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Initial-Boundary Value Problems for a Motion of a Vortex Filament with Axial Flow

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Abstract

We consider initial-boundary value problems for a nonlinear third order dispersive equation describing the motion of a vortex filament with axial flow. We prove new existence theorems for the related linear problems and apply it to the nonlinear problems.

1 Introduction

In this paper, we prove the unique solvability locally in time of the following initial-boundary value problems. For $\alpha < 0$,

$$(1.1)\quad \begin{cases} \boldsymbol{x}_t = \boldsymbol{x}_s \times \boldsymbol{x}_{ss} + \alpha\big\{\boldsymbol{x}_{sss} + \frac{3}{2}\boldsymbol{x}_{ss} \times \big(\boldsymbol{x}_s \times \boldsymbol{x}_{ss}\big)\big\}, & s>0, t>0, \\ \boldsymbol{x}(s,0) = \boldsymbol{x}_0(s), & s>0, \\ \boldsymbol{x}_{ss}(0,t) = \mathbf{0}, & t>0. \end{cases}$$

For $\alpha > 0$,

$$(1.2)\quad \begin{cases} \boldsymbol{x}_t = \boldsymbol{x}_s \times \boldsymbol{x}_{ss} + \alpha\big\{\boldsymbol{x}_{sss} + \frac{3}{2}\boldsymbol{x}_{ss} \times \big(\boldsymbol{x}_s \times \boldsymbol{x}_{ss}\big)\big\}, & s>0, t>0, \\ \boldsymbol{x}(s,0) = \boldsymbol{x}_0(s), & s>0, \\ \boldsymbol{x}_s(0,t) = \boldsymbol{e}_3, & t>0, \\ \boldsymbol{x}_{ss}(0,t) = \mathbf{0}, & t>0. \end{cases}$$

Here, $\boldsymbol{x}(s,t) = (x^1(s,t), x^2(s,t), x^3(s,t))$ is the position vector of the vortex filament parameterized by its arc length s at time t, $\times$ is the exterior product in the three dimensional Euclidean space, α is a non-zero constant that describes the magnitude of the effect of axial flow, $\boldsymbol{e}_3 = (0,0,1)$, and subscripts denote derivatives with their respective variables. Later in this paper, we will also use ∂_s and ∂_t to denote partial derivatives as well. We will refer to the equation in (1.1) and (1.2) as the vortex filament equation. We note here that the number of boundary conditions imposed changes depending on the sign of α. This is because the number of characteristic roots with a negative real part of the linearized equation, $\boldsymbol{x}_t = \alpha \boldsymbol{x}_{sss}$, changes depending on the sign of α.

Our motivation for considering (1.1) and (1.2) comes from analyzing the motion of a tornado. This paper is our humble attempt to model the motion of a tornado. While it is obvious that a vortex filament is not the same as a tornado and such modeling is questionable, many aspects of tornadoes are still unknown and we hope that our research can serve as a small step towards the complete analysis of the motion of a tornado.

To this end, in an earlier paper [1], the authors considered an initial-boundary value problem for the vortex filament equation with $\alpha = 0$, which is called the Localized Induction Equation (LIE). The LIE is a simplified model equation describing the motion of a vortex filament without axial flow. Other results considering the LIE can be found in Nishiyama and Tani [8] and Koiso [7].

Many results are known for the Cauchy problem for the vortex filament equation with non-zero α, where the filament extends to spacial infinity or the filament is closed. For example, in Nishiyama and Tani [8], they proved the unique solvability globally in time in Sobolev spaces. Onodera [9, 10] proved the unique solvability for a geometrically generalized equation. Segata [12] proved the unique solvability and showed the asymptotic behavior in time of the solution to the Hirota equation, given by

$$\mathrm{i}q_t = q_{xx} + \frac{1}{2}|q|^2 q + \mathrm{i}\alpha\big(q_{xxx} + |q|^2 q_x\big), \tag{1.3}$$

which can be obtained by applying the generalized Hasimoto transformation to the vortex filament equation. Since there are many results regarding the Cauchy problem for the Hirota equation and other Schrödinger type equations, it may feel more natural to see if the available theories from these results can be utilized to solve the initial-boundary value problem for (1.3), instead of considering (1.1) and (1.2) directly. Admittedly, problem (1.1) and (1.2) can be transformed into an initial-boundary value problem for the Hirota equation. But, in light of the possibility that a new boundary condition may be considered for the vortex filament equation in the future, we thought that it would be helpful to develop the analysis of the vortex filament equation itself because the Hasimoto transformation may not be applicable depending on the new boundary condition. For example, (1.1) and (1.2) model a vortex filament moving in the three dimensional half space, but if we consider a boundary that is not flat, it is non-trivial as to if we can apply the Hasimoto transformation or not, so we decided to work with the vortex filament equation directly.

For convenience, we introduce a new variable $\boldsymbol{v}(s,t) := \boldsymbol{x}_s(s,t)$ and rewrite the problems in terms of $\boldsymbol{v}$. Setting $\boldsymbol{v}_0(s) := \boldsymbol{x}_{0s}(s)$, we have for $\alpha < 0$,

$$
(1.4)\quad \begin{cases} \boldsymbol{v}_t = \boldsymbol{v}\times\boldsymbol{v}_{ss} + \alpha\{\boldsymbol{v}_{sss} + \frac{3}{2}\boldsymbol{v}_{ss}\times(\boldsymbol{v}\times\boldsymbol{v}_s) \\ \qquad\qquad + \frac{3}{2}\boldsymbol{v}_s\times(\boldsymbol{v}\times\boldsymbol{v}_{ss})\}, & s>0, t>0, \\ \boldsymbol{v}(s,0) = \boldsymbol{v}_0(s), & s>0, \\ \boldsymbol{v}_s(0,t) = \boldsymbol{0}, & t>0. \end{cases}
$$

For $\alpha > 0$,

$$
(1.5)\quad \begin{cases} \boldsymbol{v}_t = \boldsymbol{v}\times\boldsymbol{v}_{ss} + \alpha\{\boldsymbol{v}_{sss} + \frac{3}{2}\boldsymbol{v}_{ss}\times(\boldsymbol{v}\times\boldsymbol{v}_s) \\ \qquad\qquad + \frac{3}{2}\boldsymbol{v}_s\times(\boldsymbol{v}\times\boldsymbol{v}_{ss})\}, & s>0, t>0, \\ \boldsymbol{v}(s,0) = \boldsymbol{v}_0(s), & s>0, \\ \boldsymbol{v}(0,t) = \boldsymbol{e}_3, & t>0, \\ \boldsymbol{v}_s(0,t) = \boldsymbol{0}, & t>0. \end{cases}
$$

Once we obtain a solution for (1.4) and (1.5), we can reconstruct $\boldsymbol{x}(s,t)$ from the formula

$$
\boldsymbol{x}(s,t) = \boldsymbol{x}_0(s) + \int_0^t \{\boldsymbol{v}\times\boldsymbol{v}_s + \alpha\boldsymbol{v}_{ss} + \frac{3}{2}\alpha\boldsymbol{v}_s\times(\boldsymbol{v}\times\boldsymbol{v}_s)\}(s,\tau)\mathrm{d}\tau,
$$

and $\boldsymbol{x}(s,t)$ will satisfy (1.1) and (1.2) respectively, in other words, (1.1) is equivalent to (1.4) and (1.2) is equivalent to (1.5). Hence, we will concentrate on the solvability of (1.4) and (1.5) from now on. Our approach for solving (1.4) and (1.5) is to consider the associated linear problem. Linearizing the equation around a function $\boldsymbol{w}$ and neglecting lower order terms yield

$$
\boldsymbol{v}_t = \boldsymbol{w}\times\boldsymbol{v}_{ss} + \alpha\{\boldsymbol{v}_{sss} + \frac{3}{2}\boldsymbol{v}_{ss}\times(\boldsymbol{w}\times\boldsymbol{w}_s) + \frac{3}{2}\boldsymbol{w}_s\times(\boldsymbol{w}\times\boldsymbol{v}_{ss})\}.
$$

Directly considering the initial-boundary value problem for the above equation seems hard. When we try to estimate the solution in Sobolev spaces, the term $\boldsymbol{w}_s\times(\boldsymbol{w}\times\boldsymbol{v}_{ss})$ causes a loss of regularity because of the form of the coefficient. We were able to overcome this by using the fact that if the initial datum is parameterized by its arc length, i.e. $|\boldsymbol{v}_0| = 1$, a sufficiently smooth solution of (1.4) and (1.5) satisfies $|\boldsymbol{v}| = 1$, and this allows us to make the transformation

$$
\boldsymbol{v}_s\times(\boldsymbol{v}\times\boldsymbol{v}_{ss}) = \boldsymbol{v}_{ss}\times(\boldsymbol{v}\times\boldsymbol{v}_s) - |\boldsymbol{v}_s|^2\boldsymbol{v}_s.
$$

Linearizing the equation in (1.4) and (1.5) after the above transformation yields

$$
(1.6)\qquad \boldsymbol{v}_t = \boldsymbol{w}\times\boldsymbol{v}_{ss} + \alpha\{\boldsymbol{v}_{sss} + 3\boldsymbol{v}_{ss}\times(\boldsymbol{w}\times\boldsymbol{w}_s)\}.
$$

The term that was causing the loss of regularity is gone, but still, the existence of a solution to the initial-boundary value problem of the above third order dispersive equation is not trivial.

