

# Mathematical justification of the penalty method for viscous incompressible fluid flows\*

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*Dedicated to the memory of Professor Seiji Ukai*

## 1. Introduction

The purpose of this paper is to give a rigorous justification of the *penalty method* for the Navier-Stokes equations in  $\mathbb{R}^d$  ( $d \geq 2$ ).

First of all we shall explain the penalty method which we will discuss and we shall introduce motivation of the present paper. The motion of viscous incompressible fluid is governed by the Navier-Stokes equations.

$$\frac{\partial \mathbf{u}}{\partial t} - \Delta \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = 0, \quad x \in \Omega, t > 0, \quad (1.1a)$$

$$\operatorname{div} \mathbf{u} = 0, \quad x \in \Omega, t \geq 0, \quad (1.1b)$$

where  $\mathbf{u} = (u^1(x, t), \dots, u^d(x, t))$  and  $p = p(x, t)$  denote the velocity field and pressure, respectively;  $\Omega \subseteq \mathbb{R}^d$  ( $d \geq 2$ ) is filled with viscous incompressible fluid. If  $\partial\Omega \neq \emptyset$ , we impose some boundary condition for  $\mathbf{u}$  and  $p$  on the boundary, e.g., non-slip, perfect slip, stress free etc.

In (1.1a) the pressure term does not have time evolutionary structure. This fact is one of the main points of the Navier-Stokes equations. To overcome difficulty caused by such a fact, in mathematical analysis of the Navier-Stokes equations by semigroup approach, we are due to the Helmholtz decomposition and associated projection. In fact, applying the Helmholtz projection  $P$  to (1.1a), (1.1) can be formulated as an abstract evolution equation in some Banach space (e.g.,  $L^2_\sigma$ ,  $L^p_\sigma$ , etc) with solenoidal condition (see e.g., Fujita & Kato [2] and Lemarié-Rieusset [6]).

In numerical computation of the Navier-Stokes equations, we may encounter similar difficulties caused by presence of the pressure term. As an example, we

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consider semi-discretization of (1.1). Applying forward Euler approximation to  $\partial \mathbf{u} / \partial t$  in (1.1a), one can obtain the following difference-differential equations concerning  $\mathbf{U}^n$  and  $P^n$ :

$$\mathbf{U}^{n+1} = \mathbf{U}^n + h(\Delta \mathbf{U}^n - \mathbf{U}^n \cdot \nabla \mathbf{U}^n - \nabla P^n), \quad x \in \Omega, n \geq 0, \quad (1.2a)$$

$$\operatorname{div} \mathbf{U}^n = 0, \quad x \in \Omega, n \geq 0. \quad (1.2b)$$

Here  $(\mathbf{U}^n, P^n)$  is semi-discretized approximation of  $(\mathbf{u}(x, t_n), p(x, t_n))$ , where  $t_n = nh$  and  $h > 0$  is temporal step size of time-discretization. Since the pressure does not have time evolutionary structure in (1.1), we have no rule to compute  $P^n$  from the previous steps directly in (1.2). Hence if we use the above formulation, we need to compute  $P^n$  by (1.2) with some resources. By (1.2b), we see that  $P^n$  satisfies the Poisson type equation:

$$-\Delta P^n = \operatorname{div}(\mathbf{U}^n \cdot \nabla \mathbf{U}^n), \quad x \in \Omega. \quad (1.3)$$

Therefore  $P^n$  is formally given by  $P^n = (-\Delta_\Omega)^{-1} \operatorname{div}(\mathbf{U}^n \cdot \nabla \mathbf{U}^n)$ . However, this representation is non-local one and the boundary condition of the pressure is unclear in general. Thus, such a method requires quite complicate treatment of the pressure term.

In order to compute numerical solution of the Navier-Stokes equations without using complicate treatment of the pressure, the pressure term must be eliminated from (1.1a) as in mathematical analysis. The penalty method introduced by Temam [8] is one of the standard ways to remove pressure term from (1.1a) and is widely used in numerical computation of viscous incompressible fluid flows.

In the penalty method, the equation of continuity (1.1b) is replaced by the following one concerning  $\mathbf{u}$  and  $p$ .

$$\operatorname{div} \mathbf{u} = -\frac{p}{\eta}, \quad \eta > 0, \quad (1.4)$$

where  $\eta$  is assumed to be very large. Substituting  $p = -\eta \operatorname{div} \mathbf{u}$  into (1.1a), we have an approximate problem of the Navier-Stokes equations only in terms of the velocity  $\mathbf{u}$ . Therefore numerical treatment for such an approximate problem is much easier than that for original problem, because we are not required to treat the pressure term directly.

Letting  $\eta \rightarrow \infty$  in (1.4), we formally have (1.1b). So we expect that the system of (1.1a) and (1.4) gives a good approximate solution of (1.1). However the above argument is nothing but formal one, we have to justify the penalty method by mathematical viewpoints.

For such a problem, Temam [8] studied stationary flow in bounded domain and gave a rigorous justification. For nonstationary flow Shen [7] gave a justification for  $L^2$  strong solution in bounded domain with nonslip boundary condition. However as far as the author knows there are no results for unbounded domain cases. Our main problem is to justify the penalty method in the case of  $\Omega$  is unbounded domain. As a starting point of this study, we mainly consider the Cauchy problem of the Navier-Stokes equations.

This paper is organized as follows. In Section 2 we will state our main results of the present paper. In Section 3 we will consider the Stokes flow with the penalty method which is linearized problem of penalized Navier-Stokes equations. We will establish key estimates in this paper and show error becomes small when  $\eta$  goes to large for the Stokes flow. In Section 4 we will discuss the Navier-Stokes flow and show our main theorem with the aid of key estimates will be shown in Section 3.

## 2. Main results

### 2.1. Notation and the Helmholtz decomposition

Before stating our main results in the present paper, we shall introduce notation and the Helmholtz decomposition in  $\mathbb{R}^d$ .  $C_0^\infty(\mathbb{R}^d)$  denotes the set of all infinitely differentiable function with compact support in  $\mathbb{R}^d$ . For  $1 \leq r \leq \infty$ ,  $L^r(\mathbb{R}^d)$  denotes usual Lebesgue space. To denote function spaces for vector field, we use the following symbols:  $C_0^\infty(\mathbb{R}^d)^d$ ,  $L^r(\mathbb{R}^d)^d$ , etc.

To denote various constants, we use the same letters  $C$  and  $C_{a,b,c,\dots}$  which means that the constant depends on  $a, b, c, \dots$ . The constants  $C$  and  $C_{a,b,c,\dots}$  may change one line to another lines.

