={}^T(\bar{\rho}_p,\bar{v}_p)$ satisfying

$$\overline{\rho}_p = \overline{\rho}_p(\widetilde{x}_n) \geq \underline{\widetilde{\rho}}, \quad \frac{1}{\ell}\int_0^\ell \overline{\rho}_p(\widetilde{x}_n)\, d\widetilde{x}_n = \rho_*,$$

$$\overline{v}_p = {}^T(\overline{v}_p^1(\widetilde{x}_n, \widetilde{t}), 0, \ldots, 0), \quad \overline{v}_p^1(\widetilde{x}_n, \widetilde{t} + \widetilde{T}) = \overline{v}_p^1(\widetilde{x}_n, \widetilde{t}),$$

for a positive constant $\underline{\widetilde{\rho}}$.

We are interested in large time behavior of solutions to problem (1.1)–(1.4) when the initial value $(\widetilde{\rho}_0, \widetilde{v}_0)$ is sufficiently close to the value of time-periodic solution $\overline{u}_p = {}^T(\overline{\rho}_p, \overline{v}_p)$ at some fixed time. We study the asymptotic behavior of these solutions with respect to the time-periodic solution $\overline{u}_p$.

In the case $\widetilde{g}^1$ and $\widetilde{V}^1$ are independent of t, problem (1.1)–(1.3) has a stationary parallel flow. The stability of stationary parallel flows were investigated in [5, 6, 7, 9].

Iooss and Padula ([5]) studied the linearized stability of a stationary parallel flow in a cylindrical domain under the perturbations periodic in the unbounded direction of the domain. It was shown that the linearized operator generates a C_0-semigroup in L^2-space on the periodic box under vanishing average condition for the density-component. In particular, if the Reynolds number is suitably small, then the semigroup decays exponentially. Furthermore, by using the Fourier series expansion, it was shown that the semigroup is decomposed into a direct sum of an analytic semigroup and an exponentially decaying C_0-semigroup, which correspond to low and high frequency parts of the semigroup, respectively. It was also proved that the essential spectrum of the linearized operator lies in the left-half plane strictly away from the imaginary axis and the part of the spectrum lying in the right-half to the line $\{\mathrm{Re}\,\lambda = -c\}$ for some number $c > 0$ consists of finite number of eigenvalues with finite multiplicities.

The stability of stationary parallel flows in the infinite layer Ω under the perturbations in some L^2-Sobolev space on Ω were studied in [6, 7, 9]. By using the Fourier transform in x', it was shown in [9] that the linearized problem generates C_0-semigroup with low frequency part behaving like $n-1$ dimensional heat kernel and the high frequency part decaying exponentially as $t \to \infty$, provided that the Reynolds and Mach numbers are sufficiently small and the density of the parallel flow is sufficiently close to a positive constant. The nonlinear problem was then studied in [6, 7]; it was shown that the stationary parallel flow is asymptotically stable under sufficiently small initial perturbations in $(H^m \cap L^1)(\Omega)$ with $m \geq [n/2] + 1$. Furthermore, the asymptotic behavior of perturbations from the stationary parallel flow is described by $n-1$ dimensional linear heat equation in the case $n \geq 3$ ([6]) and by one-dimensional viscous Burgers equation in the case $n = 2$ ([7]).

Whereas [9] are concerned with the stability of the stationary parallel flows, in [4] the diffusive stability of oscillations in reaction-diffusion systems is treated. A similar asymptotic state arises in the large time dynamics around spatially homogeneous oscillations in reaction-diffusion systems ([4]).

Result presented in this paper is an extension of previous results on the stationary case [6, 7, 9] to the case of time-periodic external force and time-periodic boundary conditions.

Problem (1.1)–(1.4) with $\widetilde{\boldsymbol{g}} = (\widetilde{g}^1(x_n, t), 0, \ldots, 0, \widetilde{g}^n(x_n))$ and $\widetilde{V}^1(t)$ covers particularly interesting problem. Let us for a moment consider problem (1.1)–(1.4) together with $\widetilde{\boldsymbol{g}} = (0, \ldots, 0, \widetilde{g}^n(x_n))$ and $\widetilde{V}^1(t)$. This problem is a natural extension of Stokes' second problem from half space to infinite strip for compressible fluid. The motion of a fluid is caused by the periodic oscillation of the lower boundary plate. The study of the flow of a viscous fluid over an oscillating plate is not only of theoretical interest, but it also occurs in many applied problems and since Stokes (1851) it has received much attention under various settings.

This paper is organized as follows. In the rest of Section 1 we present the existence of the time-periodic parallel flow $\overline{u}_p$, introduce the governing equations of the perturbations from $\overline{u}_p$ and nondimensional form of these equations. At the end we show some properties of the underlying nondimensional parallel flow u_p. In Section 2 we focus on the linear problem, i.e., we neglect nonlinearities. We introduce spectral properties of the solution operator for the linear problem and later develop a Floquet theory for certain part of the solution. Finally, in Section 3 we introduce the results on the nonlinear problem.

1.1 Existence of parallel flows

Let us state the conditions, under which the time-peridic parallel flow $\overline{u}_p = {}^T(\overline{\rho}_p, \overline{v}_p)$ exists.

Substituting $(\widetilde{\rho}, \widetilde{v}) = (\overline{\rho}_p(\widetilde{x}_n), \overline{v}_p^1(\widetilde{x}_n, \widetilde{t})e_1)$ into (1.1)–(1.3), we have

$$\partial_{\widetilde{t}}\overline{v}_p^1 - \frac{\mu}{\overline{\rho}_p}\partial_{\widetilde{x}_n}^2 \overline{v}_p^1 = \widetilde{g}^1, \tag{1.5}$$

$$\partial_{\widetilde{x}_n}(\widetilde{P}(\overline{\rho}_p)) = \overline{\rho}_p \widetilde{g}^n, \tag{1.6}$$

$$\overline{v}_p^1|_{\widetilde{x}_n=0} = \widetilde{V}^1(t), \quad \overline{v}_p^1|_{\widetilde{x}_n=\ell} = 0. \tag{1.7}$$

Let ρ_* be the given positive number, recall that

$$\widetilde{P}'(\rho_*) > 0.$$

We state the existence of a time-periodic solution to (1.5)–(1.7) with

$$\rho_* = \frac{1}{\ell}\int_0^\ell \overline{\rho}_p(\widetilde{x}_n)\, d\widetilde{x}_n. \tag{1.8}$$

Lemma 1.1 *Assume that $\widetilde{P}'(\widetilde{\rho}) > 0$ for $\rho_1 \le \widetilde{\rho} \le \rho_2$ with some $0 < \rho_1 < \rho_* < \rho_2 < 2\rho_*$. Let $\Phi(\widetilde{\rho}) = \int_{\rho_*}^{\rho} \frac{\widetilde{P}'(\eta)}{\eta}\, d\eta$ for $\rho_1 \le \widetilde{\rho} \le \rho_2$ and let $\Psi(r) = \Phi^{-1}(r)$ for $r_1 \le r \le r_2$. Here Φ^{-1} denotes the inverse function of Φ and $r_j = \Phi(\rho_j)$ $(j = 1, 2)$. If*

$$|\widetilde{g}^n|_{C([0,\ell])} \le C \min\left\{ |r_1|, r_2, \frac{\rho_*}{4\widetilde{P}'(\rho_*)|\Psi''|_{C([r_1,r_2])}} \right\} \le C,$$

then there exists a smooth time-periodic solution $(\overline{\rho}_p, \overline{v}_p) = (\overline{\rho}_p(\widetilde{x}_n), \overline{v}_p^1(\widetilde{x}_n, \widetilde{t})e_1)$ of (1.5)–(1.8) satisfying

$$\rho_1 \le \overline{\rho}_p(\widetilde{x}_n) \le \rho_2, \quad |\overline{\rho}_p - \rho_*|_{C([0,\ell])} \le C\frac{\rho_*\ell}{\widetilde{P}'(\rho_*)}|\widetilde{g}^n|_{C([0,\ell])},$$

$$\overline{v}_p^1(\widetilde{x}_n, \widetilde{t}) = \frac{1}{\ell}(\ell - \widetilde{x}_n)\widetilde{V}^1(\widetilde{t}) + \int_{-\infty}^{\widetilde{t}} e^{-\mu\widetilde{A}(\widetilde{t}-z)}\{\widetilde{g}^1(\widetilde{x}_n, z) - \frac{1}{\ell}(\ell - \widetilde{x}_n)\partial_z\widetilde{V}^1(z)\}\, dz,$$

where $\widetilde{A}$ denotes the uniformly elliptic operator on $L^2(0,\ell)$ with domain $D(\widetilde{A}) = (H^2 \cap H_0^1)(0,\ell)$ and $\widetilde{A}v = -\frac{1}{\overline{\rho}_p(x_n)}\partial_{x_n}^2 v$ for $v \in D(\widetilde{A})$.

1.2 Equations of perturbation

As the next step we linearize (1.1)–(1.4) around the parallel flow $\overline{u}_p = {}^T(\overline{\rho}_p, \overline{v}_p)$.