One may wonder if we could treat the second order derivative terms as a perturbation of the linear KdV or the KdV-Burgers equation to avoid the above difficulties all together. This seems impossible, because as far as the authors know, the estimates obtained for the linear KdV and KdV-Burgers equation is insufficient to consider a second order term as a regular perturbation. See, for example, Hayashi and Kaikina [5], Hayashi, Kaikina, and Ruiz Paredes [6], or Bona and Zhang [4] for known results on the initial-boundary value problems for the KdV and KdV-Burgers equations. To this end, we consider initial-boundary value problems for a more general linear equation of the form

$$\boldsymbol{u}_t = \alpha \boldsymbol{u}_{xxx} + \mathrm{A}(\boldsymbol{w}, \partial_x)\boldsymbol{u} + \boldsymbol{f}, \tag{1.7}$$

where $\boldsymbol{u}(x,t) = (u^1(x,t), u^2(x,t), \ldots, u^m(x,t))$ is the unknown vector valued function, $\boldsymbol{w}(x,t) = (w^1(x,t), w^2(x,t), \ldots, w^k(x,t))$ and $\boldsymbol{f}(x,t) = (f^1(x,t), f^2(x,t), \ldots, f^m(x,t))$ are known vector valued functions, and $\mathrm{A}(\boldsymbol{w}, \partial_x)$ is a second order differential operator of the form $\mathrm{A}(\boldsymbol{w}, \partial_x) = \mathrm{A}_0(\boldsymbol{w})\partial_x^2 + \mathrm{A}_1(\boldsymbol{w})\partial_x + \mathrm{A}_2(\boldsymbol{w})$. A_0, A_1, A_2 are smooth matrices and $\mathrm{A}(\boldsymbol{w}, \partial_x)$ is strongly elliptic in the sense that for any bounded domain E in $\mathbf{R}^k$, there is a positive constant δ such that for any $\boldsymbol{w} \in E$

$$\mathrm{A}_0(\boldsymbol{w}) + \mathrm{A}_0(\boldsymbol{w})^* \geq \delta \mathrm{I},$$

where I is the unit matrix and $*$ denotes the adjoint of a matrix. We prove the unique solvability of initial-boundary value problems of the above equation in Sobolev spaces, and the precise statement we prove will be addressed later. This result can be applied to (1.6) after we regularize it with a second order viscosity term $\delta \boldsymbol{v}_{ss}$ with $\delta > 0$.

The contents of this paper are as follows. In section 2, we introduce function spaces and the associated notations. In section 3, we consider a linear third order dispersive equation which includes the linearized equation of the vortex filament equation and state the main theorems for the linear problems. In section 4, we consider the compatibility conditions for the linear problems and the required corrections of the given data. Since the new parabolic regularization causes the compatibility conditions to become non-standard, we give a detailed analysis of this issue. In section 5, we briefly explain the construction of the solution and the rest of the proof of the existence theorem. In section 6, we state and prove the existence theorems for (1.1) and (1.2) by applying the results for the linear problems. This section will focus on how to obtain the estimate of the solution in the case $\alpha > 0$, where the known approach for estimating the solution in the initial value problem is insufficient.

2 Function Spaces and Notations

We define some function spaces that will be used throughout this paper, and notations associated with the spaces.

For an open interval Ω, a non-negative integer m, and $1 \le p \le \infty$, $W^{m,p}(\Omega)$ is the Sobolev space containing all real-valued functions that have derivatives in the sense of distribution up to order m belonging to $L^p(\Omega)$. We set $H^m(\Omega) := W^{m,2}(\Omega)$ as the Sobolev space equipped with the usual inner product. We will particularly use the cases $\Omega = \mathbf{R}$ and $\Omega = \mathbf{R}_+$, where $\mathbf{R}_+ = \{x \in \mathbf{R}; x > 0\}$. When $\Omega = \mathbf{R}_+$, the norm in $H^m(\Omega)$ is denoted by $\|\cdot\|_m$ and we simply write $\|\cdot\|$ for $\|\cdot\|_0$. Otherwise, for a Banach space X, the norm in X is written as $\|\cdot\|_X$. The inner product in $L^2(\mathbf{R}_+)$ is denoted by $(\cdot,\cdot)$.

For $0 < T < \infty$ and a Banach space X, $C^m([0,T];X)$ denotes the space of functions that are m times continuously differentiable in t with respect to the norm of X.

For any function space described above, we say that a vector valued function belongs to the function space if each of its components does.

Finally, we define some auxiliary function spaces used for the linear problems. Let l be a non-negative integer. X^l is the function space that we are constructing the solution in, specifically,

$$X^l := \bigcap_{j=0}^{l} \Big(C^j\big([0,T]; H^{2+3(l-j)}(\mathbf{R}_+)\big) \cap H^j\big(0,T; H^{3+3(l-j)}(\mathbf{R}_+)\big) \Big).$$

As a consequence, $\boldsymbol{u}_0$ will be required to belong in $H^{2+3l}(\mathbf{R}_+)$. Y^l is the function space that $\boldsymbol{f}$ will be required to belong in, and is defined by

$$Y^l := \Big\{ f;\ f \in \bigcap_{j=0}^{l-1} C^j\big([0,T]; H^{2+3(l-1-j)}(\mathbf{R}_+)\big),\ \partial_t^l f \in L^2\big(0,T; H^1(\mathbf{R}_+)\big) \Big\}.$$

Z^l is the function space that $\boldsymbol{w}$ will belong in and is defined as

$$Z^l := \Big\{ w;\ w \in \bigcap_{j=0}^{l-1} C^j\big([0,T]; H^{2+3(l-1-j)}(\mathbf{R}_+)\big),\ \partial_t^l w \in L^\infty\big(0,T; H^1(\mathbf{R}_+)\big) \Big\}.$$

3 Associated Linear Problems

We prove the solvability of the following problems. For $\alpha < 0$,

$$\text{(3.1)} \qquad \begin{cases} \boldsymbol{u}_t = \alpha \boldsymbol{u}_{xxx} + \mathrm{A}(\boldsymbol{w}, \partial_x)\boldsymbol{u} + \boldsymbol{f}, & x > 0, t > 0, \\ \boldsymbol{u}(x,0) = \boldsymbol{u}_0(x), & x > 0, \\ \boldsymbol{u}_x(0,t) = \boldsymbol{0}, & t > 0. \end{cases}$$

For $\alpha > 0$,

$$\text{(3.2)} \qquad \begin{cases} \boldsymbol{u}_t = \alpha \boldsymbol{u}_{xxx} + \mathrm{A}(\boldsymbol{w}, \partial_x)\boldsymbol{u} + \boldsymbol{f}, & x > 0, t > 0, \\ \boldsymbol{u}(x,0) = \boldsymbol{u}_0(x), & x > 0, \\ \boldsymbol{u}(0,t) = \boldsymbol{e}, & t > 0, \\ \boldsymbol{u}_x(0,t) = \boldsymbol{0}, & t > 0. \end{cases}$$

For (3.1) and (3.2), we prove the following.

Theorem 3.1 *For any $T > 0$ and an arbitrary non-negative integer l, if $\boldsymbol{u}_0 \in H^{2+3l}(\mathbf{R}_+)$, $\boldsymbol{f} \in Y^l$, and $\boldsymbol{w} \in Z^l$ satisfy the compatibility conditions up to order l, a unique solution $\boldsymbol{u}$ of* (3.1) *exists such that $\boldsymbol{u} \in X^l$. Furthermore, the solution satisfies*

$$\|\boldsymbol{u}\|_{X^l} \le C\big(\|\boldsymbol{u}_0\|_{2+3l} + \|\boldsymbol{f}\|_{Y^l}\big),$$

where the constant C depends on T, $\|\boldsymbol{w}\|_{Z^l}$, and δ.