Next we shall introduce the Helmholtz decomposition. The Helmholtz decomposition plays an essential role in our arguments. Let  $1 < r < \infty$ . Then it is well known that  $L^r(\mathbb{R}^d)^d$  admits the Helmholtz decomposition:

$$L^r(\mathbb{R}^d)^d = L_\sigma^r(\mathbb{R}^d) \oplus G^r(\mathbb{R}^d) \quad \oplus : \text{direct sum.}$$

Here and hereafter

$$\begin{aligned} L_\sigma^r(\mathbb{R}^d) &= \overline{C_{0,\sigma}^\infty(\mathbb{R}^d)}^{\|\cdot\|_{L^r(\mathbb{R}^d)}} \\ &= \{\mathbf{f} \in L^r(\mathbb{R}^d)^d \mid \operatorname{div} \mathbf{f} = 0 \text{ (in the sense of distribution)}\}, \\ G^r(\mathbb{R}^d) &= \{\mathbf{f} = \nabla \varphi \mid \varphi \in \hat{W}^{1,r}(\mathbb{R}^d)\}. \end{aligned}$$

Here  $C_{0,\sigma}^\infty(\mathbb{R}^d) = \{\mathbf{f} \in C_0^\infty(\mathbb{R}^d)^d \mid \operatorname{div} \mathbf{f} = 0\}$  and  $\hat{W}^{1,r}(\mathbb{R}^d)$  is homogeneous Sobolev space:

$$\hat{W}^{1,r}(\mathbb{R}^d) = \{\varphi \in L_{\text{loc}}^r(\mathbb{R}^d) \mid \nabla \varphi \in L^r(\mathbb{R}^d)^d\}.$$

Let  $P = P_{r,\mathbb{R}^d}$  be a continuous projection from  $L^r(\mathbb{R}^d)^d$  into  $L_\sigma^r(\mathbb{R}^d)$  ( $1 < r < \infty$ ). It is well known that  $P_r$  is bounded linear operator from  $L^r(\mathbb{R}^d)^d$  into  $L_\sigma^r(\mathbb{R}^d)$ . To give a reformulation of the Stokes and Navier-Stokes equations, we set  $Q = Q_{r,\mathbb{R}^d} := I - P_r$ .  $Q_r$  is also bounded linear operator from  $L^r(\mathbb{R}^d)^d$  into  $G^r(\mathbb{R}^d)$ .

For the homogeneous Sobolev space  $\hat{W}^{1,r}(\mathbb{R}^d)$ , the following fact is known (see e.g., Farwig & Sohr [1], Galdi [3]).

**Lemma 2.1.**  $C_0^\infty(\mathbb{R}^d)$  is dense in  $\hat{W}^{1,r}(\mathbb{R}^d)$  with respect to the Dirichlet norm, that is, for any  $\varepsilon > 0$ , there exists  $\varphi_\varepsilon \in C_0^\infty(\mathbb{R}^d)$  such that  $\|\nabla(\varphi - \varphi_\varepsilon)\|_r < \varepsilon$  for any  $\varphi \in \hat{W}^{1,r}(\mathbb{R}^d)$ .

The above lemma plays a crucial role to show decay estimate of the solution to penalized Stokes flow in terms of  $\eta$ .

## 2.2. Results

We are now in a position to state our main result of this paper. The first result is concerning the Stokes equations.

**Theorem 2.2.** Let  $1 < r < \infty$ . Let  $(\mathbf{u}(t), p(t))$  be solution to the Stokes equations with initial data  $\mathbf{u}_0 \in L_\sigma^r(\mathbb{R}^d)$  and let  $\mathbf{u}^\eta(t)$  be solution to the penalized Stokes equations with initial data  $\mathbf{u}_0^\eta \in L^r(\mathbb{R}^d)^d$ . Then there holds that

$$\lim_{\eta \rightarrow \infty} \|\mathbf{u}^\eta(t) - \mathbf{u}(t)\|_r \leq C \|P\mathbf{u}_0^\eta - \mathbf{u}_0\|_r, \quad (2.1)$$

$$\lim_{\eta \rightarrow \infty} \|\nabla(p^\eta(t) - p(t))\|_r = 0 \quad (2.2)$$

for any  $t > 0$ , where  $p^\eta(t) = -\eta \operatorname{div} \mathbf{u}^\eta(t)$ . In particular, if we take initial data for the penalized Stokes equations in such a way that  $\mathbf{u}_0^\eta = \mathbf{u}_0 \in L_\sigma^r(\mathbb{R}^d)$ , we have

$$\lim_{\eta \rightarrow \infty} \|\mathbf{u}^\eta(t) - \mathbf{u}(t)\|_r = 0 \quad (2.3)$$

for any  $t > 0$ .

Next result is our main result concerning the Navier-Stokes initial value problem of this paper.

**Theorem 2.3.** Let  $\mathbf{u}(t) \in C([0, \infty); L^d_\sigma(\mathbb{R}^d))$  be global-in-time mild solution of the Navier-Stokes initial value problem with initial velocity  $\mathbf{u}_0 \in L^d_\sigma(\mathbb{R}^d)$  ( $\|\mathbf{u}_0\|_d \ll 1$ ) and let  $\mathbf{u}^\eta(t) \in C([0, \infty); L^d(\mathbb{R}^d)^d)$  be global-in-time mild solution of the penalized Navier-Stokes initial value problem with initial data  $\mathbf{u}_0^\eta \in L^d(\mathbb{R}^d)^d$  ( $\|\mathbf{u}_0^\eta\|_d \ll 1$ ). Then the following estimate holds.

$$\lim_{\eta \rightarrow \infty} \|\mathbf{u}^\eta(t) - \mathbf{u}(t)\|_d \leq C \|P\mathbf{u}_0^\eta - \mathbf{u}_0\|_d \quad (2.4)$$

for any  $t > 0$ . In particular, if we take  $\mathbf{u}_0^\eta = \mathbf{u}_0 \in L^d_\sigma(\mathbb{R}^d)$ , we have

$$\lim_{\eta \rightarrow \infty} \|\mathbf{u}^\eta(t) - \mathbf{u}(t)\|_d = 0 \quad (2.5)$$

for any  $t > 0$ .

### 3. Linearized problem (the Stokes flow)

For a justification of the penalty method for the Cauchy problem of the Navier-Stokes equations, we shall justify the penalty method for the linearized problem (the Stokes equations) and establish some key estimates which will be used later.