Setting $\widetilde{\rho} = \overline{\rho}_p + \widetilde{\phi}$ and $\widetilde{v} = \overline{v}_p + \widetilde{w}$ in (1.1)–(1.4) we obtain the following governing equations for the perturbation $(\widetilde{\phi}, \widetilde{w})$:

$$\partial_{\widetilde{t}}\widetilde{\phi} + \overline{v}_p^1 \partial_{\widetilde{x}_1}\widetilde{\phi} + \operatorname{div}(\overline{\rho}_p \widetilde{w}) = \widetilde{f}^0, \tag{1.9}$$

$$\begin{aligned}\partial_{\widetilde{t}}\widetilde{w} - \frac{\mu}{\overline{\rho}_p}\Delta\widetilde{w} - \frac{\mu+\mu'}{\overline{\rho}_p}\nabla\operatorname{div}\widetilde{w} + \overline{v}_p^1\partial_{\widetilde{x}_1}\widetilde{w} + (\partial_{\widetilde{x}_n}\overline{v}_p^1)\widetilde{w}^n \boldsymbol{e}_1 \\ + \frac{\mu}{\overline{\rho}_p^2}(\partial_{\widetilde{x}_n}^2 \overline{v}_p^1)\widetilde{\phi}\boldsymbol{e}_1 + \nabla\left(\frac{\widetilde{P}'(\overline{\rho}_p)}{\overline{\rho}_p}\widetilde{\phi}\right) = \widetilde{\boldsymbol{f}},\end{aligned} \tag{1.10}$$

$$\widetilde{w}|_{\partial\Omega_\ell} = 0, \tag{1.11}$$

$$(\widetilde{\phi}, \widetilde{w})|_{t=0} = (\widetilde{\phi}_0, \widetilde{w}_0), \tag{1.12}$$

where $\widetilde{f}^0$ and $\widetilde{\boldsymbol{f}} = {}^T(\widetilde{f}^1, \cdots, \widetilde{f}^n)$, denote the nonlinearities:

$$\widetilde{f}^0 = -\operatorname{div}(\widetilde{\phi}\widetilde{w}),$$

$$\begin{aligned}\widetilde{\boldsymbol{f}} &= -\widetilde{w}\cdot\nabla\widetilde{w} + \tfrac{\mu\widetilde{\phi}}{(\widetilde{\phi}+\overline{\rho}_p)\overline{\rho}_p}\left(-\Delta\widetilde{w} + \left(\tfrac{1}{\overline{\rho}_p}\Delta\overline{v}_p\right)\widetilde{\phi}\right) - \tfrac{(\mu+\mu')\widetilde{\phi}}{(\widetilde{\phi}+\overline{\rho}_p)\overline{\rho}_p}\nabla\operatorname{div}\widetilde{w} \\ &\quad + \tfrac{\widetilde{\phi}}{\overline{\rho}_p}\nabla\left(\tfrac{\widetilde{P}'(\overline{\rho}_p)}{\overline{\rho}_p}\widetilde{\phi}\right) - \tfrac{1}{2\overline{\rho}_p}\nabla\left(\widetilde{P}''(\overline{\rho}_p)\widetilde{\phi}^2\right) + \widetilde{P}^3(\overline{\rho}_p, \widetilde{\phi}, \partial_{\widetilde{x}}\widetilde{\phi}),\end{aligned}$$

where

$$\begin{aligned}\widetilde{P}^3 &= \tfrac{\widetilde{\phi}^3}{(\widetilde{\phi}+\overline{\rho}_p)\overline{\rho}_p^3}\nabla\widetilde{P}(\overline{\rho}_p) - \tfrac{1}{2(\widetilde{\phi}+\overline{\rho}_p)}\nabla\left(\widetilde{\phi}^3\widetilde{P}_3(\overline{\rho}_p, \widetilde{\phi})\right) + \tfrac{\widetilde{\phi}}{2\overline{\rho}_p^2}\nabla(\widetilde{P}''(\overline{\rho}_p)\widetilde{\phi}^2) \\ &\quad - \tfrac{\widetilde{\phi}^2}{(\widetilde{\phi}+\overline{\rho}_p)\overline{\rho}_p^2}\nabla\left(\widetilde{P}'(\overline{\rho}_p)\widetilde{\phi} + \tfrac{1}{2}\widetilde{P}''(\overline{\rho}_p)\widetilde{\phi}^2\right)\end{aligned}$$

with

$$\widetilde{P}_3(\overline{\rho}_p, \widetilde{\phi}) = \int_0^1 (1-\theta)^2 \widetilde{P}'''(\theta\widetilde{\phi} + \overline{\rho}_p)\, d\theta.$$

1.3 Governing equations for dimensionless problem

Now, we introduce dimensionless variables and scale (1.9)–(1.12) to nondimensional form.

We use the following dimensionless variables:

$$\widetilde{x} = \ell x, \ \ \widetilde{t} = \frac{\ell}{V}t, \ \ \widetilde{w} = Vw, \ \ \widetilde{\phi} = \rho_*\phi, \ \ \widetilde{P} = \rho_* V^2 P,$$

with

$$\widetilde{v} = Vv, \ \ \widetilde{\rho} = \rho_*\rho, \ \ \widetilde{V}^1 = VV^1, \ \ \widetilde{\boldsymbol{g}} = \frac{\mu V}{\rho_*\ell^2}\boldsymbol{g},$$

where

$$\gamma = \frac{\sqrt{\widetilde{P}'(\rho_*)}}{V}, \quad V = \frac{\rho_* \ell^2}{\mu}\left\{|\partial_{\widetilde{t}}\widetilde{V}^1|_{C(\mathbb{R})} + |\widetilde{g}^1|_{C(\mathbb{R}\times[0,\ell])}\right\} + |\widetilde{V}^1|_{C(\mathbb{R})} > 0.$$

In this paper we assume $V > 0$. Under this change of variables the domain Ω_ℓ is transformed into $\Omega = \mathbb{R}^{n-1} \times (0,1)$; and $g^1(x_n, t)$, $V^1(t)$ are periodic in t with period $T > 0$ defined as

$$T = \frac{V}{\ell}\widetilde{T}.$$

The time-periodic solution $\overline{u}_p$ is transformed into $u_p = {}^T(\rho_p, v_p)$ satisfying

$$\rho_p = \rho_p(x_n) > 0, \quad \int_0^1 \rho_p(x_n)\, dx_n = 1,$$

$$v_p = {}^T(v_p^1(x_n,t), 0, \dots, 0), \quad v_p^1(x_n, t+T) = v_p^1(x_n, t).$$

It then follows that the perturbation

$$u(t) = {}^T(\phi(t), w(t)) \equiv {}^T(\gamma^2(\rho(t) - \rho_p), v(t) - v_p(t)),$$

is governed by the following system of equations

$$\partial_t \phi + v_p^1 \partial_{x_1}\phi + \gamma^2 \operatorname{div}(\rho_p w) = f^0, \tag{1.13}$$

$$\begin{aligned}\partial_t w - \frac{\nu}{\rho_p}\Delta w - \frac{\widetilde{\nu}}{\rho_p}\nabla \operatorname{div} w + v_p^1 \partial_{x_1} w + (\partial_{x_n} v_p^1) w^n \boldsymbol{e}_1 \\ + \frac{\nu}{\gamma^2 \rho_p^2}(\partial_{x_n}^2 v_p^1)\phi \boldsymbol{e}_1 + \nabla\left(\frac{P'(\rho_p)}{\gamma^2 \rho_p}\phi\right) = \boldsymbol{f},\end{aligned} \tag{1.14}$$

$$w|_{\partial\Omega} = 0, \tag{1.15}$$

$$(\phi, w)|_{t=0} = (\phi_0, w_0), \tag{1.16}$$

where f^0 and $\boldsymbol{f} = {}^T(f^1, \cdots, f^n)$ denote nonlinearities, i.e.,

$$f^0 = -\operatorname{div}(\phi w),$$

$$\begin{aligned}\boldsymbol{f} = &-w\cdot\nabla w + \frac{\nu\phi}{\gamma^2\rho_p^2}\left(-\Delta w + \frac{\partial_{x_n}^2 v_p^1}{\rho_p \gamma^2}\phi \boldsymbol{e}_1\right) - \frac{\nu\phi^2}{\gamma^2\rho_p^2(\gamma^2\rho_p + \phi)}\left(-\Delta w + \frac{\partial_{x_n}^2 v_p^1}{\rho_p\gamma^2}\phi \boldsymbol{e}_1\right) \\ &- \frac{\widetilde{\nu}\phi}{\rho_p(\gamma^2\rho_p+\phi)}\nabla \operatorname{div} w + \frac{\phi}{\gamma^2\rho_p}\nabla\left(\frac{P'(\rho_p)}{\gamma^2\rho_p}\phi\right) - \frac{1}{2\gamma^4\rho_p}\nabla(P''(\rho_p)\phi^2) + \widetilde{P}_3(\rho_p, \phi, \partial_x\phi),\end{aligned}$$

$$\begin{aligned}\widetilde{P}_3(\rho_p, \phi, \partial_x\phi) &= \frac{\phi^3}{\gamma^4(\gamma^2\rho_p+\phi)\rho_p^3}\nabla P(\rho_p) + \frac{\phi\nabla(P''(\rho_p)\phi^2)}{2\gamma^4\rho_p(\gamma^2\rho_p+\phi)} \\ &\quad - \frac{\phi^2\nabla(P'(\rho_p)\phi)}{\gamma^4\rho_p^2(\gamma^2\rho_p+\phi)} - \frac{1}{2\gamma^4(\gamma^2\rho_p+\phi)}\nabla(\phi^3 P_3(\rho_p, \phi)),\end{aligned}$$

with

$$P_3(\rho_p, \phi) = \int_0^1 (1-\theta)^2 P'''(\theta\gamma^{-2}\phi + \rho_p)d\theta.$$

Here, div , ∇ and Δ denote the usual divergence, gradient and Laplacian with respect to x; ν, ν' and $\widetilde{\nu}$ are the non-dimensional parameters:

$$\nu = \frac{\mu}{\rho_* \ell V}, \quad \nu' = \frac{\mu'}{\rho_* \ell V}, \quad \widetilde{\nu} = \nu + \nu'.$$

In the rest of this paper we study the asymptotic behavior of $u(t) = {}^T(\phi(t), w(t))$ solution of (1.13)–(1.16).

Remark 1.2 We note that the Reynolds number Re and Mach number Ma are given by

$$Re = \nu^{-1} \text{ and } Ma = \gamma^{-1},$$

respectively.

1.4 Properties of the dimensionless parallel flow u_p

As the last step in this section, we show some regularity properties of $u_p(x_n, t)$.

It is straightforward to calculate that (ρ_p, v_p^1) solve the following equations:

$$\partial_t v_p^1 - \frac{\nu}{\rho_p}\partial_{x_n}^2 v_p^1 = \nu g^1, \tag{1.17}$$

$$\partial_{x_n}(P(\rho_p)) = \nu\rho_p g^n, \tag{1.18}$$

$$v_p^1|_{x_n=0} = V^1(t), \quad v_p^1|_{x_n=1} = 0, \tag{1.19}$$

$$1 = \int_0^1 \rho_p(x_n)\, dx_n, \tag{1.20}$$

Therefore, we can rewrite Lemma 1.1 as follows.