Theorem 3.2 *For any $T > 0$ and an arbitrary non-negative integer l, if $\boldsymbol{u}_0 \in H^{2+3l}(\mathbf{R}_+)$, $\boldsymbol{f} \in Y^l$, and $\boldsymbol{w} \in Z^l$ satisfy the compatibility conditions up to order l, a unique solution $\boldsymbol{u}$ of* (3.2) *exists such that $\boldsymbol{u} \in X^l$. Furthermore, the solution satisfies*

$$\|\boldsymbol{u}\|_{X^l} \le C\big(\|\boldsymbol{u}_0\|_{2+3l} + \|\boldsymbol{f}\|_{Y^l}\big),$$

where the constant C depends on T, $\|\boldsymbol{w}\|_{Z^l}$, and δ.

Since the proof for the case $\alpha > 0$ is relatively standard, we focus on the case $\alpha < 0$. Our method for constructing the solution is parabolic regularization. When $\alpha < 0$, a standard regularization using $-\partial_x^4 \boldsymbol{u}$ is inapplicable because we can impose only one boundary condition to our original problem, where as the regularized problem requires two boundary conditions to be well-posed. Thus, we will construct the solution of (3.1) by taking the limit $\varepsilon \to 0$ in the following new regularized system.

$$\text{(3.3)} \qquad \begin{cases} \boldsymbol{u}_t = \alpha\left(\boldsymbol{u}_{xx} - \varepsilon \boldsymbol{u}_t\right)_x + \mathrm{A}(\boldsymbol{w}, \partial_x)\boldsymbol{u} + \boldsymbol{g}, & x > 0, t > 0, \\ \boldsymbol{u}(x,0) = \boldsymbol{u}_0(x), & x > 0, \\ \boldsymbol{u}_x(0,t) = \boldsymbol{0}, & t > 0, \end{cases}$$

where $\varepsilon > 0$. To construct the solution of the above system, we first consider the following problem.

$$
\begin{cases} \boldsymbol{u}_t = \alpha \left(\boldsymbol{u}_{xx} - \varepsilon \boldsymbol{u}_t\right)_x + \boldsymbol{g}, & x > 0, t > 0, \\ \boldsymbol{u}(x,0) = \boldsymbol{u}_0(x), & x > 0, \\ \boldsymbol{u}_x(0,t) = \boldsymbol{0}, & t > 0. \end{cases} \tag{3.4}
$$

(3.3) is a parabolic regularization of (3.1) and the principal terms are the terms in parenthesis. In fact if we substitute $\boldsymbol{u}(x,t) = \mathrm{e}^{\tau t + \mathrm{i}\xi x}\boldsymbol{C}$ into $\boldsymbol{u}_t = \alpha \left(\boldsymbol{u}_{xx} - \varepsilon \boldsymbol{u}_t\right)_x$, we obtain the dispersion relation $\tau = -\alpha(\xi^2 + \varepsilon\tau)\mathrm{i}\xi$, so that for a non-trivial solution to exist, we need

$$
\Re\tau = -\frac{\alpha^2\varepsilon\xi^4}{1+\alpha^2\varepsilon^2\xi^2},
$$

which indicates that the equation is parabolic in nature. This allows us to regularize the problem without changing the number of boundary conditions needed for the problem to be well-posed. The main difficulty caused by this regularization is deriving the compatibility conditions and making the necessary corrections to the given data.

4 Compatibility Conditions for the Case $\alpha < 0$

As stated before, we will construct the solution of (3.1) by taking the limit $\varepsilon \to 0$ in the following regularized system.

$$
\begin{cases} \boldsymbol{u}_t = -\alpha\varepsilon \boldsymbol{u}_{tx} + \alpha \boldsymbol{u}_{xxx} + \mathrm{A}(\boldsymbol{w}, \partial_x)\boldsymbol{u} + \boldsymbol{g}, & x > 0, t > 0, \\ \boldsymbol{u}(x,0) = \boldsymbol{u}_0(x), & x > 0, \\ \boldsymbol{u}_x(0,t) = \boldsymbol{0}, & t > 0. \end{cases} \tag{4.1}
$$

Since the derivation of the compatibility conditions for the regularized system is complicated and the required corrections for the given data is not standard, we devote this section to clarify these matters.

4.1 Compatibility Conditions for (3.1)

We first define the compatibility condition for the original system (3.1). We denote the right-hand side of the equation in (3.1) as

$$
\boldsymbol{Q}_1(\boldsymbol{u}, \boldsymbol{f}, \boldsymbol{w}) = \alpha \boldsymbol{u}_{xxx} + \mathrm{A}(\boldsymbol{w}, \partial_x)\boldsymbol{u} + \boldsymbol{f}, \tag{4.2}
$$

and we also use the notation $\boldsymbol{Q}_1(x,t) := \boldsymbol{Q}_1(\boldsymbol{u}, \boldsymbol{f}, \boldsymbol{w})$ and sometimes omit the (x,t) for simplicity. We successively define

$$
\boldsymbol{Q}_n := \alpha\partial_x^3 \boldsymbol{Q}_{n-1} + \sum_{j=0}^{n-1} \binom{n-1}{j} \mathrm{B}_j \boldsymbol{Q}_{n-1-j} + \partial_t^{n-1}\boldsymbol{f}, \tag{4.3}
$$

where $\mathrm{B}_j = \left(\partial_t^j \mathrm{A}_0(\boldsymbol{w})\right)\partial_x^2 + \left(\partial_t^j \mathrm{A}_1(\boldsymbol{w})\right)\partial_x + \partial_t^j \mathrm{A}_2(\boldsymbol{w})$. The above definition gives the formula for the expression of $\partial_t^n \boldsymbol{u}$ which only contains x derivatives of $\boldsymbol{u}$ and mixed derivatives of $\boldsymbol{w}$ and $\boldsymbol{f}$. From the boundary condition in (3.1), we arrive at the following definition for the compatibility conditions.

Definition 4.1 (*Compatibility conditions for* (3.1)). *For $n \in \mathbf{N} \cup \{0\}$, we say that $\boldsymbol{u}_0$, $\boldsymbol{f}$, and $\boldsymbol{w}$ satisfy the n-th order compatibility condition for* (3.1) *if*

$$\boldsymbol{u}_{0x}(0,0) = \mathbf{0}$$

when $n = 0$, and

$$\left(\partial_x \boldsymbol{Q}_n\right)(0,0) = \mathbf{0}$$

when $n \geq 1$. We also say that the data satisfy the compatibility conditions for (3.1) *up to order n if they satisfy the k-th order compatibility condition for all k with $0 \leq k \leq n$.*

Now that we have defined the compatibility conditions, we discuss an approximation of the data via smooth functions which keep the compatibility conditions. Recall that X^l, Y^l, and Z^l are function spaces defined in section 2 that we consider the solution and given data in. Data belonging to these function spaces with index l are smooth enough for the l-th order compatibility condition to have meaning in a point-wise sense, but the $(l+1)$-th order compatibility condition does not. By utilizing the method in [11] used by Rauch and Massey, we can get the following.

Lemma 4.2 *Fix non-negative integers l and N with $N > l$. For any $\boldsymbol{u}_0 \in H^{2+3l}(\mathbf{R}_+)$, $\boldsymbol{f} \in Y^l$, and $\boldsymbol{w} \in Z^l$ satisfying the compatibility conditions for* (3.1) *up to order l, there exist sequences $\{\boldsymbol{u}_{0n}\}_{n\geq 1} \subset H^{2+3N}(\mathbf{R}_+)$, $\{\boldsymbol{f}_n\}_{n\geq 1} \subset Y^N$, and $\{\boldsymbol{w}_n\}_{n\geq 1} \subset Z^N$ such that for any $n \geq 1$, $\boldsymbol{u}_{0n}$, $\boldsymbol{f}_n$, and $\boldsymbol{w}_n$ satisfy the compatibility conditions for* (3.1) *up to order N and*

$$\boldsymbol{u}_{0n} \to \boldsymbol{u}_0 \text{ in } H^{2+3l}(\mathbf{R}_+), \quad \boldsymbol{f}_n \to \boldsymbol{f} \text{ in } Y^l, \text{ and } \boldsymbol{w}_n \to \boldsymbol{w} \text{ in } Z^l.$$

From Lemma 4.2, we can assume that the given data are arbitrarily smooth and satisfy the necessary compatibility conditions in the proceeding arguments.