In order to do so, first we shall give a reformulation of the penalized Stokes equations.

$$\frac{\partial \mathbf{u}^\eta}{\partial t} - \Delta \mathbf{u}^\eta - \nabla \operatorname{div} \mathbf{u}^\eta = \mathbf{0}, \quad x \in \mathbb{R}^d, t > 0, \quad (3.1a)$$

$$\mathbf{u}^\eta(x, 0) = \mathbf{u}_0^\eta, \quad x \in \mathbb{R}^d. \quad (3.1b)$$

Here and in what follows  $\mathbf{u}_0^\eta = \mathbf{u}_0^\eta(x)$  is given initial velocity.

To give a reformulation of (3.1), we are due to the Helmholtz decomposition of  $L^r$ -vector fields. By the Helmholtz decomposition  $\mathbf{u}^\eta \in L^r(\mathbb{R}^d)^d$  ( $1 < r < \infty$ ) is decomposed into  $\mathbf{u}^\eta = \mathbf{v}^\eta + \mathbf{w}^\eta$ , where  $\mathbf{v}^\eta = P\mathbf{u}^\eta \in L^r_\sigma(\mathbb{R}^d)$  and  $\mathbf{w}^\eta = \nabla \varphi^\eta \in G^r(\mathbb{R}^d)$ ,  $\varphi^\eta \in \hat{W}^{1,r}(\mathbb{R}^d)$ .

Applying  $P_r$  and  $Q_r$  to (3.1a), (3.1) is decoupled into the following two initial value problems in  $L^r_\sigma(\mathbb{R}^d)$  and  $G^r(\mathbb{R}^d)$ , respectively.

$$\frac{\partial \mathbf{v}^\eta}{\partial t} - \Delta \mathbf{v}^\eta = \mathbf{0}, \quad (3.2a)$$

$$\frac{\partial \mathbf{w}^\eta}{\partial t} - (1 + \eta)\Delta \mathbf{w}^\eta = \mathbf{0}, \quad (3.2b)$$

$$\mathbf{v}^\eta(x, 0) = \mathbf{v}_0^\eta := P\mathbf{u}_0^\eta, \quad \mathbf{w}^\eta(x, 0) = \mathbf{w}_0^\eta := Q\mathbf{u}_0^\eta. \quad (3.2c)$$

Here we have used the facts that  $P$ ,  $Q$  and spatial derivative  $\partial_{x_j}$  commutes each other in  $\mathbb{R}^d$  and  $\nabla \operatorname{div} \mathbf{w}^\eta = \nabla \operatorname{div} \nabla \varphi^\eta = \nabla \Delta \varphi^\eta = \Delta \nabla \varphi^\eta = \Delta \mathbf{w}^\eta$ .

**Remark 3.1.** The above reformulation does not work in general domains, because we used the fact that  $P$ ,  $Q$  and  $\partial_{x_j}$  are commutable.

### 3.1. Estimate of solution

To justify the penalty method, we need a good error estimate between  $(\mathbf{u}(t), p(t))$  and  $(\mathbf{u}^\eta(t), p^\eta(t))$ , where  $p^\eta(t) = -\eta \operatorname{div} \mathbf{u}^\eta(t)$ .

For such a purpose, we observe  $\eta$ -dependence of solution to (3.1). Of course it suffices to get such one for (3.2a) and (3.2b), respectively. In order to get  $\eta$ -dependence of solution to (3.2a) and (3.2b), we consider the following initial value problem of the linear diffusion equation as a model problem.

$$\frac{\partial y}{\partial t} - \nu \Delta y = 0 \quad x \in \mathbb{R}^d, t > 0 \quad (3.3a)$$

$$y(x, 0) = y_0, \quad x \in \mathbb{R}^d. \quad (3.3b)$$

Here  $y = y(x, t; \nu)$  is unknown and  $y_0 = y_0(x)$  is given initial datum.  $\nu > 0$  denotes the diffusivity. It is well known that the solution of (3.3) is given by

$$y(x, t; \nu) = e^{\nu t \Delta} y_0(x) := \frac{1}{(4\pi \nu t)^{d/2}} \int_{\mathbb{R}^d} \exp\left(-\frac{|x - \xi|^2}{4\nu t}\right) y_0(\xi) d\xi \quad (3.4)$$

(see e.g., Giga, Giga & Saal [4]).  $e^{\nu t \Delta}$  is standard notation of the heat semigroup. For the heat semigroup  $e^{\nu t \Delta}$ , the following  $L^r$ - $L^q$  estimates follows from Hausdorff-Young's inequality.

**Lemma 3.2** ( $L^r$ - $L^q$  estimates). Let  $1 \leq r \leq q \leq \infty$ . Then the following  $L^r$ - $L^q$  type estimate holds for any  $t > 0$ .

$$\|\partial_t^j \partial_x^\alpha y^\nu(\cdot, t)\|_q \leq C_{q,r} t^{-\frac{d}{2}(\frac{1}{r}-\frac{1}{q})-\frac{|\alpha|}{2}-j} \nu^{-\frac{d}{2}(\frac{1}{r}-\frac{1}{q})-\frac{|\alpha|}{2}} \|y_0\|,$$

where  $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$  is multi-index and  $j \in \mathbb{N}_0$ .

As a consequence of Lemma 3.2, we have the following estimates for  $\mathbf{v}^\eta(t)$  and  $\mathbf{w}^\eta(t)$ .

**Lemma 3.3.** Let  $1 < r \leq q \leq \infty, r \neq \infty$ . Then there hold the following estimates.

$$\|\partial_t^j \partial_x^\alpha \mathbf{v}^\eta(t)\|_q \leq C_{r,q,\alpha,j} t^{-\frac{d}{2}(\frac{1}{r}-\frac{1}{q})-\frac{|\alpha|}{2}-j} \|\mathbf{v}_0^\eta\|_r, \quad (3.5)$$

$$\|\partial_t^j \partial_x^\alpha \mathbf{w}^\eta(t)\|_q \leq C_{r,q,\alpha,j} (1 + \eta)^{-\frac{d}{2}(\frac{1}{r}-\frac{1}{q})-\frac{|\alpha|}{2}} t^{-\frac{d}{2}(\frac{1}{r}-\frac{1}{q})-\frac{|\alpha|}{2}-j} \|\mathbf{w}_0^\eta\|_r \quad (3.6)$$

for any  $t > 0$ , where  $j \in \mathbb{N}_0$  and  $\alpha \in \mathbb{N}_0^d$ .

**Remark 3.4.** Taking  $q = r, j = 0, \alpha = \{0\}^d$  in (3.6), we have only the boundedness of  $\mathbf{w}^\eta(t)$ :  $\|\mathbf{w}^\eta(t)\|_r \leq C_r \|\mathbf{w}_0^\eta\|_r$ . This boundedness is not enough to guarantee the penalty method for the Stokes equations.