Lemma 1.3 *Assume that $P'(\rho) > 0$ for $\rho_1 \le \rho \le \rho_2$ with some $0 < \rho_1 < 1 < \rho_2 < 2$. Let $\Phi(\rho) = \int_1^\rho \frac{P'(\eta)}{\eta}\, d\eta$ for $\rho_1 \le \rho \le \rho_2$ and let $\Psi(r) = \Phi^{-1}(r)$ for $r_1 \le r \le r_2$. Here Φ^{-1} denotes the inverse function of Φ and $r_j = \Phi(\rho_j)$ $(j = 1, 2)$. If*

$$\nu|g^n|_{C([0,1])} \le C \min\left\{|r_1|, r_2, \frac{1}{4\gamma^2|\Psi''|_{C([r_1,r_2])}}\right\} \le C,$$

then there exists a smooth time-periodic solution $(\rho_p, v_p) = (\rho_p(x_n), v_p^1(x_n, t)e_1)$ of (1.17)–(1.20) satisfying

$$\rho_1 \le \rho_p(x_n) \le \rho_2, \quad |\rho_p - 1|_\infty \le C\frac{\nu}{\gamma^2}|g^n|_{C([0,1])},$$

$$v_p^1(x_n, t) = (1 - x_n)V^1(t) + \int_{-\infty}^t e^{-\nu A(t-s)}\{\nu g^1(x_n, s) - (1 - x_n)\partial_s V^1(s)\}\, ds,$$

where A denotes the uniformly elliptic operator on $L^2(0,1)$ with domain $D(A) = (H^2 \cap H_0^1)(0,1)$ and

$$Av = -\frac{1}{\rho_p(x_n)}\partial_{x_n}^2 v, \tag{1.21}$$

for $v \in D(A)$. Additionally, if $\nu|g^n|_{C^{k-1}([0,1])} \leq \eta$, then

$$|\partial_{x_n}^k \rho_p|_{C([0,1])} \leq C_k \nu |g^n|_{C^{k-1}([0,1])} \textit{ for } k = 1, 2, \ldots$$

Here, C_k are positive constants depending on $k, \eta, |\Psi|_{C^k([r_1,r_2])}$, ρ_2 and being independent of ν and γ. In particular,

$$|\partial_{x_n}\rho_p|_{C([0,1])} \leq C\frac{\nu}{\gamma^2}|g^n|_{C([0,1])},$$

$$|P'(\rho_p) - \gamma^2|_{C([0,1])} \leq C|P''|_{C([\rho_1,\rho_2])}\frac{\nu}{\gamma^2}|g^n|_{C([0,1])}.$$

Next, let us introduce some higher regularity assumptions.

Assumptions 1.4 For a given integer $m \geq 2$ assume that $\widetilde{g} = {}^T(\widetilde{g}^1(\widetilde{x}_n, \widetilde{t}), 0, \ldots, 0, \widetilde{g}^n(\widetilde{x}_n))$ and $\widetilde{V}^1(\widetilde{t})$ belong to the following spaces:

$$\widetilde{g}^1 \in \bigcap_{j=0}^{[\frac{m}{2}]} C^j(\mathbb{R}; H^{m-2j}(0,\ell)), \quad \widetilde{g}^n \in C^m([0,\ell]),$$

$$\widetilde{V}^1 \in C^{[\frac{m+1}{2}]}(\mathbb{R}).$$

Furthermore, assume

$$\widetilde{P}(\cdot) \in C^{m+1}(\mathbb{R}).$$

It is straightforward to see that under Assumptions 1.4 dimensionless quantities g and V^1 belong to similar spaces as $\widetilde{g}$ and $\widetilde{V}^1$.

The following lemma shows higher regularity of the time-periodic parallel flow u_p under Assumptions 1.4.

Lemma 1.5 *Let Assumptions 1.4 hold true for some $m \geq 2$. There exists $\delta_0 > 0$ such that if*

$$\nu|g^n|_{C^m([0,1])} \leq \delta_0,$$

then the following assertions hold true. The time-periodic solution $u_p = {}^T(\rho_p(x_n), v_p(x_n, t))$ of (1.17)–(1.20) given by Lemma 1.3 satisfies

$$v_p \in \bigcap_{j=0}^{[\frac{m+2}{2}]} C^j(\mathbb{R}; H^{m+2-2j}(0,1)), \quad \rho_p \in C^{m+1}([0,1]),$$

and

$$0 < \underline{\rho} \le \rho_p(x_n) \le \overline{\rho}, \int_0^1 \rho_p(x_n)dx_n = 1, \; v_p(x_n, t) = {}^T(v_p^1(x_n, t), 0),$$

with

$$P'(\rho) > 0 \textit{ for } \underline{\rho} \le \rho \le \overline{\rho},$$

$$|1 - \rho_p|_{C^{k+1}([0,1])} \le \frac{C}{\gamma^2}\nu(|P''|_{C^{k-1}([\underline{\rho},\overline{\rho}])} + |g^n|_{C^k([0,1])}), \quad k = 1, \dots, m,$$

$$|P'(\rho_p) - \gamma^2|_{C([0,1])} \le \frac{C}{\gamma^2}\nu|g^n|_{C([0,1])},$$

for some constants $0 < \underline{\rho} < 1 < \overline{\rho}$.

Proofs of Lemmas 1.1, 1.3 and 1.5 can be found in [1].

2 Linear problem

Let us write (1.13)–(1.16) in the form

$$\begin{aligned} &\partial_t u + L(t)u = \boldsymbol{F}, \\ &w|_{\partial\Omega} = 0, \; u|_{t=0} = u_0. \end{aligned} \tag{2.1}$$

Here, $u = {}^T(\phi, w)$; $\boldsymbol{F} = {}^T(f^0, \boldsymbol{f})$ with $\boldsymbol{f} = {}^T(f^1, \cdots, f^n)$ is the nonlinearity; and $L(t)$ is operator of the form

$$L(t) = \begin{pmatrix} v_p^1(t)\partial_{x_1} & \gamma^2 \mathrm{div}\,(\rho_p \,\cdot\,) \\ \nabla\left(\frac{P'(\rho_p)}{\gamma^2\rho_p}\,\cdot\right) & -\frac{\nu}{\rho_p}\Delta I_n - \frac{\tilde{\nu}}{\rho_p}\nabla \mathrm{div} \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & 0 \\ \frac{\nu}{\gamma^2\rho_p^2}\partial_{x_n}^2 v_p^1(t)\boldsymbol{e}_1 & v_p^1(t)\partial_{x_1}I_n + (\partial_{x_n}v_p^1(t))\boldsymbol{e}_1{}^T\boldsymbol{e}_n \end{pmatrix}.$$

Here, $\boldsymbol{e}_n = {}^T(0, \dots, 0, 1) \in \mathbb{R}^n$. Note that $L(t)$ satisfies $L(t) = L(t+T)$.

In this section we discuss the spectral properties of the linearized problem, i.e., (2.1) with $\boldsymbol{F} = 0$. These results were established in [1, 2] and we omit their proofs here. The nonlinear problem (2.1) is treated in Section 3.

2.1 Spectral properties of the linear problem

Now, let us consider the linear problem

$$\partial_t u + L(t)u = 0, \; t > s, \; w|_{\partial\Omega} = 0, \; u|_{t=s} = u_0. \tag{2.2}$$

We introduce space Z_s defined by

$$Z_s = \{u = {}^T(\phi, w); \phi \in C_{loc}([s,\infty); H^1(\Omega)),$$
$$\partial_{x'}^{\alpha'} w \in C_{loc}([s,\infty); L^2(\Omega)) \cap L^2_{loc}([s,\infty); H^1_0(\Omega))\ (|\alpha'| \le 1),$$
$$w \in C_{loc}((s,\infty); H^1_0(\Omega))\}.$$

In [1] it was shown that for any initial data $u_0 = {}^T(\phi_0, w_0)$ satisfying $u_0 \in (H^1 \cap L^2)(\Omega)$ with $\partial_{x'} w_0 \in L^2(\Omega)$ there exists a unique solution $u(t)$ of the linear problem (2.2) in Z_s. We denote $\mathscr{U}(t,s)$ the solution operator for (2.2) given by

$$u(t) = \mathscr{U}(t,s)u_0.$$

We study the spectral properties of the solution operator $\mathscr{U}(t,s)$. To do so, we consider the Fourier transform of (2.2). We thus obtain

$$\frac{d}{dt}\widehat{u} + \widehat{L}_{\xi'}(t)\widehat{u} = 0,\ t > s,\ \ \widehat{u}|_{t=s} = \widehat{u}_0. \tag{2.3}$$

Here, $\widehat{\phi} = \widehat{\phi}(\xi', x_n, t)$ and $\widehat{w} = \widehat{w}(\xi', x_n, t)$ are the Fourier transforms of $\phi = \phi(x', x_n, t)$ and $w = w(x', x_n, t)$ in $x' \in \mathbb{R}^{n-1}$ with $\xi' \in \mathbb{R}^{n-1}$ being the dual variable; $\widehat{L}_{\xi'}(t)$ is an operator on $X_0 \equiv (H^1 \times L^2)(0,1)$ with domain $D(\widehat{L}_{\xi'}(t)) = H^1(0,1) \times (H^2 \cap H^1_0)(0,1)$, which takes the form

$$\widehat{L}_{\xi'}(t) = \begin{pmatrix} i\xi_1 v_p^1(t) & i\gamma^2 \rho_p {}^T\xi' & \gamma^2 \partial_{x_n}(\rho_p\,\cdot\,) \\ i\xi' \frac{P'(\rho_p)}{\gamma^2\rho_p} & \frac{\nu}{\rho_p}(|\xi'|^2 - \partial_{x_n}^2) I_{n-1} + \frac{\tilde{\nu}}{\rho_p}\xi'^T\xi' & -i\frac{\tilde{\nu}}{\rho_p}\xi'\partial_{x_n} \\ \partial_{x_n}\left(\frac{P'(\rho_p)}{\gamma^2\rho_p}\,\cdot\right) & -i\frac{\tilde{\nu}}{\rho_p}{}^T\xi'\partial_{x_n} & \frac{\nu}{\rho_p}(|\xi'|^2 - \partial_{x_n}^2) - \frac{\tilde{\nu}}{\rho_p}\partial_{x_n}^2 \end{pmatrix}$$
$$+ \begin{pmatrix} 0 & 0 & 0 \\ \frac{\nu}{\gamma^2\rho_p^2}(\partial_{x_n}^2 v_p^1(t)) e_1' & i\xi_1 v_p^1(t) I_{n-1} & \partial_{x_n}(v_p^1(t)) e_1' \\ 0 & 0 & i\xi_1 v_p^1(t) \end{pmatrix}.$$

Here, $e_1' = {}^T(1,0,\ldots,0) \in \mathbb{R}^{n-1}$. Let us note that $\widehat{L}_{\xi'}(t)$ is sectorial uniformly with respect to $t \in \mathbb{R}$ for each $\xi' \in \mathbb{R}^{n-1}$. As for the evolution operator $\widehat{U}_{\xi'}(t,s)$ for (2.3) we have the following results.