4.2 Compatibility Conditions for (4.1)

Now, we define the compatibility conditions for (4.1). We write the equation in (4.1) as

$$\boldsymbol{u}_t = -\alpha\varepsilon \boldsymbol{u}_{tx} + \boldsymbol{P}_1(\boldsymbol{u}, \boldsymbol{g}, \boldsymbol{w}), \tag{4.4}$$

in other words, $\boldsymbol{P}_1(\boldsymbol{u},\boldsymbol{g},\boldsymbol{w}) = \alpha\boldsymbol{u}_{xxx} + \mathrm{A}(\boldsymbol{w},\partial_x)\boldsymbol{u} + \boldsymbol{g}$. We use the notations $\boldsymbol{P}_1(x,t)$ and $\boldsymbol{P}_1$ as we did with $\boldsymbol{Q}_1$ in the last subsection. Setting $\boldsymbol{\phi}_1(x) := \boldsymbol{u}_t(x,0)$ and taking the trace $t=0$ of the equation we have

$$\alpha\varepsilon\boldsymbol{\phi}_1' + \boldsymbol{\phi}_1 = \boldsymbol{P}_1(\cdot,0). \tag{4.5}$$

A prime denotes a derivative with respect to x. Note that $\boldsymbol{P}_1(x,0)$ is expressed using given data only. Solving the above ordinary differential equation for $\boldsymbol{\phi}_1$ we have

$$\boldsymbol{\phi}_1(x) = \mathrm{e}^{-\frac{x}{\alpha\varepsilon}}\left\{\boldsymbol{\phi}_1(0) + \frac{1}{\alpha\varepsilon}\int_0^x \mathrm{e}^{\frac{y}{\alpha\varepsilon}}\boldsymbol{P}_1(y,0)\mathrm{d}y\right\}.$$

Since we are looking for solutions that are square integrable, we impose that $\lim\limits_{x\to\infty}\boldsymbol{\phi}_1(x) = \boldsymbol{0}$, so we have

$$\boldsymbol{\phi}_1(0) = -\frac{1}{\alpha\varepsilon}\int_0^\infty \mathrm{e}^{\frac{y}{\alpha\varepsilon}}\boldsymbol{P}_1(y,0)\mathrm{d}y,$$

which gives

$$\boldsymbol{\phi}_1(x) = -\frac{1}{\alpha\varepsilon}\int_x^\infty \mathrm{e}^{-\frac{1}{\alpha\varepsilon}(x-y)}\boldsymbol{P}_1(y,0)\mathrm{d}y.$$

By direct calculation, we see that

$$\boldsymbol{\phi}_1'(x) = -\frac{1}{\alpha\varepsilon}\int_x^\infty \mathrm{e}^{-\frac{1}{\alpha\varepsilon}(x-y)}\boldsymbol{P}_1'(y,0)\mathrm{d}y,$$

where we have used integration by parts. We also note here that $\boldsymbol{\phi}_1$ is expressed with given data only. From the boundary condition in (4.1), we see that the first order compatibility condition is

$$\int_0^\infty \mathrm{e}^{\frac{y}{\alpha\varepsilon}}\boldsymbol{P}_1'(y,0)\mathrm{d}y = \boldsymbol{0}.$$

In the same manner, we will derive the n-th order compatibility condition for $n \geq 2$. Taking the t derivative of the equation in (4.1) $(n-1)$ times, taking the trace $t=0$, and setting $\boldsymbol{\phi}_n(x) := \partial_t^n\boldsymbol{u}(x,0)$, we have

$$\alpha\varepsilon\boldsymbol{\phi}_n' + \boldsymbol{\phi}_n = \partial_t^{n-1}\boldsymbol{P}_1.$$

We denote

$$\boldsymbol{P}_n := \partial_t^{n-1}\boldsymbol{P}_1.$$

We will prove by induction that ϕ_n and $\boldsymbol{P}_n(x,0)$ are expressed using given data only. Since $\boldsymbol{P}_n = \partial_t^{n-1}\boldsymbol{P}_{n-1} = \partial_t^{n-1}(\alpha \boldsymbol{u}_{xxx} + \mathrm{A}(\boldsymbol{w})\boldsymbol{u} + \boldsymbol{g})$, it holds that

$$\boldsymbol{P}_n(\cdot,0) = \alpha\phi_{n-1}''' + \sum_{j=0}^{n-1}\binom{n-1}{j}\mathrm{B}_j\phi_{n-1-j} + \partial_t^{n-1}\boldsymbol{g}(\cdot,0). \tag{4.6}$$

For a $n \geq 2$, assume that ϕ_k and $\boldsymbol{P}_k(x,0)$ are expressed with given data for $1 \leq k \leq n-1$. Formula (4.6) implies that $\boldsymbol{P}_n(\cdot,0)$ is expressed with given data. Solving for ϕ_n yields

$$\phi_n(x) = -\frac{1}{\alpha\varepsilon}\int_x^\infty \mathrm{e}^{-\frac{1}{\alpha\varepsilon}(x-y)}\boldsymbol{P}_n(y,0)\mathrm{d}y.$$

This proves that ϕ_n is also expressed using given data only.

Again by direct calculation, we have

$$\phi_n'(x) = -\frac{1}{\alpha\varepsilon}\int_x^\infty \mathrm{e}^{-\frac{1}{\alpha\varepsilon}(x-y)}\boldsymbol{P}_n'(y,0)\mathrm{d}y,$$

and arrive at the n-th order compatibility condition

$$\int_0^\infty \mathrm{e}^{\frac{y}{\alpha\varepsilon}}\boldsymbol{P}_n'(y,0)\mathrm{d}y = \boldsymbol{0}.$$

Now we can define the following.

Definition 4.3 (*Compatibility conditions for* (4.1)). *For $n \in \mathbf{N}\cup\{0\}$, we say that $\boldsymbol{u}_0$, $\boldsymbol{g}$, and $\boldsymbol{w}$ satisfy the n-th order compatibility condition for* (4.1) *if*

$$\boldsymbol{u}_{0x}(0) = \boldsymbol{0}$$

when $n = 0$, and

$$\int_0^\infty \mathrm{e}^{\frac{y}{\alpha\varepsilon}}\boldsymbol{P}_n'(y,0)\mathrm{d}y = \boldsymbol{0}$$

when $n \geq 1$. We also say that the data satisfy the compatibility conditions for (4.1) *up to order n if the data satisfy the k-th order compatibility condition for all k with $0 \leq k \leq n$. For the definition of $\boldsymbol{P}_n$, see* (4.4) *and* (4.6).

We note that for $\boldsymbol{u}_0 \in H^{2+3l}(\mathbf{R}_+)$, $\boldsymbol{f} \in Y^l$, and $\boldsymbol{w} \in Z^l$, the compatibility conditions up to order l have meaning in the point-wise sense, but the $(l+1)$-th order compatibility condition does not.

4.3 Corrections to the Data

Since we regularized the equation, we must make corrections to the data to assure that the compatibility conditions continue to hold. Fix a large integer N and suppose that $\boldsymbol{u}_0 \in H^{2+3N}(\mathbf{R}_+)$, $\boldsymbol{f} \in Y^N$, and $\boldsymbol{w} \in Z^N$ satisfy the compatibility conditions for (3.1) up to order N. We will make corrections to the forcing term so that the data satisfy the compatibility conditions for (4.1) up to order N. More specifically, we prove the following

Proposition 4.4 *Fix a positive integer N. For $\boldsymbol{u}_0 \in H^{2+3N}(\mathbf{R}_+)$, $\boldsymbol{f} \in Y^N$, and $\boldsymbol{w} \in Z^N$ satisfying the compatibility conditions for* (3.1) *up to order N, we can define $\boldsymbol{g} \in Y^N$ in the form $\boldsymbol{g} = \boldsymbol{f} + \boldsymbol{h}_\varepsilon$ such that $\boldsymbol{u}_0$, $\boldsymbol{g}$, and $\boldsymbol{w}$ satisfy the compatibility conditions for* (4.1) *up to order N and $\boldsymbol{h}_\varepsilon \to \boldsymbol{0}$ in Y^N as $\varepsilon \to 0$.*