In order to guarantee the penalty method for the Stokes equations, we need to refine the above estimate. To refine the estimate, we are due to the density argument. For any  $\varepsilon > 0$ , there exists  $\varphi_{0,\varepsilon} \in C_0^\infty(\mathbb{R}^d)$  such that

$$\|\mathbf{w}_0^\eta - \nabla \varphi_{0,\varepsilon}\|_r = \|\nabla(\varphi_0^\eta - \varphi_{0,\varepsilon})\|_r < \varepsilon \quad (3.7)$$

for any  $r \in (1, \infty)$ . Such a fact follows from Lemma 2.1.

By triangle inequality with (3.7), Lemma 3.3 and analytic semigroup property of the heat semigroup  $e^{t(1+\eta)\Delta}$ , we have

$$\begin{aligned} \|\mathbf{w}^\eta(t)\|_r &= \|e^{t(1+\eta)\Delta} \mathbf{w}_0^\eta\|_r \\ &\leq \|e^{t(1+\eta)\Delta} (\nabla \varphi_0^\eta - \varphi_{0,\varepsilon})\|_r + \|e^{t(1+\eta)\Delta} \nabla \varphi_{0,\varepsilon}\|_r \\ &\leq C\varepsilon + \|\nabla e^{t(1+\eta)\Delta} \varphi_{0,\varepsilon}\|_r \\ &\leq C\varepsilon + C(1+\eta)^{-\frac{d}{2}(\frac{1}{s}-\frac{1}{r})-\frac{1}{2}t-\frac{d}{2}(\frac{1}{s}-\frac{1}{r})-\frac{1}{2}} \|\varphi_{0,\varepsilon}\|_s \end{aligned}$$

for  $t > 0$ , where  $s \in (1, r]$ . Here we have used the fact that  $\varphi_{0,\varepsilon} \in C_0^\infty(\mathbb{R}^d) \subset L^s(\mathbb{R}^d)$ . Let us fix  $t_0 > 0$ . Then we have  $\limsup_{\eta \rightarrow \infty} \|\mathbf{w}^\eta(t)\|_r \leq C\varepsilon$  for any  $t \geq t_0 > 0$ .

Since  $\varepsilon > 0$  can be chosen arbitrary, we have desired result.

$$\lim_{\eta \rightarrow \infty} \|\mathbf{w}^\eta(t)\|_r = 0 \quad (3.8)$$

for any  $t \geq t_0 > 0$  and  $r \in (1, \infty)$ , provided that  $\mathbf{u}_0^\eta \in L^r(\mathbb{R}^d)^d$ .

### 3.2. Error estimate (proof of Theorem 2.2)

We are now in a position to show error estimate for the Stokes equations. Let  $(\mathbf{u}, p)$  be solution to the Stokes equations with initial datum  $\mathbf{u}_0 \in L^r_\sigma(\mathbb{R}^d)$  and  $\mathbf{u}^\eta$  be solution to the penalized Stokes equations with initial datum  $\mathbf{u}_0^\eta \in L^r(\mathbb{R}^d)^d$ , where  $r \in (1, \infty)$ .

Set  $\mathbf{U}^\eta := \mathbf{u}^\eta - \mathbf{u}$  and  $\Pi^\eta := p^\eta - p$ . Then  $(\mathbf{U}^\eta, \Pi^\eta)$  satisfies

$$\frac{\partial \mathbf{U}^\eta}{\partial t} - \Delta \mathbf{U}^\eta + \nabla \Pi^\eta = \mathbf{0}, \quad x \in \mathbb{R}^d, t > 0, \quad (3.9a)$$

$$\operatorname{div} \mathbf{U}^\eta = \operatorname{div} \mathbf{u}^\eta = -\frac{p^\eta}{\eta}, \quad x \in \mathbb{R}^d, t > 0, \quad (3.9b)$$

$$\mathbf{U}^\eta(x, 0) = \mathbf{U}_0^\eta := \mathbf{u}_0^\eta - \mathbf{u}_0, \quad x \in \mathbb{R}^d. \quad (3.9c)$$

Applying  $P_r$  and  $Q_r$ , (3.9) is decoupled into two initial value problems:

$$\frac{\partial \mathcal{E}^\eta}{\partial t} - \Delta \mathcal{E}^\eta = \mathbf{0}, \quad \operatorname{div} \mathcal{E}^\eta = 0, \quad (3.10a)$$

$$\mathcal{E}^\eta(x, 0) = \mathcal{E}_0^\eta := \mathbf{v}_0^\eta - \mathbf{u}_0 \in L^r_\sigma(\mathbb{R}^d) \quad (3.10b)$$

and (3.2b), because  $\nabla p = 0$  in  $G^r(\mathbb{R}^d)$ . By the triangle inequality and the boundedness of heat semigroup,  $\|U^\eta(t)\|_r$  is estimated by

$$\|U^\eta(t)\|_r \leq \|\mathcal{E}^\eta(t)\|_r + \|\mathbf{w}^\eta(t)\|_r \leq C \|\mathcal{E}_0^\eta\|_r + \|\mathbf{w}^\eta(t)\|_r.$$

Hence by (3.8), we obtain

$$\lim_{\eta \rightarrow \infty} \|U^\eta(t)\|_r \leq C \|\mathcal{E}_0^\eta\|_r \quad (3.11)$$

for any  $t \geq t_0 > 0$ .

Next we shall estimate  $L^r$ -norm of the pressure gradient  $\nabla \Pi^\eta$ . Since  $\operatorname{div} \mathbf{u}^\eta = \operatorname{div} \mathbf{w}^\eta$  and  $\nabla p = 0$  in  $G^r(\mathbb{R}^d)$ ,  $\nabla \Pi^\eta = \nabla p^\eta = -\eta \nabla \operatorname{div} \mathbf{w}^\eta$ . Hence, by virtue of (3.6) and semigroup property of  $e^{t(1+\eta)\Delta}$ , we have

$$\begin{aligned} \|\nabla \Pi^\eta(t)\|_r &\leq \eta \|\nabla^2 \mathbf{w}^\eta(t)\|_r = \eta \|\nabla^2 e^{\frac{(1+\eta)t\Delta}{2}} e^{\frac{(1+\eta)t\Delta}{2}} \mathbf{w}_0^\eta\|_r \\ &\leq C \frac{\eta}{1+\eta} t^{-1} \left\| \mathbf{w}^\eta \left( \frac{t}{2} \right) \right\|_r \end{aligned}$$

Combining the above estimate and (3.8), we have

$$\lim_{\eta \rightarrow \infty} \|\nabla \Pi^\eta(t)\|_r = 0 \quad (3.12)$$

for any  $t \geq t_0 > 0$ .