Lemma 2.1 *For each $\xi' \in \mathbb{R}^{n-1}$ and for all $t \ge s$ there exists unique evolution operator $\widehat{U}_{\xi'}(t,s)$ for (2.3) that satisfies*

$$|\widehat{L}_{\xi'}(t)\widehat{U}_{\xi'}(t,s)|_{L(X_0)} \le C_{t_1 t_2},\ t_1 \le s < t \le t_2.$$

Furthermore, for $u_0 \in X_0$, $f \in C^\alpha([s,\infty); X_0), \alpha \in (0,1]$ there exists unique classical solution u of inhomogeneous problem

$$\frac{d}{dt}u + \widehat{L}_{\xi'}(t)u = f,\ t > s,\ \ u|_{t=s} = u_0,$$

satisfying $u \in C_{loc}([s,\infty);X_0)\cap C^1(s,\infty;X_0)\cap C(s,\infty;(H^1\times(H^2\cap H_0^1))(0,1))$*; and the solution* u *is given by*

$$u(t)=(\phi(t),w(t))=\widehat{U}_{\xi'}(t,s)u_0+\int_s^t \widehat{U}_{\xi'}(t,z)f(z)dz.$$

The solution operator $\mathscr{U}(t,s)$ satisfies

$$\mathscr{U}(t,s)u_0=\mathscr{F}^{-1}\{\widehat{U}_{\xi'}(t,s)\widehat{u}_0\},$$

for $u_0\in(H^1\cap L^2)(\Omega)$ with $\partial_{x'}w_0\in L^2(\Omega)$.

Definition 2.2 *For* $u_j={}^T(\phi_j,w_j)\in L^2(0,1)$ *with* $w_j={}^T(w_j^1,\ldots,w_j^n)$ $(j=1,2)$*, we define a weighted inner product* $\langle u_1,u_2\rangle$ *by*

$$\langle u_1,u_2\rangle=\int_0^1 \phi_1\overline{\phi}_2\frac{P'(\rho_p)}{\gamma^4\rho_p}\,dx_n+\int_0^1 w_1\overline{w}_2\rho_p\,dx_n.$$

Here, $\overline{g}$ *denotes the complex conjugate of* g.

Next, let us introduce *adjoint problem* to

$$\partial_t u+\widehat{L}_{\xi'}(t)u=0,\ t>s,\ \ u|_{t=s}=u_0.$$

Lemma 2.3 *For each* $\xi'\in\mathbb{R}^{n-1}$ *and for all* $s\le t$ *there exists unique evolution operator* $\widehat{U}^*_{\xi'}(s,t)$ *for adjoint problem*

$$-\partial_s u+\widehat{L}^*_{\xi'}(s)u=0,\ s<t,\ u|_{s=t}=u_0,$$

on X_0*. Here,* $\widehat{L}^*_{\xi'}(s)$ *is an operator on* X_0 *with domain* $D(\widehat{L}^*_{\xi'}(s))=(H^1\times(H^2\cap H_0^1))(0,1)$*, which takes the form*

$$\widehat{L}^*_{\xi'}(s)=\begin{pmatrix} -i\xi_1 v_p^1(s) & -i\gamma^2\rho_p{}^T\xi' & -\gamma^2\partial_{x_n}(\rho_p\,\cdot\,) \\ -i\xi'\frac{P'(\rho_p)}{\gamma^2\rho_p} & \frac{\nu}{\rho_p}(|\xi'|^2-\partial_{x_n}^2)I_{n-1}+\frac{\tilde{\nu}}{\rho_p}\xi'{}^T\xi' & -i\frac{\tilde{\nu}}{\rho_p}\xi'\partial_{x_n} \\ -\partial_{x_n}\left(\frac{P'(\rho_p)}{\gamma^2\rho_p}\,\cdot\right) & -i\frac{\tilde{\nu}}{\rho_p}{}^T\xi'\partial_{x_n} & \frac{\nu}{\rho_p}(|\xi'|^2-\partial_{x_n}^2)-\frac{\tilde{\nu}}{\rho_p}\partial_{x_n}^2 \end{pmatrix}$$

$$+\begin{pmatrix} 0 & \frac{\nu\gamma^2}{P'(\rho_p)}(\partial_{x_n}^2 v_p^1(s))^T e_1' & 0 \\ 0 & -i\xi_1 v_p^1(s)I_{n-1} & 0 \\ 0 & \partial_{x_n}(v_p^1(s))^T e_1' & -i\xi_1 v_p^1(s) \end{pmatrix}.$$

Moreover, $\widehat{L}^*_{\xi'}(s)$ *satisfies* $\langle\widehat{L}_{\xi'}(s)u,v\rangle=\langle u,\widehat{L}^*_{\xi'}(s)v\rangle$ *for* $s\in\mathbb{R}$ *and* $u,v\in(H^1\times(H^2\cap H_0^1))(0,1)$ *and*

$$|\widehat{L}^*_{\xi'}(s)\widehat{U}^*_{\xi'}(s,t)|_{L(X_0)}\le C_{t_1t_2},\ t_1\le s<t\le t_2.$$

Furthermore, for $u_0\in X_0$*,* $f\in C^\alpha((-\infty,t];X_0),\alpha\in(0,1]$ *there exists unique classical solution* u *of inhomogeneous problem*

$$-\partial_s u + \widehat{L}^*_{\xi'}(s)u = f,\ s < t,\ \ u|_{s=t} = u_0,$$

satisfying $u \in C_{loc}((-\infty, t]; X_0) \cap C^1(-\infty, t; X_0) \cap C(-\infty, t; (H^1 \times (H^2 \cap H^1_0))(0,1))$*; and the solution* u *is given by*

$$u(s) = (\phi(s), w(s)) = \widehat{U}^*_{\xi'}(s,t)u_0 + \int_s^t \widehat{U}^*_{\xi'}(s,z)f(z)dz.$$

Note that $\widehat{U}_{\xi'}(t,s)$ and $\widehat{U}^*_{\xi'}(s,t)$ are defined for all $t \geq s$ and

$$\widehat{U}_{\xi'}(t+T, s+T) = \widehat{U}_{\xi'}(t,s),\ \widehat{U}^*_{\xi'}(s+T, t+T) = \widehat{U}^*_{\xi'}(s,t).$$

The operator $\widehat{U}_{\xi'}(t,s)$ has different characters between cases $|\xi'| \ll 1$ and $|\xi'| \gg 1$. We thus decompose the solution operator $\mathscr{U}(t,s)$ associated with (2.2) into three parts:

$$\mathscr{U}(t,s) = \mathscr{F}^{-1}\left(\widehat{U}_{\xi'}(t,s)|_{|\xi'|\leq r}\right) + \mathscr{F}^{-1}\left(\widehat{U}_{\xi'}(t,s)|_{r\leq|\xi'|\leq R}\right) + \mathscr{F}^{-1}\left(\widehat{U}_{\xi'}(t,s)|_{|\xi'|\geq R}\right),$$

for $0 < r \ll 1 \ll R$, where $\mathscr{F}^{-1}$ denotes the inverse Fourier transform. Let us first discuss

$$\mathscr{U}_0(t,s) = \mathscr{F}^{-1}\left(\widehat{U}_{\xi'}(t,s)|_{|\xi'|\leq r}\right).$$

Since $\widehat{L}_{\xi'}(t)$ is T-time periodic, we have for $t - s \geq 2T$ that

$$\widehat{U}_{\xi'}(t,s) = \widehat{U}_{\xi'}(t, t-\tau_1)\widehat{U}^m_{\xi'}(T,0)\widehat{U}_{\xi'}(s+\tau_1, s),$$

where $\tau_1, \tau_2 \in [0,T)$ and $t - s = \tau_1 + mT + \tau_2$. Thus the spectrum of $\widehat{U}_{\xi'}(T,0)$ plays an important role in the study of large time behavior. We investigate the operator $\widehat{U}_{\xi'}(T,0)$ for $|\xi'| \leq r \ll 1$ as in [4] and we regard $\widehat{U}_{\xi'}(T,0)$ as a perturbation from $\widehat{U}_0(T,0) = \widehat{U}_{\xi'}(T,0)|_{\xi'=0}$.