Proof. We write the equation in (4.1) as

$$\boldsymbol{u}_t = -\alpha\varepsilon \boldsymbol{u}_{tx} + P(x,t,\partial_x)\boldsymbol{u} + \boldsymbol{g}.$$

Setting $\boldsymbol{\phi}_1(x) := \boldsymbol{u}_t(x,0)$ and taking the trace $t=0$ of the equation we have

$$\alpha\varepsilon\boldsymbol{\phi}_1' + \boldsymbol{\phi}_1 = P(x,0,\partial_x)\boldsymbol{u}_0 + \boldsymbol{f}(x,0) + \boldsymbol{h}_\varepsilon(x,0). \tag{4.7}$$

Using the notations in (4.2) we have $P(x,0,\partial_x)\boldsymbol{u}_0 + \boldsymbol{f}(x,0) = \boldsymbol{Q}_1(x,0)$. As before, solving the above ordinary differential equation for $\boldsymbol{\phi}_1$ under the constraint $\lim_{x\to\infty} \boldsymbol{\phi}_1(x) = \boldsymbol{0}$ we have

$$\boldsymbol{\phi}_1(x) = -\frac{1}{\alpha\varepsilon}\int_x^\infty \mathrm{e}^{-\frac{1}{\alpha\varepsilon}(x-y)}\{\boldsymbol{Q}_1(y,0) + \boldsymbol{h}_\varepsilon(y,0)\}\mathrm{d}y.$$

We give an ansatz for the form of $\boldsymbol{h}_\varepsilon$, namely

$$\boldsymbol{h}_\varepsilon(x,t) = \left(\sum_{j=0}^N \boldsymbol{C}_{j,\varepsilon}\frac{t^j}{j!}\right)\mathrm{e}^{-x},$$

where $\boldsymbol{C}_{j,\varepsilon}$, $j = 0,1,\dots,N$, are constant vectors depending on ε to be determined later. From Definition 4.3 the first order compatibility condition is

$$\int_0^\infty \mathrm{e}^{\frac{y}{\alpha\varepsilon}}\{\boldsymbol{Q}_1'(y,0) + \boldsymbol{h}_\varepsilon'(y,0)\}\mathrm{d}y = \boldsymbol{0}.$$

Substituting the ansatz for $\boldsymbol{h}_\varepsilon(x,t)$, we have

$$\boldsymbol{C}_{0,\varepsilon}\left(1-\frac{1}{\alpha\varepsilon}\right)^{-1} = \int_0^\infty \mathrm{e}^{\frac{y}{\alpha\varepsilon}}\boldsymbol{Q}_1'(y,0)\mathrm{d}y.$$

Since $\boldsymbol{Q}_1'(0,0) = \mathbf{0}$ from the compatibility condition for (3.1), we have by integration by parts

$$\boldsymbol{C}_{0,\varepsilon} = (\alpha\varepsilon - 1)\int_0^\infty e^{\frac{y}{\alpha\varepsilon}}\boldsymbol{Q}_1''(y,0)\mathrm{d}y.$$

So if we limit ourselves to $0 < \varepsilon < \min\{1, 1/|\alpha|\}$, from

$$e^{\frac{y}{\alpha\varepsilon}}|\boldsymbol{Q}_1''(y,0)| \le e^{-y}|\boldsymbol{Q}_1''(y,0)|,$$

and for $y > 0$

$$e^{\frac{y}{\alpha\varepsilon}}|\boldsymbol{Q}_1''(y,0)| \to 0 \text{ as } \varepsilon \to 0,$$

we see that $\boldsymbol{C}_{0,\varepsilon} \to \mathbf{0}$ as $\varepsilon \to 0$. We will show by induction that $\boldsymbol{C}_{j,\varepsilon}$ can be chosen so that $\boldsymbol{C}_{j,\varepsilon} \to \mathbf{0}$ for $1 \le j \le N$ and $\boldsymbol{g} = \boldsymbol{f} + \boldsymbol{h}_\varepsilon$ with $\boldsymbol{u}_0$ and $\boldsymbol{w}$ satisfies the compatibility conditions for (4.1) up to order N. Suppose that the above statement holds for $0 \le j \le n-2$ for some n with $2 \le n \le N$.

We define $\boldsymbol{P}_n(x,0)$ and $\boldsymbol{\phi}_n(x)$ as before and we have

$$\boldsymbol{\phi}_n(x) = -\frac{1}{\alpha\varepsilon}\int_x^\infty e^{-\frac{1}{\alpha\varepsilon}(x-y)}\boldsymbol{P}_n(y,0)\mathrm{d}y, \tag{4.8}$$

and the n-th order compatibility condition for (4.1) is

$$\int_0^\infty e^{\frac{y}{\alpha\varepsilon}}\boldsymbol{P}_n'(y,0)\mathrm{d}y = \mathbf{0}.$$

We rewrite this condition as

$$-\boldsymbol{P}_n'(0,0) + \int_0^\infty e^{\frac{y}{\alpha\varepsilon}}\boldsymbol{P}_n''(y,0)\mathrm{d}y = \mathbf{0} \tag{4.9}$$

by integration by parts. We recall that $\boldsymbol{P}_n(x,0)$ was successively defined by

$$\boldsymbol{P}_n(\cdot,0) = \alpha\boldsymbol{\phi}_{n-1}''' + \sum_{j=0}^{n-1}\binom{n-1}{j}\mathrm{B}_j\boldsymbol{\phi}_{n-1-j} + \partial_t^{n-1}\boldsymbol{g}(\cdot,0),$$

with $\boldsymbol{P}_1(x,0) = \alpha\boldsymbol{u}_{0xxx} + \mathrm{A}(\boldsymbol{w}(x,0),\partial_x)\boldsymbol{u}_0 + \boldsymbol{g}(x,0)$. Substituting (4.8) with $n = j$ for $\boldsymbol{\phi}_j$ and using integration by parts, we have

$$\boldsymbol{P}_n(\cdot,0) = \alpha\boldsymbol{P}_{n-1}''' + \sum_{j=0}^{n-1}\binom{n-1}{j}\mathrm{B}_j\boldsymbol{P}_{n-1-j} + \partial_t^{n-1}\boldsymbol{g}(\cdot,0)$$
$$- \alpha\varepsilon\Big\{\alpha\boldsymbol{\phi}_{n-1}'''' + \sum_{j=0}^{n-1}\binom{n-1}{j}\mathrm{B}_j\boldsymbol{\phi}_{n-1-j}'\Big\}.$$

Also recall that

$$\boldsymbol{Q}_n = \alpha\partial_x^3\boldsymbol{Q}_{n-1} + \sum_{j=0}^{n-1}\binom{n-1}{j}\mathrm{B}_j\boldsymbol{Q}_{n-1-j} + \partial_t^{n-1}\boldsymbol{f},$$

with $\boldsymbol{Q}_1(x,0) = \alpha\boldsymbol{u}_{0xxx} + \mathrm{A}(\boldsymbol{w}(x,0),\partial_x)\boldsymbol{u}_0 + \boldsymbol{f}(x,0)$. Thus, setting $\boldsymbol{R}_n := \boldsymbol{P}_n - \boldsymbol{Q}_n$, we have

$$\boldsymbol{R}_n(x,0) = \alpha\boldsymbol{R}_{n-1}''' + \sum_{j=0}^{n-1}\binom{n-1}{j}\mathrm{B}_j\boldsymbol{R}_{n-1-j} + \partial_t^{n-1}\boldsymbol{h}_\varepsilon(\cdot,0)$$
$$- \alpha\varepsilon\left\{\alpha\boldsymbol{\phi}_{n-1}'''' + \sum_{j=0}^{n-1}\binom{n-1}{j}\mathrm{B}_j\boldsymbol{\phi}_{n-1-j}'\right\},$$

with $\boldsymbol{R}_1(x,0) = \boldsymbol{h}_\varepsilon(x,0)$. We prove by induction that $\boldsymbol{R}_n(x,0) = \partial_t^{n-1}\boldsymbol{h}_\varepsilon(x,0) + o(1)$, where $o(1)$ are terms that tend to zero as $\varepsilon \to 0$. The case $n = 1$ is obvious from the definition of $\boldsymbol{R}_1(x,0)$. Suppose that it holds for $\boldsymbol{R}_k(x,0)$ for $1 \le k \le n-1$. From the above expression for $\boldsymbol{R}_n(x,0)$, the assumption of induction on $\boldsymbol{R}_n$, and the assumption of induction that $\boldsymbol{C}_{j,\varepsilon} \to \boldsymbol{0}$ for $0 \le j \le n-2$, we see that

$$\boldsymbol{R}_n(x,0) = \partial_t^{n-1}\boldsymbol{h}_\varepsilon + o(1) - \alpha\varepsilon\left\{\alpha\boldsymbol{\phi}_{n-1}'''' + \sum_{j=0}^{n-1}\binom{n-1}{j}\mathrm{B}_j\boldsymbol{\phi}_{n-1-j}'\right\}.$$