(3.11) implies that if  $\|\mathbf{v}_0^\eta - \mathbf{u}_0\|_r$  is small enough, then error between  $\mathbf{u}^\eta(t)$  and  $\mathbf{u}(t)$  is also small enough. Therefore (3.11) and (3.12) give us a mathematical justification of the penalty method for the Stokes equations.

#### 4. Proof of main results

This section is devoted to the proof of Theorem 2.3. We consider the penalized Navier-Stokes initial value problem.

$$\frac{\partial \mathbf{u}^\eta}{\partial t} - \Delta \mathbf{u}^\eta - \eta \nabla \operatorname{div} \mathbf{u}^\eta + \mathbf{u}^\eta \cdot \nabla \mathbf{u}^\eta = \mathbf{0}, \quad x \in \mathbb{R}^d, t > 0, \quad (4.1a)$$

$$\mathbf{u}^\eta(x, 0) = \mathbf{u}_0^\eta, \quad x \in \mathbb{R}^d. \quad (4.1b)$$



Let  $\mathcal{L}_\eta \mathbf{u} := -\Delta \mathbf{u} - \eta \nabla \operatorname{div} \mathbf{u}$  for  $\mathbf{u} \in D(\mathcal{L}_\eta) = W^{2,r}(\mathbb{R}^d)^d$  ( $1 < r < \infty$ ).  $\mathcal{L}_\eta$  is called Lamé operator. It is well known that  $-\mathcal{L}_\eta$  generates an analytic semigroup  $(e^{-t\mathcal{L}_\eta})_{t \geq 0}$  on  $L^r(\mathbb{R}^d)^d$  and  $e^{-t\mathcal{L}_\eta}$  enjoys usual  $L^r$ - $L^q$  estimates like the heat semigroup  $e^{t\Delta}$ . Furthermore (4.1) has the same scaling property as original Navier-Stokes equations. Therefore one can construct global-in-time mild solution for the penalized Navier-Stokes equations, provided that the initial velocity  $\mathbf{u}_0^\eta$  satisfies suitable smallness condition:  $\|\mathbf{u}_0\|_d \ll 1$  (By similar argument, one can show that local in time existence for large initial data if we choose existence time  $T > 0$  small enough. In what follows, we only consider global mild solution).

By using  $e^{-t\mathcal{L}_\eta}$ ,  $\eta$ -dependence of  $\mathbf{u}^\eta(t)$  may be hidden. In order to show that the penalty method works well for the Navier-Stokes initial value problem, careful analysis on the  $\eta$ -dependence of solution  $\mathbf{u}^\eta$  is important.

#### 4.1. Construction of mild solutions

To know  $\eta$ -dependence of solution, we shall construct mild solution of the penalized Navier-Stokes equations without using  $\mathcal{L}_\eta$ . In what follows, we consider the following system of abstract evolution equations.

$$\frac{d\mathbf{v}^\eta}{dt} = \Delta \mathbf{v}^\eta - P(\mathbf{u}^\eta \cdot \nabla \mathbf{u}^\eta), \quad (4.2a)$$

$$\frac{d\mathbf{w}^\eta}{dt} = (1 + \eta)\Delta \mathbf{w}^\eta - Q(\mathbf{u}^\eta \cdot \nabla \mathbf{u}^\eta), \quad (4.2b)$$

where  $\mathbf{u}^\eta(t) = P\mathbf{u}^\eta(t) + (I - P)\mathbf{u}^\eta(t) = \mathbf{v}^\eta(t) + \mathbf{w}^\eta(t)$ .

By Duhamel's principle, (4.2) is converted into the following system of integral equations.

$$\begin{aligned} \mathbf{v}^\eta(t) &= e^{t\Delta} \mathbf{v}_0^\eta - \int_0^t e^{(t-s)\Delta} P(\mathbf{u}^\eta(s) \cdot \nabla \mathbf{u}^\eta(s)) ds \\ &=: \mathbf{v}^0(t) + N_1(\mathbf{u})(t), \end{aligned} \quad (4.3a)$$

$$\begin{aligned} \mathbf{w}^\eta(t) &= e^{t(1+\eta)\Delta} \mathbf{w}_0^\eta - \int_0^t e^{(t-s)(1+\eta)\Delta} Q(\mathbf{u}^\eta(s) \cdot \nabla \mathbf{u}^\eta(s)) ds \\ &=: \mathbf{w}^0(t) + N_2(\mathbf{u})(t). \end{aligned} \quad (4.3b)$$

For (4.3), we have a result on small data global existence.

**Lemma 4.1.** Let  $\mathbf{u}_0^\eta \in L^d(\mathbb{R}^d)^d$ , that is,  $(\mathbf{v}_0^\eta, \mathbf{w}_0^\eta) \in L_\sigma^r(\mathbb{R}^d) \times G^r(\mathbb{R}^d)$ . Then there exists a  $\delta > 0$  such that if  $\|\mathbf{u}_0^\eta\|_d < \delta$  then there exists a unique mild solution

of (4.3)  $(\mathbf{v}^\eta(t), \mathbf{w}^\eta(t)) \in C([0, \infty); L_\sigma^d(\mathbb{R}^d) \times G^d(\mathbb{R}^d))$  which enjoys

$$\begin{aligned} \lim_{t \rightarrow +0} \|(\mathbf{v}^\eta(t), \mathbf{w}^\eta(t)) - (\mathbf{v}_0^\eta, \mathbf{w}_0^\eta)\|_d &= 0, \\ \|(\mathbf{v}^\eta(t), \mathbf{w}^\eta(t))\|_r &= O\left(t^{-\frac{1}{2} + \frac{d}{2r}}\right), \quad d \leq r < \infty, \\ \|\nabla(\mathbf{v}^\eta(t), \mathbf{w}^\eta(t))\|_d &= O\left(t^{-\frac{1}{2}}\right) \end{aligned}$$

as  $t \rightarrow +\infty$  for any fixed  $\eta > 0$ . Furthermore, the above mild solution satisfies

$$\|\mathbf{w}^\eta(t)\|_r = O\left(\eta^{-\frac{1}{2} + \frac{d}{2r}}\right), \quad d \leq r < \infty \quad (4.4)$$

as  $\eta \rightarrow \infty$  for any fixed  $t \geq 0$ .

To show Lemma 4.1, we are due to Banach's fixed point theorem with the aid of  $L^r$ - $L^q$  estimates for linearized problem (such an argument is essentially the same as Kato's iteration scheme [5]).