Lemma 2.4 *There exist positive numbers* ν_0 *and* γ_0 *such that if* $\nu \geq \nu_0$ *and* $\gamma^2/(\nu + \widetilde{\nu}) \geq \gamma_0^2$ *then there exists* $r_0 > 0$ *such that for each* ξ' *with* $|\xi'| \leq r_0$ *there hold the following statements.*

(i) The spectrum of operator $\widehat{U}_{\xi'}(T,0)$ *on* $(H^1 \times H^1_0)(0,1)$ *satisfies*

$$\sigma(\widehat{U}_{\xi'}(T,0)) \subset \{\mu_{\xi'}\} \cup \{\mu : |\mu| \leq q_0\},$$

with constant $q_0 < \mathrm{Re}\,\mu_{\xi'} < 1$*. Here,* $\mu_{\xi'} = e^{\lambda_{\xi'}T}$ *is simple eigenvalue of* $\widehat{U}_{\xi'}(T,0)$ *and* $\lambda_{\xi'}$ *has an expansion*

$$\lambda_{\xi'} = -i\kappa_0\xi_1 - \kappa_1\xi_1^2 - \kappa''|\xi''|^2 + O(|\xi'|^3), \tag{2.4}$$

where $\kappa_0 \in \mathbb{R}$ *and* $\kappa_1 > 0$*,* $\kappa'' > 0$*. Here,* $\mathrm{Re}\,\lambda$ *denotes the real part of* $\lambda \in \mathbb{C}$*.*
Moreover, let $\widehat{\Pi}_{\xi'}$ *denote the eigenprojection associated with* $\mu_{\xi'}$*. There holds*

$$|\widehat{U}_{\xi'}(t,s)(I - \widehat{\Pi}_{\xi'})u|_{H^1(0,1)} \leq Ce^{-d(t-s)}|(I - \widehat{\Pi}_{\xi'})u|_{X_0},$$

for $u \in X_0$ *and* $T \leq t - s$*. Here,* d *is a positive constant depending on* r_0*.*

(ii) The spectrum of operator $\widehat{U}^*_{\xi'}(0,T)$ *on* $(H^1 \times H^1_0)(0,1)$ *satisfies*

$$\sigma(\widehat{U}^*_{\xi'}(0,T)) \subset \{\overline{\mu}_{\xi'}\} \cup \{\mu : |\mu| \leq q_0\}.$$

Here, $\overline{\mu}_{\xi'}$ *is simple eigenvalue of* $\widehat{U}^*_{\xi'}(0,T)$.

On the other hand, if $|\xi'| \geq R \gg 1$, one can derive the exponential decay property of the corresponding part of the solution operator $\mathscr{U}(t,s)$ by the Fourier transformed version of Matsumura-Nishida's energy method (e.g., see [10]), provided that Re and Ma are sufficiently small. As for the bounded frequency part $r \leq |\xi'| \leq R$, one can employ a certain time-dependent decomposition argument and apply a variant of Matsumura-Nishida's energy method as in [9] to show the exponential decay.

Let us denote

$$\mathscr{U}_1(t,s) = \mathscr{F}^{-1}\left(\widehat{U}_{\xi'}(t,s)|_{r \leq |\xi'| \leq R}\right), \quad \mathscr{U}_\infty(t,s) = \mathscr{F}^{-1}\left(\widehat{U}_{\xi'}(t,s)|_{|\xi'| \geq R}\right).$$

Next two theorems show that $\mathscr{U}_j(t,s)u_0$ $(j = 1, \infty)$ decay exponentially in time.

Theorem 2.5 *There exist constants* $R_0 > 1$, $\nu_0 > 0$ *and* $\gamma_0 > 0$ *such that if* $\nu \geq \nu_0$ *and* $\gamma^2/(\nu + \widetilde{\nu}) \geq \gamma_0^2$ *then there exists a constant* $d > 0$ *such that the estimate*

$$\|\mathscr{U}_\infty(t,s)u_0\|_{H^1(\Omega)} \leq Ce^{-d(t-s-4T)}(\|u_0\|_{(H^1 \times L^2)(\Omega)} + \|\partial_{x'} w_0\|_{L^2(\Omega)}),$$

holds uniformly in $t - s \geq 4T$, $s \geq 0$.

Theorem 2.6 *There exist constants* $\nu_0 > 0$ *and* $\gamma_0 > 0$ *such that if* $\nu \geq \nu_0$ *and* $\gamma^2/(\nu + \widetilde{\nu}) \geq \gamma_0^2$ *then for any* $0 < r < R_0$ *there exists a constant* $d(r) > 0$ *such that the estimate*

$$\|\mathscr{U}_1(t,s)u_0\|_{H^1(\Omega)} \leq Ce^{-d(t-s-4T)}(\|u_0\|_{(H^1 \times L^2)(\Omega)} + \|\partial_{x'} w_0\|_{L^2(\Omega)}),$$

holds uniformly in $t - s \geq 4T$, $s \geq 0$.

Therefore, we see from Theorem 2.5 and Theorem 2.6 that the interesting part of solution is given by $\mathscr{U}_0(t,s)u_0$. To investigate $\mathscr{U}_0(t,s)u_0$, we introduce the following Floquet theory in a Fourier space.

Definition 2.7 *Let* $k = 1, 2, \ldots$. *Let us define spaces* Y^k_{per} *as*

$$Y^1_{per} = L^2_{per}([0,T]; X_0),$$

$$Y^k_{per} = \bigcap_{j=0}^{[\frac{k}{2}]} H^j_{per}([0,T]; H^{k-2j}(0,1) \times H^{k-1-2j}(0,1)), \text{ for } k \geq 2.$$

Here, for Banach space X *and* $j = 0, \ldots$ *spaces* $L^2_{per}([0,T]; X)$ *and* $H^j_{per}([0,T]; X)$ *consist of functions from* $L^2([0,T]; X)$ *and* $H^j([0,T]; X)$, *respectively, that are restrictions of* T*-periodic functions.*

Definition 2.8 *We define operator $B_{\xi'}$ on space Y^1_{per} with domain*

$$D(B_{\xi'}) = H^1_{per}([0,T]; X_0) \cap L^2_{per}([0,T]; (H^1 \times (H^2 \cap H^1_0))(0,1)),$$

in the following way

$$B_{\xi'} v = \partial_t v + \widehat{L}_{\xi'}(\cdot) v,$$

for $v \in D(B_{\xi'})$. Moreover, we define formal adjoint operator $B^*_{\xi'}$ with respect to inner product $\frac{1}{T}\int_0^T \langle \cdot, \cdot \rangle dt$ as

$$B^*_{\xi'} v = -\partial_t v + \widehat{L}^*_{\xi'}(\cdot) v,$$

for $v \in D(B^*_{\xi'}) = D(B_{\xi'})$.

Remark 2.9 Operators $B_{\xi'}$ and $B^*_{\xi'}$ are closed, densely defined on Y^1_{per} for each fixed $\xi' \in \mathbb{R}^{n-1}$.

Lemma 2.10 *Let Assumptions 1.4 be satisfied for $m \geq 2$. There exist positive numbers $\nu_1 \geq \nu_0$ and $\gamma_1 \geq \gamma_0$ such that if $\nu \geq \nu_1$ and $\gamma^2/(\nu + \widetilde{\nu}) \geq \gamma_1^2$ then there exists $0 < r_1 \leq 1$ such that for each $|\xi'| \leq r_1$ there hold the following statements.*

(i) Let $1 \leq k \leq m$. There exists $q_1 > 0$ such that spectrum of operator $B_{\xi'}$ on Y^k_{per} satisfies

$$\sigma(B_{\xi'}) \subset \bigcup_{l \in \mathbb{Z}} \{-\lambda_{\xi'} + i\frac{2l\pi}{T}\} \cup \{\lambda : \mathrm{Re}\,\lambda \geq q_1\},$$

with $0 \leq |\lambda_{\xi'}| \leq \frac{1}{2}q_1$ uniform for all k. Here, $-\lambda_{\xi'} + i\frac{2l\pi}{T}$, $l \in \mathbb{Z}$ are simple eigenvalues of $B_{\xi'}$.

(ii) Let $1 \leq k \leq m$. Spectrum of operator $B^*_{\xi'}$ on Y^k_{per} satisfies

$$\sigma(B^*_{\xi'}) \subset \bigcup_{l \in \mathbb{Z}} \{-\overline{\lambda}_{\xi'} - i\frac{2l\pi}{T}\} \cup \{\lambda : \mathrm{Re}\,\lambda \geq q_1\}.$$

Here, $-\overline{\lambda}_{\xi'} - i\frac{2l\pi}{T}$, $l \in \mathbb{Z}$ are simple eigenvalues of $B^*_{\xi'}$.

(iii) There exist $u_{\xi'}$ and $u^*_{\xi'}$ eigenfunctions associated with $-\lambda_{\xi'}$ and $-\overline{\lambda}_{\xi'}$, respectively, with the following properties:

$$\langle u_{\xi'}(t), u^*_{\xi'}(t) \rangle = 1,$$

$$u_{\xi'}(t) = u^{(0)}(t) + i\xi' \cdot u^{(1)}(t) + |\xi'|^2 u^{(2)}(\xi', t),$$

$$u^*_{\xi'}(t) = u^{*(0)} + i\xi' \cdot u^{*(1)}(t) + |\xi'|^2 u^{*(2)}(\xi', t),$$

for $t \in \mathbb{R}$. Here, all functions

$$u_{\xi'}, u^*_{\xi'}, u^{(0)}, u^{(0)*}, u^{(1)}, u^{(1)*}, u^{(2)}(\xi'), u^{(2)*}(\xi'),$$

are T-periodic in t,

$$u \in \bigcap_{j=0}^{[\frac{m}{2}]} C^j([0,T]; (H^{m-2j} \times (H^{m-2j} \cap H^1_0))(0,1)),$$

$$\phi \in \bigcap_{j=0}^{[\frac{m}{2}]} H^{j+1}(0,T; H^{m-2j}(0,1)),\ w \in \bigcap_{j=0}^{[\frac{m+1}{2}]} H^j(0,T; (H^{m+1-2j} \cap H^1_0)(0,1)),$$

and we have estimate

$$\sup_{z\in[0,T]} \sum_{j=0}^{[\frac{m}{2}]} |\partial_z^j u(z)|^2_{H^{m-2j}(0,1)} + \int_0^T \sum_{j=0}^{[\frac{m-1}{2}]} |\partial_z^{j+1} u|^2_{(H^{m-2j}\times H^{m-1-2j})(0,1)}$$

$$+|\partial_z^{[\frac{m+2}{2}]} Q_0 u|^2_{L^2(0,1)} + |u|^2_{(H^m\times H^{m+1})(0,1)} dz \le C,$$

for $u = {}^T(\phi, w) \in \{u_{\xi'}, u^*_{\xi'}, u^{(2)}(\xi'), u^{(2)*}(\xi')\}$ and a constant $C > 0$ depending on r_1.

As for $u^{(0)}(t)$, we have the following result.