Again, from (4.8) and Lebesgue's dominated convergence theorem, we see that the last two terms are $o(1)$, which proves $\boldsymbol{R}_n(x,0) = \boldsymbol{P}_n(x,0) - \boldsymbol{Q}_n(x,0) = \partial_t^{n-1}\boldsymbol{h}_\varepsilon(x,0) + o(1)$. Here, we have used the fact that $\boldsymbol{P}_k(x,0)$ for $1 \le k \le n-1$ are uniformly bounded with respect to ε. We note that from the expressions of $\boldsymbol{R}_n(x,0)$ and $\boldsymbol{h}_\varepsilon$, the terms in $o(1)$ are composed of terms such that their x derivative are also $o(1)$. Substituting for $\boldsymbol{P}_n(x,0)$ and the ansatz for $\boldsymbol{h}_\varepsilon$ in (4.9) yields,

$$\begin{aligned}\boldsymbol{C}_{n-1,\varepsilon} &= \boldsymbol{Q}_n'(0,0) + \int_0^\infty \mathrm{e}^{\frac{y}{\alpha\varepsilon}}\boldsymbol{Q}_n''(y,0)\mathrm{d}y + o(1)\\ &= \int_0^\infty \mathrm{e}^{\frac{y}{\alpha\varepsilon}}\boldsymbol{Q}_n''(y,0)\mathrm{d}y + o(1),\end{aligned}$$

where we have used the assumption of induction that $\boldsymbol{u}_0$, $\boldsymbol{f}$, and $\boldsymbol{w}$ satisfy the n-th order compatibility condition for (3.1), i.e. $\boldsymbol{Q}_n'(0,0) = \boldsymbol{0}$. By using the above expression to define $\boldsymbol{C}_{n-1,\varepsilon}$, we see that $\boldsymbol{C}_{n-1,\varepsilon} \to \boldsymbol{0}$ as $\varepsilon \to 0$ and $\boldsymbol{u}_0$, $\boldsymbol{g}$, and $\boldsymbol{w}$ satisfy the compatibility conditions for (4.1) up to order n. Furthermore, from the explicit form we see that $\boldsymbol{h}_\varepsilon \to \boldsymbol{0}$ in Y^N. This finishes the proof of the proposition. □

The corrections to the data associated with (3.4) can be treated the same way.

5 Construction of the Solution

5.1 The Case $\alpha < 0$

We first construct the solution to (3.4) as a sum of two functions $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ which are defined as the solutions of the following systems. $\boldsymbol{u}_1$ is defined as the solution to the initial value problem

$$\begin{cases} \boldsymbol{u}_{1t} = \alpha\left(\boldsymbol{u}_{1xx} - \varepsilon \boldsymbol{u}_{1t}\right)_x + \boldsymbol{G}, & x \in \mathbf{R}, t > 0, \\ \boldsymbol{u}_1(x,0) = \boldsymbol{U}_0, & x \in \mathbf{R}, \end{cases}$$

and $\boldsymbol{u}_2$ is defined as the solution to the initial-boundary value problem

$$\begin{cases} \boldsymbol{u}_{2t} = \alpha\left(\boldsymbol{u}_{2xx} - \varepsilon \boldsymbol{u}_{2t}\right)_x, & x > 0, t > 0, \\ \boldsymbol{u}_2(x,0) = \boldsymbol{0}, & x > 0, \\ \boldsymbol{u}_{2x}(0,t) = -\boldsymbol{u}_{1x}(0,t) =: \boldsymbol{\Phi}(t), & t > 0. \end{cases}$$

Here, $\boldsymbol{G}$ and $\boldsymbol{U}_0$ are smooth extensions of $\boldsymbol{g}$ and $\boldsymbol{u}_0$ to $x < 0$, respectively. We can construct $\boldsymbol{u}_1$ by Fourier transform with respect to x, and $\boldsymbol{u}_2$ by Laplace transform with respect to t. When solving the ODE in x for $\boldsymbol{u}_2$, we make use of the following lemma concerning the characteristic roots.

Lemma 5.1 *For $h > 0$ and $\varepsilon > 0$, the characteristic equation, $\lambda^3 - \varepsilon\tau\lambda - \frac{\tau}{\alpha} = 0$, has exactly one root λ satisfying $\Re\lambda < 0$. We will denote this root as μ. Furthermore, there are positive constants η_0 and C such that for $|\eta| \geq \eta_0$ the following holds.*

$$\left|\mu + \sqrt{\frac{\varepsilon}{2}}\,(1+\mathrm{i})\,|\eta|^{1/2}\right| \leq C.$$

We note here that the leading order term of μ tells us that the solution of our new regularized equation is parabolic in nature. In case of the heat equation, the corresponding characteristic root would be equal to $-\sqrt{\frac{\varepsilon}{2}}\,(1+\mathrm{i})\,|\eta|^{1/2}$ so the solution to our regularized problem behaves asymptotically the same as the solution to the heat equation. Also, the fact that there is exactly one root with a negative real part insures that only one boundary condition is needed for the problem to be well-posed.

Through these arguments, for any fixed non-negative integer l, we can construct the solution to (3.4) such that

$$\boldsymbol{u} \in \bigcap_{j=0}^{l} C^j\left([0,T]; H^{2(l-j)}(\mathbf{R}_+)\right).$$

We can also construct the solution to (4.1) by a standard iteration scheme in the same function space. Finally we must obtain estimates uniform in ε to take the limit $\varepsilon \to +0$. We use a standard energy method combined with interpolation inequalities. We use energies of the form

$$\|\partial_x^j \boldsymbol{u}\|^2 + \alpha^2\varepsilon^2\|\partial_x^{j+1}\boldsymbol{u}\|^2$$

with $j = 0, 1, 2$ for our basic estimate, and obtain

$$\sup_{0\le t\le T} \|\boldsymbol{u}(t)\|_2^2 + \int_0^T \left(\|\boldsymbol{u}_{xxx}(t)\|^2 + \varepsilon\|\boldsymbol{u}_{tx}(t)\|^2 + |\boldsymbol{u}_{xx}(0,t)|^2 + |\boldsymbol{u}_{xxx}(0,t)|^2\right)\mathrm{d}t$$
$$\le C\left(\|\boldsymbol{u}_0\|_2^2 + \int_0^T \|\boldsymbol{g}(t)\|_1^2 \mathrm{d}t\right)$$

for sufficiently small ε. Here, C is a positive constant independent of ε. Using the above estimate as a starting point, the higher order estimates can be obtained by estimating the t derivatives of $\boldsymbol{u}$ in the same way, yielding uniform estimates in X^l. Finally, we can take the limit $\varepsilon \to +0$ and this proves Theorem 3.1.

5.2 Remark on the Case $\alpha > 0$

The case $\alpha > 0$ can be treated by a standard argument. We start by considering the following regularized problem for $\varepsilon > 0$.

$$\begin{cases} \boldsymbol{u}_t = -\varepsilon \boldsymbol{u}_{xxxx} + \boldsymbol{g}, & x > 0, t > 0, \\ \boldsymbol{u}(x,0) = \boldsymbol{u}_0(x), & x > 0, \\ \boldsymbol{u}(0,t) = \boldsymbol{e}, & t > 0, \\ \boldsymbol{u}_x(0,t) = \boldsymbol{0}, & t > 0. \end{cases}$$

The construction of the solution can be done explicitly via Fourier and Laplace transforms. After an iteration argument, we can construct the solution to

$$\begin{cases} \boldsymbol{u}_t = \alpha \boldsymbol{u}_{xxx} - \varepsilon \boldsymbol{u}_{xxxx} + \mathrm{A}(\boldsymbol{w}, \partial_x)\boldsymbol{u} + \boldsymbol{f}, & x > 0, t > 0, \\ \boldsymbol{u}(x,0) = \boldsymbol{u}_0(x), & x > 0, \\ \boldsymbol{u}(0,t) = \boldsymbol{e}, & t > 0, \\ \boldsymbol{u}_x(0,t) = \boldsymbol{0}, & t > 0. \end{cases}$$

The uniform estimate can be obtained by using the standard Sobolev norm as the energy. This allows us to take the limit $\varepsilon \to +0$, proving Theorem 3.2.