Let  $\Phi$  be defined by

$$\Phi(\mathbf{u}^\eta) := \begin{bmatrix} \mathbf{v}^0(t) \\ \mathbf{w}^0(t) \end{bmatrix} + \begin{bmatrix} N_1(\mathbf{u}^\eta)(t) \\ N_2(\mathbf{u}^\eta)(t) \end{bmatrix} \quad (4.5)$$

and let us set

$$\begin{aligned} |\mathbf{u}^\eta|_{\ell, q, t} &:= \sup_{0 < s \leq t} s^\ell (\|\mathbf{v}^\eta(s)\|_q + (1 + \eta)^\ell \|\mathbf{w}^\eta(s)\|_q), \\ [\mathbf{u}^\eta]_t &:= |\mathbf{u}^\eta|_{\frac{1}{2} - \frac{d}{2r}, r, t} + |\nabla \mathbf{u}^\eta|_{\frac{1}{2}, d, t}, \\ |||\mathbf{u}^\eta|||_t &:= |\mathbf{u}^\eta|_{0, d, t} + [\mathbf{u}^\eta]_t. \end{aligned}$$

Our first task is to show unique existence of the fixed point of mapping  $\Phi$ . As an underlying space, let us introduce  $X_R$  as follows.

$$X_R := \{(\mathbf{v}^\eta(t), \mathbf{w}^\eta(t)) \in C([0, \infty); L_\sigma^d(\mathbb{R}^d) \times G^d(\mathbb{R}^d)) \mid \lim_{t \rightarrow +0} \|\mathbf{v}^\eta(t) - \mathbf{v}_0^\eta\|_d = 0, \quad \lim_{t \rightarrow +0} \|\mathbf{w}^\eta(t) - \mathbf{w}_0^\eta\|_d = 0, \quad (4.6)$$

$$\lim_{t \rightarrow +0} |\mathbf{u}^\eta|_{\frac{1}{2} - \frac{d}{2r}, r, t} = 0, \quad \lim_{t \rightarrow +0} |\nabla \mathbf{u}^\eta|_{\frac{1}{2}, d, t} = 0, \quad (4.7)$$

$$\sup_{t > 0} |||\Phi(\mathbf{v}^\eta, \mathbf{w}^\eta)|||_t \leq 2R \|\mathbf{u}_0^\eta\|_d\}, \quad (4.8)$$

where  $r \in (d, \infty)$  and  $R > 0$  will be determined later.

In order to find a fixed point of  $\Phi$  on  $X_R$ , we first consider initial flows  $\mathbf{v}^0(t)$  and  $\mathbf{w}^0(t)$ . Since  $C_{0,\sigma}^\infty(\mathbb{R}^d)$  is dense in  $L_\sigma^d(\mathbb{R}^d)$ , for any  $\varepsilon > 0$  there exists  $\mathbf{v}_{0,\varepsilon} \in C_{0,\sigma}^\infty(\mathbb{R}^d)$  such that  $\|\mathbf{v}_0^\eta - \mathbf{v}_{0,\varepsilon}\|_d < \varepsilon$ . Hence by  $L^d$ -boundedness of the heat semigroup  $e^{t\Delta}$  and triangle inequality, we obtain

$$\begin{aligned} \|\mathbf{v}^\eta(t) - \mathbf{v}_0^\eta\|_d &\leq \|e^{t\Delta}(\mathbf{v}_0^\eta - \mathbf{v}_{0,\varepsilon})\|_d + \|e^{t\Delta}\mathbf{v}_{0,\varepsilon} - \mathbf{v}_{0,\varepsilon}\|_d + \|\mathbf{v}_{0,\varepsilon} - \mathbf{v}_0^\eta\|_d \\ &\leq C_d\varepsilon + \int_0^t \left\| \frac{d}{ds} e^{s\Delta} \mathbf{v}_{0,\varepsilon} \right\|_d ds \\ &\leq C_d\varepsilon + Ct \|\mathbf{v}_{0,\varepsilon}\|_{W^{2,d}(\mathbb{R}^d)}. \end{aligned}$$

This implies that  $\lim_{t \rightarrow +0} \sup_{0 < s < t} \|\mathbf{v}^\eta(t) - \mathbf{v}_0^\eta\|_d \leq C\varepsilon$ . Since  $\varepsilon > 0$  can be chosen arbitrary, we can conclude that  $\lim_{t \rightarrow +0} \|\mathbf{v}^\eta(t) - \mathbf{v}_0^\eta\|_d = 0$ . By similar manners, (4.6) and (4.7) can be verified.

Next we shall estimate the Duhamel terms  $N_1(\mathbf{u})(t)$  and  $N_2(\mathbf{u})(t)$ . Let  $r > d$  and  $q$  satisfy  $1/q = 1/r + 1/d$ . Then by using  $L^r$ - $L^q$  estimate (Lemma 3.3) and the Hölder inequality, we have the following estimate for  $N_1(\mathbf{u})(t)$ .

$$\begin{aligned} \|N_1(\mathbf{u}^\eta)(t)\|_r &\leq \int_0^t \|e^{(t-s)\Delta} P(\mathbf{u}^\eta(s) \cdot \nabla \mathbf{u}^\eta(s))\|_r ds \\ &\leq C_{r,d} \int_0^t (t-s)^{-\frac{1}{2}} \|\mathbf{u}^\eta(s)\|_r \|\nabla \mathbf{u}^\eta(s)\|_d ds \\ &\leq C_{r,d} \int_0^t (t-s)^{-\frac{1}{2}} s^{-1+\frac{d}{2r}} ds [\mathbf{u}^\eta]_t^2 \\ &\leq C_{r,d} B\left(\frac{1}{2}, \frac{d}{2r}\right) t^{-\frac{1}{2}+\frac{d}{2r}} [\mathbf{u}^\eta]_t^2. \end{aligned}$$

Here and hereafter  $B(\alpha, \beta)$  denotes Euler's beta function. By similar manners, we obtain

$$\|N_1(\mathbf{u}^\eta)(t)\|_d \leq C[\mathbf{u}^\eta]_t^2, \quad \|\nabla N_1(\mathbf{u}^\eta)(t)\|_d \leq Ct^{-\frac{1}{2}}[\mathbf{u}^\eta]_t^2.$$