Lemma 2.11 *Function $u^{(0)}(t)$ satisfies $\partial_t u^{(0)} + \widehat{L}_0(t)u^{(0)} = 0$ and $u^{(0)}(t) = u^{(0)}(t+T)$ for all $t \in \mathbb{R}$. Function $u^{(0)}(t)$ is given as*

$$u^{(0)}(x_n, t) = {}^T(\phi^{(0)}(x_n), w^{(0),1}(x_n, t), 0).$$

Here,

$$\phi^{(0)}(x_n) = \alpha_0 \frac{\gamma^2 \rho_p(x_n)}{P'(\rho_p(x_n))}, \qquad \alpha_0 = \left(\int_0^1 \frac{\gamma^2 \rho_p(x_n)}{P'(\rho_p(x_n))} dx_n\right)^{-1},$$

$$w^{(0),1}(x_n, t) = -\frac{1}{\gamma^2} \int_{-\infty}^t e^{-(t-s)\nu A} \nu \frac{\alpha_0 \gamma^2}{P'(\rho_p)\rho_p} (\partial^2_{x_n} v^1_p(s))\, ds,$$

where A is given by (1.21). Moreover, function $w^{(0),1}$ satisfies

$$\partial_t w^{(0),1}(t) - \frac{\nu}{\rho_p(x_n)} \partial^2_{x_n} w^{(0),1}(t) = -\frac{\nu}{\gamma^2} \frac{\alpha_0 \gamma^2}{P'(\rho_p)\rho_p} (\partial^2_{x_n} v^1_p(t)),$$

for all $t \in \mathbb{R}$ and under Assumptions 1.4 there holds

$$\|w^{(0),1}(t)\|_{C^{m+1}(\Omega)} = O(\frac{1}{\gamma^2}).$$

2.2 Floquet theory for $\mathscr{P}(t)u(t)$

In this subsection we assume that $\nu \geq \nu_1$ and $\gamma^2/(\nu+\widetilde{\nu}) \geq \gamma_1^2$ and Assumptions 1.4 hold for an integer m, $m \geq 2$. We introduce time-peridic operators and projection based on spectrum of $B_{\xi'}$ and $B_{\xi'}^*$, which are used to decompose the solution of the nonlinear problem (2.1) in Section 3. We also give a summary of their properties.

Definition 2.12 *We define $\widehat{\chi}_1$ by*

$$\widehat{\chi}_1(\xi') = \mathbf{1}_{[0,r_1)}(|\xi'|) = \begin{cases} 1, & 0 \leq |\xi'| < r_1, \\ 0, & |\xi'| \geq r_1, \end{cases}$$

for $\xi' \in \mathbb{R}^{n-1}$, where r_1 is given by Lemma 2.10.

Now, we introduce time-periodic operators based on eigenfunctions $u_{\xi'}$ and $u_{\xi'}^*$.

Definition 2.13 *We define operators $\mathscr{P}(t) : L^2(\Omega) \to L^2(\mathbb{R}^{n-1})$ by*

$$\mathscr{P}(t)u = \mathscr{F}^{-1}\{\widehat{\mathscr{P}}_{\xi'}(t)\widehat{u}\}, \quad \widehat{\mathscr{P}}_{\xi'}(t)\widehat{u} = \widehat{\chi}_1\langle \widehat{u}, u_{\xi'}^*(t)\rangle;$$

operators $\mathscr{Q}(t) : L^2(\mathbb{R}^{n-1}) \to L^2(\Omega)$ by

$$\mathscr{Q}(t)\sigma = \mathscr{F}^{-1}\{\widehat{\chi}_1\widehat{\mathscr{Q}}_{\xi'}(t)\widehat{\sigma}\}, \quad \widehat{\mathscr{Q}}_{\xi'}(t)\widehat{\sigma} = u_{\xi'}(\cdot,t)\widehat{\sigma};$$

multiplier $\Lambda : L^2(\mathbb{R}^{n-1}) \to L^2(\mathbb{R}^{n-1})$ by

$$\Lambda\sigma = \mathscr{F}^{-1}\{\widehat{\chi}_1\lambda_{\xi'}\widehat{\sigma}\};$$

and projections $\mathbb{P}(t)$ on $L^2(\Omega)$ as

$$\mathbb{P}(t)u = \mathscr{Q}(t)\mathscr{P}(t)u = \mathscr{F}^{-1}\{\widehat{\chi}_1\langle \widehat{u}, u_{\xi'}^*(t)\rangle u_{\xi'}(\cdot,t)\},$$

for $t \in [0,\infty)$ and $u \in L^2(\Omega)$, $\sigma \in L^2(\mathbb{R}^{n-1})$.

One can see that $\mathbb{P}(t)^2 = \mathbb{P}(t)$. Moreover, Λ is bounded linear operator on $L^2(\mathbb{R}^{n-1})$. It then follows that Λ generates uniformly continuous group $\{e^{t\Lambda}\}_{t\in\mathbb{R}}$. Furthermore, if $\sigma \in L^p(\Omega)$, $1 \leq p \leq 2$ then

$$\|\partial_{x'}^k e^{t\Lambda}\sigma\|_{L^2(\mathbb{R}^{n-1})} \leq C(1+t)^{-\frac{n-1}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k}{2}}\|\sigma\|_{L^p(\mathbb{R}^{n-1})}, \ k = 0,1,\ldots.$$

In terms of $\mathbb{P}(t)$ we have the following decomposition of $\mathscr{U}(t,s)$.

Theorem 2.14 *$\mathbb{P}(t)$ satisfies the following:*

(i)

$$\mathbb{P}(t)(\partial_t + L(t))u(t) = (\partial_t + L(t))\mathbb{P}(t)u(t) = \mathscr{Q}(t)[(\partial_t - \Lambda)\mathscr{P}(t)u(t)],$$

for $u \in L^2([0,T];(H^1 \times (H^2 \cap H_0^1))(\Omega)) \cap H^1([0,T];L^2(\Omega))$.

(ii)

$$\mathbb{P}(t)\mathscr{U}(t,s) = \mathscr{U}(t,s)\mathbb{P}(s) = \mathscr{Q}(t)e^{(t-s)\Lambda}\mathscr{P}(s).$$

If $u \in L^1(\Omega)$, then

$$\|\partial_t^j\partial_{x'}^k\partial_{x_n}^l\mathbb{P}(t)\mathscr{U}(t,s)u\|_{L^2(\Omega)} \le C(1+t-s)^{-\frac{n-1}{4}-\frac{k}{2}}\|u\|_{L^1(\Omega)},$$

for $0 \le 2j+l \le m$, $k = 0, \ldots$.

(iii) $(I-\mathbb{P}(t))\mathscr{U}(t,s) = \mathscr{U}(t,s)(I-\mathbb{P}(s))$ satisfies

$$\|(I-\mathbb{P}(t))\mathscr{U}(t,s)u_0\|_{H^1(\Omega)} \le Ce^{-d(t-s)}(\|u_0\|_{(H^1\times L^2)(\Omega)} + \|\partial_{x'}w_0\|_{L^2(\Omega)}),$$

for $t-s \ge T$. Here d is a positive constant.

Let us consider the following inhomogeneous problem:

$$\partial_t u + L(t)u = f(t),\ t > 0,\ u|_{t=0} = u_0. \tag{2.5}$$

One can show that if $u_0 \in (H^1 \times H_0^1)(\Omega)$ and $f \in L^2_{loc}([0,\infty);(H^1\times L^2)(\Omega))$, then there exists unique $u(t) = {}^T(\phi(t), w(t))$,

$$u \in C_{loc}([0,\infty);(H^1\times H_0^1)(\Omega)),\ \phi \in H^1_{loc}([0,\infty);L^2(\Omega)),\ w \in \bigcap_{j=0}^{1} H^j_{loc}([0,\infty);H_*^{2-2j}(\Omega)), \tag{2.6}$$

that satisfies (2.5).

Theorem 2.15 *Let $u_0 \in (H^1 \times H_0^1)(\Omega)$, $f \in L^2_{loc}([0,\infty);(H^1\times L^2)(\Omega))$ and let $u(t) = {}^T(\phi(t), w(t))$ is unique solution of* (2.5) *in the class* (2.6). *Then*

(i) $\mathscr{P}(t)u(t)$ satisfies

$$\mathscr{P}(t)u(t) = e^{t\Lambda}\mathscr{P}(0)u_0 + \int_0^t e^{(t-z)\Lambda}\mathscr{P}(z)f(z)dz,\ t\in[0,\infty). \tag{2.7}$$

(ii) $u_\infty(t) = {}^T(\phi_\infty(t), w_\infty(t)) = (I-\mathbb{P}(t))u(t)$ belongs to class (2.6) *and satisfies*

$$\partial_t u_\infty + L(t)u_\infty = (I-\mathbb{P}(t))f,\ t>0,\ u_\infty|_{t=0} = (I-\mathbb{P}(0))u_0.$$

Next, let us show the asymptotic properties of $\mathscr{U}(t,s)$. First, let us define a semigroup $\mathscr{H}(t)$ on $L^2(\mathbb{R}^{n-1})$ associated with a linear heat equation with a convective term:

$$\partial_t\sigma - \kappa_1\partial_{x_1}^2\sigma - \kappa''\Delta''\sigma + \kappa_0\partial_{x_1}\sigma = 0.$$

Definition 2.16 *We define operator $\mathscr{H}(t)$ as*

$$\mathscr{H}(t)\sigma = \mathscr{F}^{-1}[e^{-(i\kappa_0\xi_1+\kappa_1\xi_1^2+\kappa''|\xi''|^2)t}\hat{\sigma}],$$

for $\sigma \in L^2(\mathbb{R}^{n-1})$. Here, κ_0, κ_1 and κ'' are given by (2.4).

Theorem 2.17 *There hold the following estimates for* $1 \le p \le 2$ *and* $k = 0, 1, \ldots$.

(i)

$$\|\partial_{x'}^k(\mathscr{H}(t)\sigma)\|_{L^2(\mathbb{R}^{n-1})} \le Ct^{-\frac{n-1}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k}{2}}\|\sigma\|_{L^p(\mathbb{R}^{n-1})},$$

for $\sigma \in L^p(\mathbb{R}^{n-1})$.