6 Vortex Filament with Axial Flow

We utilize Theorems 3.1 and 3.2 to prove the following.

Theorem 6.1 (The case $\alpha > 0$) *For a natural number k, if $\boldsymbol{x}_{0ss} \in H^{2+3k}(\mathbf{R}_+)$, $|\boldsymbol{x}_{0s}| = 1$, and $\boldsymbol{x}_{0s}$ satisfies the compatibility conditions for* (1.5) *up to order k, then* *there exists $T>0$ such that* (1.2) *has a unique solution $\boldsymbol{x}$ satisfying*

$$\boldsymbol{x}_{ss} \in \bigcap_{j=0}^{k} W^{j,\infty}\big([0,T]; H^{2+3j}(\mathbf{R}_+)\big)$$

and $|\boldsymbol{x}_s| = 1$. Here, T depends on $\|\boldsymbol{x}_{0ss}\|_2$.

Theorem 6.2 (The case $\alpha < 0$) *For a natural number k, if $\boldsymbol{x}_{0ss} \in H^{1+3k}(\mathbf{R}_+)$, $|\boldsymbol{x}_{0s}| = 1$, and $\boldsymbol{x}_{0s}$ satisfies the compatibility conditions for* (1.4) *up to order k, then there exists $T>0$ such that* (1.1) *has a unique solution $\boldsymbol{x}$ satisfying*

$$\boldsymbol{x}_{ss} \in \bigcap_{j=0}^{k} W^{j,\infty}\big([0,T]; H^{1+3j}(\mathbf{R}_+)\big)$$

and $|\boldsymbol{x}_s| = 1$. Here, T depends on $\|\boldsymbol{x}_{0ss}\|_3$.

6.1 Compatibility Conditions

We derive the compatibility conditions for (1.4) and (1.5). We set $\boldsymbol{Q}_{(0)}(\boldsymbol{v}) = \boldsymbol{v}$ and we denote the right-hand side of the equation in (1.4) and (1.5) as

$$\boldsymbol{Q}_{(1)}(\boldsymbol{v}) = \boldsymbol{v} \times \boldsymbol{v}_{ss} + \alpha\big\{\boldsymbol{v}_{sss} + \frac{3}{2}\boldsymbol{v}_{ss} \times (\boldsymbol{v} \times \boldsymbol{v}_s) + \frac{3}{2}\boldsymbol{v}_s \times (\boldsymbol{v} \times \boldsymbol{v}_{ss})\big\}.$$

We will also use the notation $\boldsymbol{Q}_{(1)}(s,t)$ and $\boldsymbol{Q}_{(1)}$ instead of $\boldsymbol{Q}_{(1)}(\boldsymbol{v})$ for convenience. For $n \geq 2$, we successively define $\boldsymbol{Q}_{(n)}$ by

$$\begin{aligned}
\boldsymbol{Q}_{(n)} &= \sum_{j=0}^{n-1} \binom{n-1}{j} \boldsymbol{Q}_{(j)} \times \boldsymbol{Q}_{(n-1-j)ss} + \alpha \boldsymbol{Q}_{(n-1)sss} \\
&\quad + \frac{3}{2}\alpha\Big\{\sum_{j=0}^{n-1}\sum_{k=0}^{n-1-j} \binom{n-1}{j}\binom{n-1-j}{k} \boldsymbol{Q}_{(j)ss} \times \Big(\boldsymbol{Q}_{(k)} \times \boldsymbol{Q}_{(n-1-j-k)s}\Big)\Big\} \\
&\quad + \frac{3}{2}\alpha\Big\{\sum_{j=0}^{n-1}\sum_{k=0}^{n-1-j} \binom{n-1}{j}\binom{n-1-j}{k} \boldsymbol{Q}_{(j)s} \times \Big(\boldsymbol{Q}_{(k)} \times \boldsymbol{Q}_{(n-1-j-k)ss}\Big)\Big\}.
\end{aligned}$$

The above definition of $\boldsymbol{Q}_{(n)}(\boldsymbol{v})$ corresponds to giving an expression for $\partial_t^n \boldsymbol{v}$ in terms of $\boldsymbol{v}$ and its s derivatives only. It is obvious from the definition that the term with the highest order derivative in $\boldsymbol{Q}_{(n)}$ is $\alpha^n \partial_s^{3n} \boldsymbol{v}$. From the boundary conditions of (1.4) and (1.5), we arrive at the following compatibility conditions.

Definition 6.3 (*Compatibility conditions for* (1.4)) *For $n \in \mathbf{N} \cup \{0\}$, we say that $\boldsymbol{v}_0$ satisfies the n-th comaptibility condition for* (1.4) *if $\boldsymbol{v}_{0s} \in H^{1+3n}(\mathbf{R}_+)$ and*

$$\big(\partial_s \boldsymbol{Q}_{(n)}(\boldsymbol{v}_0)\big)(0) = \mathbf{0}.$$

We also say that $\boldsymbol{v}_0$ satisfies the compatibility conditions for (1.4) *up to order n if it satisfies the k-th compatibility condition for all k with $0 \leq k \leq n$.*

Definition 6.4 (*Compatibility conditions for* (1.5)) *For $n \in \mathbf{N} \cup \{0\}$, we say that $\boldsymbol{v}_0$ satisfies the n-th comaptibility condition for* (1.5) *if $\boldsymbol{v}_{0s} \in H^{2+3n}(\mathbf{R}_+)$ and*

$$\boldsymbol{v}_0(0) = \boldsymbol{e}_3, \ \ \boldsymbol{v}_{0s}(0) = \mathbf{0},$$

when $n = 0$, and

$$(\boldsymbol{Q}_{(n)}(\boldsymbol{v}_0))(0) = \mathbf{0}, \ \ \big(\partial_s \boldsymbol{Q}_{(n)}(\boldsymbol{v}_0)\big)(0) = \mathbf{0},$$

when $n \geq 1$. We also say that $\boldsymbol{v}_0$ satisfies the compatibility conditions for (1.5) *up to order n if it satisfies the k-th compatibility condition for all k with $0 \leq k \leq n$.*

Note that the regularity imposed on $\boldsymbol{v}_{0s}$ in Definition 6.4 is not the minimal regularity required for the trace at $s = 0$ to have meaning, but we defined it as above so that it corresponds to the regularity assumption in the existence theorem that we obtain later. Also note that the regularity assumption is made on $\boldsymbol{v}_{0s}$ instead of $\boldsymbol{v}_0$ because $|\boldsymbol{v}_0| = 1$ and so $\boldsymbol{v}_0$ is not square integrable.

6.2 Construction of Solutions

By setting

$$\mathrm{A}(\boldsymbol{w}, \partial_x)\boldsymbol{v} = \delta \boldsymbol{v}_{xx} + \boldsymbol{w} \times \boldsymbol{v}_{xx} + 3\alpha \boldsymbol{v}_{xx} \times (\boldsymbol{w} \times \boldsymbol{w}_x), \tag{6.1}$$

we can apply the two existence theorems for the linear problems to construct the solutions to

$$\begin{cases} \boldsymbol{v}_t = \boldsymbol{v} \times \boldsymbol{v}_{ss} + \alpha\big\{\boldsymbol{v}_{sss} + \frac{3}{2}\boldsymbol{v}_{ss} \times \big(\boldsymbol{v} \times \boldsymbol{v}_s\big) + \frac{3}{2}\boldsymbol{v}_s \times (\boldsymbol{v} \times \boldsymbol{v}_{ss})\big\} & \\ \qquad\qquad\qquad\qquad\qquad\qquad + \delta\big(\boldsymbol{v}_{ss} + |\boldsymbol{v}_s|^2\boldsymbol{v}\big), & s > 0, t > 0, \\ \boldsymbol{v}(s,0) = \boldsymbol{v}_0^\delta(s), & s > 0, \\ \boldsymbol{v}_s(0,t) = \mathbf{0}, & t > 0, \end{cases}$$

and

$$\begin{cases} \boldsymbol{v}_t = \boldsymbol{v} \times \boldsymbol{v}_{ss} + \alpha\big\{\boldsymbol{v}_{sss} + \frac{3}{2}\boldsymbol{v}_{ss} \times (\boldsymbol{v} \times \boldsymbol{v}_s) + \frac{3}{2}\boldsymbol{v}_s \times (\boldsymbol{v} \times \boldsymbol{v}_{ss})\big\} & \\ \qquad\qquad\qquad\qquad\qquad\qquad + \delta\big(\boldsymbol{v}_{ss} + |\boldsymbol{v}_s|^2\boldsymbol{v}\big), & s > 0, t > 0, \\ \boldsymbol{v}(s,0) = \boldsymbol{v}_0^\delta, & s > 0, \\ \boldsymbol{v}(0,t) = \boldsymbol{e}_3, & t > 0, \\ \boldsymbol{v}_s(0,t) = \mathbf{0}, & t > 0, \end{cases}$$

through iteration. Here, we have used $|\boldsymbol{v}| \equiv 1$ to rewrite the nonlinear term.