Estimates for  $N_2(\mathbf{u}^\eta)(t)$  are also follows from similar arguments. In fact, by using Lemma 3.3, we obtain

$$\begin{aligned} \|N_2(\mathbf{u})(t)\|_r &\leq \int_0^t \|e^{(1+\eta)(t-s)\Delta} P(\mathbf{u}^\eta(s) \cdot \nabla \mathbf{u}^\eta(s))\|_r ds \\ &\leq C_{r,d}(1+\eta)^{-\frac{1}{2}} \int_0^t (t-s)^{-\frac{1}{2}} s^{-1+\frac{d}{2r}} ds [\mathbf{u}^\eta]_t^2 \\ &\leq C_{r,d}(1+\eta)^{-\frac{1}{2}+\frac{d}{2r}} [\mathbf{u}^\eta]_t^2. \end{aligned}$$

Here we have used the fact that  $1 + \eta > 1$ . By similar arguments, the following estimates are also guaranteed.

$$\begin{aligned}\|N_2(\mathbf{u}^\eta)(t)\|_d &\leq C(1 + \eta)^{-\frac{d}{2r}} [\mathbf{u}^\eta]_t^2 \leq C[\mathbf{u}^\eta]_t^2, \\ \|\nabla N_2(\mathbf{u}^\eta)(t)\|_d &\leq C(1 + \eta)^{-\frac{1}{2} - \frac{d}{2r}} t^{-\frac{1}{2}} [\mathbf{u}^\eta]_t^2 \leq C(1 + \eta)^{-\frac{1}{2}} t^{-\frac{1}{2}} [\mathbf{u}^\eta]_t^2.\end{aligned}$$

Summing up the above estimates, if  $(\mathbf{v}^\eta, \mathbf{w}^\eta) \in X_R$  then there holds

$$\begin{aligned}\|\|\Phi(\mathbf{u}^\eta)\|\|_t &\leq R\|\mathbf{u}_0^\eta\|_d + C[\mathbf{u}^\eta]_t^2 \\ &\leq R\|\mathbf{u}_0^\eta\|_d + 4CR^2\|\mathbf{u}_0^\eta\|_d^2\end{aligned}$$

for any  $t > 0$  and  $\eta > 0$ . Therefore if we choose  $\delta > 0$  in such a way that  $4CR\delta < 1$ , then we have

$$\|\|\Phi(\mathbf{u}^\eta)\|\|_t \leq 2R\|\mathbf{u}_0^\eta\|_d$$

for any  $t > 0$  and  $\eta > 0$ . This implies that  $\Phi(\mathbf{u}^\eta) \in X_R$ , provided that  $\mathbf{u}^\eta = (\mathbf{v}^\eta, \mathbf{w}^\eta) \in X_R$ .

Since a similar argument works well for the difference  $\Phi(\mathbf{u}^\eta) - \Phi(\tilde{\mathbf{u}}^\eta)$ , we can conclude that  $\Phi$  becomes contraction mapping on  $X_R$  into itself. Therefore existence of fixed point of mapping  $\Phi$  is follows from Banach's fixed point theorem. Such a fixed point gives global mild solution of (4.3). Uniqueness of solution also follows from property of fixed point.

Let  $t_0 > 0$  be fixed arbitrary. Then by the above construction of mild solution, we see that  $\mathbf{w}^\eta(t)$  satisfies  $\lim_{\eta \rightarrow \infty} \|\mathbf{w}^\eta(t)\|_r = 0$  for  $r \in (d, \infty)$  and any  $t \geq t_0 > 0$ .

However it is not enough to guarantee the penalty method for the Navier-Stokes equations. As in the case for Stokes equations, we have to show that  $\mathbf{w}^\eta(t)$  satisfies

$$\lim_{\eta \rightarrow \infty} \|\mathbf{w}^\eta(t)\|_d = 0 \quad (4.9)$$

for any  $t \geq t_0 > 0$ . To show (4.9), we first show such a result for  $(\mathbf{v}_0^\eta, \mathbf{w}_0^\eta) \in C_{0,\sigma}^\infty(\mathbb{R}^d) \times C_0^\infty(\mathbb{R}^d)^d$ . Taking  $q \in (d/2, d)$  and set  $\sigma = d/2q - 1/2$  (i.e.,  $\sigma$  satisfies  $0 < \sigma < 1/2$ ), by Lemma 4.1,  $L^q$ - $L^d$  estimate and  $L^{\frac{d}{2}}$ - $L^d$  estimate, we have

$$\begin{aligned}\|\mathbf{v}^\eta(t)\|_d &\leq Ct^{-\sigma} \|\mathbf{v}_0^\eta\|_q + C \int_0^t (t-s)^{-\frac{1}{2}} \|\mathbf{u}^\eta(s)\|_d \|\nabla \mathbf{u}^\eta(s)\|_d ds \\ &\leq Ct^{-\sigma} \|\mathbf{v}_0^\eta\|_d + C|\mathbf{u}^\eta|_{\sigma,d,t} \|\mathbf{u}_0^\eta\|_d \int_0^t (t-s)^{-\frac{1}{2}} s^{-\gamma-\frac{1}{2}} ds \\ &\leq Ct^{-\sigma} (\|\mathbf{v}_0^\eta\|_d + \tilde{C} \|\mathbf{u}_0^\eta\|_d |\mathbf{u}^\eta|_{\sigma,d,t}).\end{aligned} \quad (4.10)$$

By similar computation, we have

$$\begin{aligned}
& \|w^\eta(t)\|_d \\
& \leq Ct^{-\sigma}(1+\eta)^{-\sigma}\|w_0\|_q + C(1+\eta)^{-\frac{1}{2}}\int_0^t(t-s)^{-\frac{1}{2}}\|u(s)\|_d\|\nabla u(s)\|_d ds \\
& \leq Ct^{-\sigma}(1+\eta)^{-\sigma}(\|w_0\|_q + \tilde{C}\|u_0\|_d|u^\eta|_{\sigma,d,t}).
\end{aligned} \tag{4.11}$$

Taking initial data so small that  $\tilde{C}\|u_0^\eta\|_d < 1/2$ , we have by (4.10) and (4.11)

$$\sup_{0 < s \leq t} s^\sigma \left( \|v^\eta(s)\|_d + \sup_{\eta > 0} (1+\eta)^\sigma \|w^\eta(s)\|_d \right) \leq 2C\|u_0^\eta\|_d. \tag{4.12}$$

This implies that (4.9) holds for any  $t \geq t_0 > 0$ . For general initial data  $(v_0^\eta, w_0^\eta) \in L_\sigma^d(\mathbb{R}^d) \times G^d(\mathbb{R}^d)$ , (4.9) follows from the density argument. We omit the details.

#### 4.2. Error estimate (proof of Theorem 2.3)

In this subsection we shall prove error estimate.