(ii) It holds the relation,

$$\mathscr{P}(t)\mathscr{U}(t,s) = e^{(t-s)\Lambda}\mathscr{P}(s).$$

Set $\sigma = [Q_0 u]$. *Then*

$$\|\partial_{x'}^k(Q(t)e^{(t-s)\Lambda}\mathscr{P}(s)u - u^{(0)}(t)\mathscr{H}(t-s)\sigma)\|_{L^2(\mathbb{R}^{n-1})} \le C(t-s)^{-\frac{n-1}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k+1}{2}}\|u\|_{L^p(\Omega)},$$

for $u \in L^p$. *Furthermore, for any* $\sigma \in L^p(\mathbb{R}^{n-1})$ *there holds*

$$\|(e^{(t-s)\Lambda} - \mathscr{H}(t-s))\partial_{x'}^k\sigma\|_{L^2(\mathbb{R}^{n-1})} \le C(t-s)^{-\frac{n-1}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k+1}{2}}\|\sigma\|_{L^p(\mathbb{R}^{n-1})}.$$

Remark 2.18 Combining (2.7) with Theorem 2.17 (ii) we see that asymptotic leading part of $\mathscr{U}(t,s)u_0$ is represented by $u^{(0)}(t)\mathscr{H}(t-s)\sigma$, where $\sigma = \int_0^1 \phi_0(x', x_n)dx_n$ and $u_0 = {}^T(\phi_0, w_0)$.

Theorems 2.14, 2.15 and 2.17 follow from the properties of $\mathscr{Q}(t)$ and $\mathscr{P}(t)$ introduced below. Next, we introduce the properties of $\mathscr{Q}(t)$ and $\mathscr{P}(t)$.

Lemma 2.19 $\mathscr{Q}(t)$ *has the following properties:*

(i)

$$\mathscr{Q}(t+T) = \mathscr{Q}(t),\ \partial_{x'}^k\mathscr{Q}(t) = \mathscr{Q}(t)\partial_{x'}^k.$$

(ii)

$$\|\partial_t^j\partial_{x'}^k\partial_{x_n}^l(\mathscr{Q}(t)\sigma)\|_{L^2(\Omega)} \le C\|\sigma\|_{L^2(\mathbb{R}^{n-1})},\ 0 \le 2j+l \le m+1,\ k = 0, 1, \ldots,$$

for $\sigma \in L^2(\mathbb{R}^{n-1})$.

(iii) $\mathscr{Q}(t)$ *is decomposed as*

$$\mathscr{Q}(t) = \mathscr{Q}^{(0)}(t) + \mathrm{div}'\mathscr{Q}^{(1)}(t) + \Delta'\mathscr{Q}^{(2)}(t).$$

Here, $\mathscr{Q}^{(0)}(t)\sigma = (\mathscr{F}^{-1}\{\hat{\chi}_1\hat{\sigma}\})u^{(0)}(\cdot,t)$, $\mathscr{Q}^{(1)}(t)$ *and* $\mathscr{Q}^{(2)}(t)$ *share the same properties given in (i) and (ii) for* $\mathscr{Q}(t)$.

Lemma 2.20 $\mathscr{P}(t)$ *has the following properties:*

(i)

$$\mathscr{P}(t+T) = \mathscr{P}(t),\ \partial_{x'}^k\mathscr{P}(t) = \mathscr{P}(t)\partial_{x'}^k,\ \partial_{x_n}\mathscr{P}(t) = 0.$$

(ii)

$$\|\partial_t^j\partial_{x'}^k(\mathscr{P}(t)u)\|_{L^2(\mathbb{R}^{n-1})} \le C\|u\|_{L^2(\Omega)},\ 0 \le 2j \le m+1,\ k = 0, 1, \ldots,$$

for $u \in L^2(\Omega)$.

Moreover,

$$\|\mathscr{P}(t)u\|_{L^2(\mathbb{R}^{n-1})} \leq C\|u\|_{L^p(\Omega)},$$

for $u \in L^p(\Omega)$ *and* $1 \leq p \leq 2$.

(iii)

$$\mathscr{P}(t)(\partial_t + L(t))u(t) = (\partial_t - \Lambda)(\mathscr{P}(t)u(t)),$$

for $u \in L^2_{loc}([0,\infty); (H^1 \times (H^2 \cap H^1_0))(\Omega)) \cap H^1_{loc}([0,\infty); L^2(\Omega))$.

(iv) $\mathscr{P}(t)$ *is decomposed as*

$$\mathscr{P}(t) = \mathscr{P}^{(0)} + \mathrm{div}'\mathscr{P}^{(1)}(t) + \Delta'\mathscr{P}^{(2)}(t).$$

Here, $u = {}^T(\phi, w)$ *and*

$$\mathscr{P}^{(0)}u = \mathscr{F}^{-1}\{\widehat{\chi}_1\langle \widehat{u}, u^{*(0)}\rangle\} = \mathscr{F}^{-1}\{\widehat{\chi}_1 \int_0^1 \widehat{\phi}(\xi', x_n)dx_n\},$$

$$\mathscr{P}^{(1)}(t)u = \mathscr{F}^{-1}\{\widehat{\chi}_1\langle \widehat{u}, u^{*(1)}(t)\rangle\},$$

$$\mathscr{P}^{(2)}(t)u = \mathscr{F}^{-1}\{-\widehat{\chi}_1\langle \widehat{u}, u^{*(2)}(\xi', t)\rangle\}.$$

$\mathscr{P}^{(p)}(t)$, $p = 0, 1, 2$, *share the same properties given in (i) and (ii) for* $\mathscr{P}(t)$.

(v) There holds

$$\|\partial^k_{x'} e^{(t-s)\Lambda}\mathscr{P}^{(q)}(s)u\|_{L^2(\mathbb{R}^{n-1})} \leq C(1+t-s)^{-\frac{n-1}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k}{2}}\|u\|_{L^p(\Omega)},\ q = 0, 1, 2,$$

for $u \in L^p(\Omega)$, $1 \leq p \leq 2$ *and* $k = 0, 1, \dots$.

Properties of $\mathscr{Q}(t)$ and $\mathscr{P}(t)$ given in Lemma 2.19 and Lemma 2.20 follow by computation from properties of eigenfunctions $u_{\xi'}$ and $u^*_{\xi'}$ introduced in Lemma 2.10.

3 Nonlinear problem

In this section we state the main results on the nonlinear problem (1.13)–(1.16). These results were established in [3] and we omit their proofs here.

First, let us introduce the local existence result. To do so, we rewrite (1.13)–(1.16) in the form

$$\partial_t\phi + v \cdot \nabla\phi = -\gamma^2 w \cdot \nabla\rho_p - \rho \mathrm{div}\, w, \tag{3.1}$$

$$\rho\partial_t w - \nu\Delta w - \widetilde{\nu}\nabla\mathrm{div}\, w = -\frac{\nu}{\gamma^2\rho_p}\partial^2_{x_n}v_p\phi - \nabla(P(\rho) - P(\rho_p)) - \rho(v \cdot \nabla v), \tag{3.2}$$

$$w|_{\partial\Omega} = 0, \tag{3.3}$$

$$(\phi, w)|_{t=0} = (\phi_0, w_0), \tag{3.4}$$

where $\rho = \rho_p + \gamma^{-2}\phi$ and $v = v_p + w$.

Here, we mention the compatibility condition for $u_0 = {}^T(\phi_0, w_0)$. We look for a solution $u = {}^T(\phi, w)$ of (3.1)–(3.4) in $\bigcap_{j=0}^{[\frac{m}{2}]} C^j([0,\infty); H^{m-2j}(\Omega))$ satisfying $\int_0^t \|\partial_x w(z)\|^2_{H^m(\Omega)} dz < \infty$ for all $t \geq 0$ with $m \geq [n/2] + 1$. Therefore, we need to require the compatibility condition for the initial value $u_0 = {}^T(\phi_0, w_0)$, which is formulated as follows.

Let $u = {}^T(\phi, w)$ be a smooth solution of (3.1)–(3.4). Then $\partial_t^j u = {}^T(\partial_t^j \phi, \partial_t^j w)$, $j \geq 1$ is inductively determined by

$$\partial_t^j \phi = -v \cdot \nabla \partial_t^{j-1}\phi - \rho \operatorname{div} \partial_t^{j-1} w - \gamma^2 \partial_t^{j-1} w \cdot \nabla \rho_p - \{[\partial_t^{j-1}, v \cdot \nabla]\phi + [\partial_t^{j-1}, \rho]\operatorname{div} w\},$$

and

$$\begin{aligned}
\partial_t^j w =& -\rho^{-1}\{-\nu\Delta\partial_t^{j-1} w - \tilde{\nu}\nabla \operatorname{div} \partial_t^{j-1} w + P'(\rho)\nabla\partial_t^{j-1}\rho\} \\
& -\rho^{-1}\{\gamma^{-2}[\partial_t^{j-1}, \phi]\partial_t w + [\partial_t^{j-1}, P'(\rho)]\nabla\rho\} \\
& -\rho^{-1}\{\tfrac{\nu}{\gamma^2\rho_p}\partial_t^{j-1}(\partial_{x_n}^2 v_p \phi) - \partial_t^{j-1}\nabla P(\rho_p)\} - \rho^{-1}\partial_t^{j-1}(\rho(v\cdot\nabla v)).
\end{aligned}$$