The final task is to obtain estimates uniform in δ, which is also equivalent to obtaining estimates for the solution of the limit systems. When $\alpha < 0$, we make use of conserved quantities. These quantities are conserved for the initial value problem with $\delta = 0$. Although they are not conserved for our initial-boundary value problems, we can still take advantage of these quantities. In fact we see that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{v}_s\|^2 &= \frac{\alpha}{2}|\boldsymbol{v}_{ss}(0)|^2 - \delta\|\boldsymbol{v}_{ss}\|^2 + \delta\|\boldsymbol{v}_s\|^4_{L^4(\mathbf{R}_+)} \\ &\leq \frac{\alpha}{2}|\boldsymbol{v}_{ss}(0)|^2 - \frac{\delta}{2}\|\boldsymbol{v}_{ss}\|^2 + C\delta\|\boldsymbol{v}_s\|^6.\end{aligned}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\|\boldsymbol{v}_{ss}\|^2 - \frac{5}{4}\||\boldsymbol{v}_s|^2\|^2\right\} \leq \alpha|\boldsymbol{v}_{sss}(0)|^2 - \frac{\delta}{4}\|\boldsymbol{v}_{sss}\|^2 + C_1.$$

Here, C_1 is a positive constant depending on $\|\boldsymbol{v}_s\|$. Thus, when $\alpha < 0$, the above give a closed estimate for $\|\boldsymbol{v}_s\|_1$ and using this as the basic estimate, a standard energy method yields the necessary higher order estimate.

When $\alpha > 0$, the boundary value appearing in the above estimates have a bad sign, and thus, we need something extra to close the estimate. To do this, we first define some notations. We set $\boldsymbol{P}_{(0)}(\boldsymbol{v}) = \boldsymbol{v}$ and define $\boldsymbol{P}_{(1)}(\boldsymbol{v})$ by

$$\boldsymbol{P}_{(1)}(\boldsymbol{v}) = \boldsymbol{v} \times \boldsymbol{v}_{ss} + \alpha\big\{\boldsymbol{v}_{sss} + \frac{3}{2}\boldsymbol{v}_{ss} \times (\boldsymbol{v} \times \boldsymbol{v}_s) + \frac{3}{2}\boldsymbol{v}_s \times (\boldsymbol{v} \times \boldsymbol{v}_{ss})\big\} + \delta\big(\boldsymbol{v}_{ss} + |\boldsymbol{v}_s|^2\boldsymbol{v}\big).$$

We successively define $\boldsymbol{P}_{(n)}$ for $n \geq 2$ by

$$\begin{aligned}\boldsymbol{P}_{(n)} &= \sum_{j=0}^{n-1}\binom{n-1}{j}\boldsymbol{P}_{(j)} \times \boldsymbol{P}_{(n-1-j)ss} + \alpha\boldsymbol{P}_{(n-1)sss} \\ &+ \frac{3}{2}\alpha\left\{\sum_{j=0}^{n-1}\sum_{k=0}^{n-1-j}\binom{n-1}{j}\binom{n-1-j}{k}\boldsymbol{P}_{(j)ss} \times \left(\boldsymbol{P}_{(k)} \times \boldsymbol{P}_{(n-1-j-k)s}\right)\right\} \\ &+ \frac{3}{2}\alpha\left\{\sum_{j=0}^{n-1}\sum_{k=0}^{n-1-j}\binom{n-1}{j}\binom{n-1-j}{k}\boldsymbol{P}_{(j)s} \times \left(\boldsymbol{P}_{(k)} \times \boldsymbol{P}_{(n-1-j-k)ss}\right)\right\} \\ &+ \delta\left\{\boldsymbol{P}_{(n-1)ss} + \sum_{j=0}^{n-1}\sum_{k=0}^{n-1-j}\binom{n-1}{j}\binom{n-1-j}{k}\left(\boldsymbol{P}_{(j)s} \cdot \boldsymbol{P}_{(k)s}\right)\boldsymbol{P}_{(n-1-j-k)}\right\}.\end{aligned}$$

The above definition of $\boldsymbol{P}_{(n)}$ corresponds to giving an expression for $\partial_t^n \boldsymbol{v}$ in terms of $\boldsymbol{v}$ and its s derivatives for the regularized nonlinear system.

To close the estimate, we use $\|\boldsymbol{v}_{sss}\|^2 + \frac{2}{\alpha}(\boldsymbol{v} \times \boldsymbol{v}_{ss}, \boldsymbol{v}_{sss})$ instead of $\|\boldsymbol{v}_{sss}\|^2$ as our next energy, yielding

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\|\boldsymbol{v}_{sss}\|^2 + \frac{2}{\alpha}(\boldsymbol{v} \times \boldsymbol{v}_{ss}, \boldsymbol{v}_{sss})\right\} \leq C\|\boldsymbol{v}_s\|_2^2(1 + \|\boldsymbol{v}_s\|_2^2),$$

which combined with the conserved quantity closes the estimate for $\|\boldsymbol{v}_s\|_2^2$. This modification is done to take care of boundary terms that give us trouble. If we directly estimate $\|\boldsymbol{v}_{sss}\|^2$, boundary term of the form $\boldsymbol{v}_{sss}(0)\cdot\partial_s^5\boldsymbol{v}(0)$ comes out and the order of derivative is too high to estimate. By adding a lower order modification term in the energy, we can cancel out this term. This kind of modification is needed every three derivatives. We use the first modification as an example to demonstrate the idea behind finding the correct modifying term. Taking the trace $s=0$ in the equation yields

$$\alpha\boldsymbol{v}_{sss}(0,t)+(\boldsymbol{v}\times\boldsymbol{v}_{ss})(0,t)=\mathbf{0}$$

for any $t>0$. Thus, replacing $\|\boldsymbol{v}_{sss}\|^2$ with $\|\boldsymbol{v}_{sss}\|^2+\frac{2}{\alpha}(\boldsymbol{v}\times\boldsymbol{v}_{ss},\boldsymbol{v}_{sss})$ changes the boundary term from $\boldsymbol{v}_{sss}(0)\cdot\partial_s^5\boldsymbol{v}(0)$ to $\big(\boldsymbol{v}_{sss}(0)+\frac{1}{\alpha}\boldsymbol{v}\times\boldsymbol{v}_{ss}(0)\big)\cdot\partial_s^5\boldsymbol{v}(0)$, which is zero.

We continue the estimate in this pattern. Suppose that we have a uniform estimate $\sup_{0\le t\le T}\|\boldsymbol{v}_s(t)\|_{2+3(i-1)}\le M$ for some $i\ge 1$. For $j=1,2$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_s^{3i+j}\boldsymbol{v}\|^2\le C(1+\|\boldsymbol{v}_s\|_{2+3i}^2),$$

where we have used $|\partial_s^{3(i+1)}\boldsymbol{v}(0)|^2\le C\|\boldsymbol{v}_s\|_{2+3i}^2$. Here, C depends on M, but not on δ. Set $\boldsymbol{W}_{(m)}(\boldsymbol{v}):=\boldsymbol{P}_{(m)}(\boldsymbol{v})-\alpha^m\partial_s^{3m}\boldsymbol{v}$, which is $\boldsymbol{P}_{(m)}(\boldsymbol{v})$ without the highest order derivative term. Then, the final estimate is

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\|\partial_s^{3(i+1)}\boldsymbol{v}\|^2+\frac{2}{\alpha^{i+1}}(\boldsymbol{W}_{(i+1)}(\boldsymbol{v}),\partial_s^{3(i+1)}\boldsymbol{v})\right\}\le C\|\boldsymbol{v}_s\|_{2+3i}^2+C,$$

where C is independent of δ. This allows us to take the limit $\delta\to+0$, which finishes the proof of Theorem 6.1 and 6.2.

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