The following integral equation is mild formulation of the Navier-Stokes initial value problem.

$$u(t) = e^{t\Delta}u_0 - \int_0^t e^{(t-s)\Delta}P(u(s) \cdot \nabla u(s)) ds. \tag{4.13}$$

If  $\|u_0\|_d$  is small enough, unique existence of global-in-time mild solution holds (see Kato [5]). In what follows, we will denote by  $u(t)$  the mild solution of (4.13) with initial velocity  $u_0$ . Let  $u^\eta(t) = v^\eta(t) + w^\eta(t)$  be global mild solution of (4.3) with initial data  $\|u_0^\eta\|_d \ll 1$ .

Our main purpose of this subsection is to show that

$$\lim_{\eta \rightarrow \infty} \|U^\eta(t)\|_d := \lim_{\eta \rightarrow \infty} \|u^\eta(t) - u(t)\|_d \leq C\|u_0^\eta - u_0\|_d \tag{4.14}$$

for any  $t \geq t_0 > 0$ . We have by triangle inequality,  $\|U^\eta(t)\|_d \leq \|\mathcal{E}^\eta(t)\|_d + \|w^\eta(t)\|_d$ , where we have set  $\mathcal{E}^\eta(t) := v^\eta(t) - u(t)$ . For  $w^\eta(t)$ , we already have the estimate (4.9). Therefore it suffices to show that  $\mathcal{E}^\eta(t)$  enjoys

$$\lim_{\eta \rightarrow \infty} \|\mathcal{E}^\eta(t)\|_d \leq C\|\mathcal{E}_0^\eta\|_d \tag{4.15}$$

for any  $t \geq t_0 > 0$ .

From (4.3) and (4.13),  $\mathcal{E}^\eta(t)$  satisfies the following integral equations.

$$\begin{aligned} \mathcal{E}(t) &= e^{t\Delta} \mathcal{E}_0^\eta - \int_0^t e^{(t-s)\Delta} P(\mathcal{E}^\eta \cdot \nabla v^\eta + \mathbf{u} \cdot \nabla \mathcal{E}^\eta)(s) ds \\ &\quad - \int_0^t e^{(t-s)\Delta} P(\mathbf{w}^\eta \cdot \nabla v^\eta + v^\eta \cdot \nabla \mathbf{w}^\eta + \mathbf{w}^\eta \cdot \nabla \mathbf{w}^\eta)(s) ds \\ &=: I_0(t) + I_1(t) + I_2(t). \end{aligned} \quad (4.16)$$

where  $\mathcal{E}_0^\eta := v_0^\eta - \mathbf{u}_0$ .

We shall estimate  $I_0(t)$ ,  $I_1(t)$  and  $I_2(t)$ , separately. By  $L^d$ -boundedness of  $e^{t\Delta}$ ,  $I_0(t)$  satisfies  $\|\mathcal{E}^\eta(t)\|_d \leq C \|\mathcal{E}_0\|_d$ .

Next we shall observe  $I_1(t)$ . Since  $\mathbf{u}(t)$  and  $v^\eta(t)$  are solenoidal,  $\mathcal{E}^\eta(t)$  also satisfies solenoidal condition. Hence by the fact that  $P(\mathbf{u} \cdot \nabla v) = P \operatorname{div}(\mathbf{u} \otimes v) = \operatorname{div} P(\mathbf{u} \otimes v)$  for solenoidal vector fields  $\mathbf{u}$  and  $v$ ,  $L^q$ -boundedness of the Helmholtz projection  $P = P_q$  ( $q \in (1, \infty)$ ), properties of the mild solutions  $\mathbf{u}(t)$  and  $\mathbf{u}^\eta(t)$  and Lemma 3.2, we have

$$\begin{aligned} \|I_1(t)\|_d &\leq \int_0^t \left\| e^{(t-s)\Delta} P(\operatorname{div}(\mathcal{E} \otimes \mathbf{u}) + \operatorname{div}(v^\eta \otimes \mathcal{E}))(s) \right\|_d ds \\ &= \int_0^t \left\| \operatorname{div} e^{(t-s)\Delta} P(\mathcal{E}^\eta \otimes \mathbf{u} + v^\eta \otimes \mathcal{E}^\eta)(s) \right\|_d ds \\ &\leq C \int_0^t (t-s)^{-\frac{d}{2r}-\frac{1}{2}} (\|v^\eta(s)\|_r + \|\mathbf{u}(s)\|_r) \|\mathcal{E}^\eta(s)\|_d ds \\ &\leq C(\|\mathbf{u}_0\|_d + \|\mathbf{u}_0^\eta\|_d) \sup_{0 < s \leq t} \|\mathcal{E}^\eta(s)\|_d. \end{aligned}$$

By  $L^q$ -boundedness of  $P_q$ , Lemma 3.2 and estimates for  $\mathbf{w}^\eta(t)$  and properties of mild solution, we have

$$\begin{aligned} \|I_2(t)\|_d &\leq C \int_0^t (t-s)^{-\frac{d}{2r}} (\|\mathbf{w}^\eta(s)\|_r \|\nabla v^\eta(s)\|_d \\ &\quad + \|v^\eta(s)\|_r \|\nabla \mathbf{w}^\eta(s)\|_d + \|\mathbf{w}^\eta(s)\|_r \|\nabla \mathbf{w}^\eta(s)\|_d) ds \\ &\leq C \|\mathbf{u}_0^\eta\|_d^2 (1 + \eta)^{-\frac{1}{2} + \frac{d}{2r}}. \end{aligned}$$

If we choose  $\|\mathbf{u}_0\|_d$  and  $\|\mathbf{u}_0^\eta\|_d$  are small enough if necessary, we have by the above estimates for  $I_0(t)$ ,  $I_1(t)$  and  $I_2(t)$ ,

$$\sup_{0 < s \leq t} \|\mathcal{E}^\eta(s)\|_d \leq C \|\mathcal{E}_0\|_d + C(1 + \eta)^{-\frac{1}{2} + \frac{d}{2r}},$$

which proves desired estimate:

$$\limsup_{\eta \rightarrow +\infty} \sup_{0 < s \leq t} \|\mathcal{E}^\eta(t)\|_d \leq C \|\mathcal{E}_0\|_d \quad (4.17)$$

for any  $t \geq t_0 > 0$ .

In particular, if we take  $\mathbf{u}_0 \equiv \mathbf{u}_0^\eta \in L_\sigma^d(\mathbb{R}^d)$ , we can conclude that solution of penalized Navier-Stokes initial value problem converges to solution of original Navier-Stokes one as  $\eta$  goes to infinity. This result is a rigorous justification of the penalty method.

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