From these relations we see that $\partial_t^j u|_{t=0} = {}^T(\partial_t^j \phi, \partial_t^j w)|_{t=0}$ is inductively given by $u_0 = {}^T(\phi_0, w_0)$ in the following way:

$$\partial_t^j u|_{t=0} = {}^T(\partial_t^j \phi, \partial_t^j w)|_{t=0} = {}^T(\phi_j, w_j) = u_j,$$

where

$$\phi_j = -v_0 \cdot \nabla\phi_{j-1} - \rho_0 \operatorname{div} w_{j-1} - \gamma^2 w_{j-1}\cdot\nabla\rho_p - \sum_{l=1}^{j-1}\begin{pmatrix} j-1 \\ l \end{pmatrix}\{v_l \cdot \nabla\phi_{j-1-l} + \gamma^{-2}\phi_l \operatorname{div} w_{j-1}\},$$

and

$$\begin{aligned}
w_j =& -\rho_0^{-1}\{-\nu\Delta w_{j-1} - \tilde{\nu}\nabla\operatorname{div} w_{j-1} + P'(\rho_0)\nabla\rho_{j-1}\} - \rho_0^{-1}\sum_{l=1}^{j-1}\begin{pmatrix} j-1 \\ l \end{pmatrix}\{\gamma^{-2}\phi_l w_{j-l} \\
& + a_l(\phi_0; \phi_1, \ldots, \phi_l)\nabla\rho_{j-1-l}\} - \rho_0^{-1}\frac{\nu}{\gamma^2\rho_p}\sum_{l=0}^{j-1}\begin{pmatrix} j-1 \\ l \end{pmatrix}\partial_t^{j-1-l}\partial_{x_n}^2 v_p(0)\phi_l + \delta_{1j}\rho_0^{-1}\nabla P(\rho_p) \\
& -\rho_0^{-1}G_{j-1}(\phi_0, w_0, \partial_x w_0; \phi_1, \ldots, \phi_{j-1}, w_1, \ldots, w_{j-1}, \partial_x w_1, \ldots, \partial_x w_{j-1}),
\end{aligned}$$

with $v_l = \partial_t^l v_p(0) + w_l$, $\rho_l = \delta_{0l}\rho_p + \gamma^{-2}\phi_l$; and $a_l(\phi_0; \phi_1, \ldots, \phi_l)$ is certain polynomial in $\phi_1, \ldots, \phi_l$; and so on. Here, δ_{jk} denotes the Kronecker's delta.

By the boundary condition $w|_{\partial\Omega} = 0$ in (3.3), we necessarily have $\partial_t^j w|_{\partial\Omega} = 0$, and hence,

$$w_j|_{\partial\Omega} = 0.$$

Assume that $u = {}^T(\phi, w)$ is a solution of (3.1) –(3.4) in $\bigcap_{j=0}^{[\frac{m}{2}]} C^j([0,\tau_0]; H^{m-2j}(\Omega))$ for some $\tau_0 > 0$. Then, from above observation, we need the regularity $u_j = {}^T(\phi_j, w_j) \in (H^{m-2j} \times H^{m-2j})(\Omega)$ for $j = 1, \dots, [m/2]$, which follows from the fact that $u_0 = {}^T(\phi_0, w_0) \in H^m(\Omega)$ with $m \geq [n/2] + 1$. Furthermore, it is necessary to require that $u_0 = {}^T(\phi_0, w_0)$ satisfies the *$\widehat{m}$-th order compatibility condition*:

$$w_j \in H_0^1(\Omega) \text{ for } j = 0, \dots, \widehat{m} = \left[\frac{m-1}{2}\right].$$

Now, using local solvability result obtained in [8] one can show the following assertion.

Proposition 3.1 *Let $n \geq 2$, Assumptions 1.4 be satisfied for an integer m, $m \geq [n/2] + 1$ and $M > 0$. Assume that $u_0 = {}^T(\phi_0, w_0) \in H^m(\Omega)$ satisfies the following conditions:*

(a) $\|u_0\|_{H^m(\Omega)} \leq M$ and u_0 satisfies the $\widehat{m}$-th compatibility condition,

(b) $-\frac{\gamma^2}{4}\underline{\rho} \leq \phi_0$.

Then there exists a positive number τ_0 depending on M and $\underline{\rho}$ such that problem (3.1)–(3.4) *has a unique solution $u(t)$ on $[0, \tau_0]$ satisfying*

$$u \in \bigcap_{j=0}^{[\frac{m}{2}]} C^j([0,\tau_0]; H^{m-2j}(\Omega)),$$

together with

$$\sup_{0 \leq z \leq \tau_0} [\![f(t)]\!]_m^2 + \int_0^{\tau_0} |||Dw(z)|||_m^2 dz < \infty.$$

Here,

$$|||Df(t)|||_m^2 = [\![\partial_x f(t)]\!]_m^2 + [\![\partial_t f(t)]\!]_{m-1}^2, \textit{ with } [\![f(t)]\!]_k^2 = \sum_{j=0}^{[\frac{k}{2}]} \|\partial_t^j f(t)\|_{H^{k-2j}(\Omega)}, \ k \geq 0.$$

Remark 3.2 It is straightforward to see that solution $u(t)$ of (3.1)–(3.4) is solution of (1.13)–(1.16). Condition (b) in the previous proposition assures that $\gamma^{-2}\phi_0 + \rho_p \geq \frac{3}{4}\underline{\rho} > 0$.

Second, we state our main results of this paper.

Theorem 3.3 *Suppose that $n \geq 2$ and Assumptions 1.4 are satisfied for an integer m, $m \geq [n/2] + 1$. There are positive numbers ν_0 and γ_0 such that if $\nu \geq \nu_0$ and $\gamma^2/(\nu + \widetilde{\nu}) \geq \gamma_0^2$ then the following assertions hold true.*

There is a positive number ε_0 such that if $u_0 = {}^T(\phi_0, w_0) \in (H^m \cap L^1)(\Omega)$ satisfies the $\widehat{m}$-th compatibility condition and $\|u_0\|_{(H^m \cap L^1)(\Omega)} \leq \varepsilon_0$, then there exists a unique global solution $u(t) = {}^T(\phi(t), w(t))$ of (1.13)–(1.16) *in $\bigcap_{j=0}^{[\frac{m}{2}]} C^j([0,\infty); H^{m-2j}(\Omega))$ which satisfies*

$$\|\partial_{x'}^k u(t)\|_{L^2(\Omega)} = O(t^{-\frac{n-1}{4}-\frac{k}{2}}),\ k = 0, 1.$$

Moreover, let $n = 2$. There holds

$$\|u(t) - (\sigma u^{(0)})(t)\|_{L^2(\Omega)} = O(t^{-\frac{3}{4}+\delta}),\ \forall\delta > 0,$$

as $t \to \infty$. Here, $u^{(0)} = u^{(0)}(x_2, t)$ is function given in Lemma 2.11; $\sigma = \sigma(x_1, t)$ is function satisfying

$$\partial_t\sigma - \kappa_1\partial_{x_1}^2\sigma + \kappa_0\partial_{x_1}\sigma + \omega_0\partial_{x_1}(\sigma^2) = 0,\ \sigma|_{t=0} = \int_0^1 \phi_0(x_1, x_2)\, dx_2,$$

with given constants $\kappa_0, \omega_0 \in \mathbb{R}$, $\kappa_1 > 0$.

Furthermore, let $n \geq 3$. There holds

$$\|u(t) - (\sigma u^{(0)})(t)\|_{L^2(\Omega)} = O(t^{-\frac{n-1}{4}-\frac{1}{2}}\eta_n(t)),$$

as $t \to \infty$. Here, $\sigma = \sigma(x', t)$ is function satisfying

$$\partial_t\sigma - \kappa_1\partial_{x_1}^2\sigma - \kappa''\Delta''\sigma + \kappa_0\partial_{x_1}\sigma = 0,\ \sigma|_{t=0} = \int_0^1 \phi_0(x', x_n)\, dx_n,$$

with given constants $\kappa_0 \in \mathbb{R}$, $\kappa_1, \kappa'' > 0$; where $\Delta'' = \partial_{x_2}^2 + \cdots + \partial_{x_{n-1}}^2$; and $\eta_n(t) = \log(1+t)$ when $n = 3$ and $\eta_n(t) = 1$ when $n \geq 4$.

Remark 3.4 As we already mentioned in Section 2, constants κ_0, κ_1 and κ'' come from the expansion (2.4) of $\lambda_{\xi'}$, where $e^{\lambda_{\xi'}T}$ is eigenvalue of $\widehat{U}_{\xi'}(T, 0)$.

In the case $n = 2$ parameter ω_0 appears. It comes from the quadratic nonlinearities:

$$-\text{div}\,(\phi w),\quad \frac{\nu\phi}{\gamma^2\rho_p^2}\left(-\partial_{x_n}^2 w^1 + \frac{\partial_{x_n}^2 v_p^1}{\rho_p\gamma^2}\phi\right),\quad -\frac{1}{2\gamma^4\rho_p}\partial_{x_n}(P''(\rho_p)\phi^2).$$

It is given as

$$\omega_0 = \frac{1}{T}\int_0^T \int_0^1 \phi^{(0)}(x', x_n) w^{(0),1}(x', x_n, z) dx_n - \langle \boldsymbol{F}_1(z), u^{*(1)}(z)\rangle_\Omega dz,$$

where $u^{(0)}(t) = {}^T(\phi^{(0)}, w^{(0),1}(t)e_1)$, properties of $u^{*(1)}(t)$ are specified in Lemma 2.10 and

$$\boldsymbol{F}_1 = {}^T\left(0,\ \frac{\phi^{(0)}(x_n)}{\gamma^2\rho_p}\partial_t w^{(0),1}(x_n, t),\ \frac{1}{2\gamma^4\rho_p(x_n)}\partial_{x_n}\left(P''(\rho_p(x_2))\{\phi^{(0)}(x_n)\}^2\right)\right).$$

The proof of Theorem 3.3 is obtained by decomposing the solution $u(t)$ of (1.13)–(1.16) into the $\mathbb{P}(t)$-part and $(I - \mathbb{P}(t))$-part. Considering the $\mathbb{P}(t)$-part, we represent $\mathbb{P}(t)u(t)$ as in (2.7) with $f(z) = \boldsymbol{F}(z)$. We then combine various estimates on $\mathbb{P}(t)$ and $\mathbb{P}(t)U(t, s)$ to obtain the necessary estimates on $\mathbb{P}(t)u(t)$. On the other hand, $(I - \mathbb{P}(t))u(t)$ can be estimated by a variant of Matsumura-Nishida energy method as in the case of the stationary parallel flow ([7]). However, in contrast to [7], the linearized operator has time-dependent coefficients. Therefore a modification of the argument in [7] is needed for the time-periodic case to aquire the necessary energy estimate.

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Graduate School of Mathematics
Kyushu University
Fukuoka 819-0395
JAPAN
E-mail address:h.brezina@gmail.com