

Convergence rates towards traveling waves for a model system of radiating gas

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1 Introduction

In the present paper, we study the initial value problem to the system of equations:

$$u_t + uu_x + q_x = 0, \quad (1.1a)$$

$$-q_{xx} + q + u_x = 0, \quad (1.1b)$$

where $u(t, x)$ and $q(t, x)$ are real-valued functions for $t \geq 0$ and $x \in \mathbb{R}$ with

$$u(0, x) = u_0(x), \quad (1.2a)$$

$$u_0(x) \rightarrow u_{\pm}, \quad q(t, x) \rightarrow 0 \quad \text{as } x \rightarrow \pm\infty. \quad (1.2b)$$

The hyperbolic-elliptic coupled system (1.1) is originally derived in [3] from equations for the polytropic gas with a radiative heat flow. Its governing equations are

$$\rho_t + (\rho u)_x = 0, \quad (1.3a)$$

$$\rho(u_t + uu_x) + p_x = 0, \quad (1.3b)$$

$$\rho\theta(s_t + us_x) + q_x = 0, \quad (1.3c)$$

$$p = \rho R\theta = A\rho^\gamma \exp((\gamma - 1)s/R), \quad (1.3d)$$

$$-q_{xx} + 3\alpha^2 q + 4\alpha\sigma(\theta^4)_x = 0. \quad (1.3e)$$

Here, R is the gas constant, $\gamma > 1$ the (constant) rate of specific heats, A, α positive constants and σ is the Stefan-Boltzmann constant. The unknown functions ρ, u, p, θ, s and q represent the density, the velocity, the pressure, the absolute temperature, the entropy and the radiative heat-flux, respectively. Now we see the derivation of (1.1) from (1.3) following [4].

Since the Stefan-Boltzmann constant σ is small, we assume it is expressed as $\sigma = \varepsilon\sigma_0$ for a dimensionless small parameter ε and a positive constant σ_0 . We easily see that a constant state $(\rho, u, s, q) = (\rho_0, 0, s_0, 0)$ where ρ_0 and s_0 are positive constants, is a solution. Then we expand a state (ρ, u, s, q) around the constant solution $(\rho_0, 0, s_0, 0)$ as

$$\begin{cases} \rho = \rho_0 + \varepsilon\bar{\rho}(\bar{t}, \bar{x}), \\ u = \varepsilon\bar{u}(\bar{t}, \bar{x}), \\ s = s_0 + \varepsilon^2\bar{s}(\bar{t}, \bar{x}), \\ q = \varepsilon^2\bar{q}(\bar{t}, \bar{x}) \end{cases} \quad (1.4)$$

where $\bar{\rho}$, \bar{u} , \bar{s} and \bar{q} are functions of $\bar{t} = \varepsilon t$ and $\bar{x} = x - C_0 t$. Here, C_0 is the acoustic velocity, given by

$$C_0 = \sqrt{\frac{\partial p}{\partial \rho}(\rho_0, s_0)} = \sqrt{\gamma \frac{p_0}{\rho_0}} = \sqrt{\gamma R \theta_0}.$$

Expanding p and θ around the equilibrium state, we have

$$p = p_0 + \varepsilon C_0^2 \bar{\rho} + \varepsilon^2 (\gamma - 1) \left(\frac{C_0^2}{2\rho_0} \bar{\rho}^2 + \rho_0 \theta_0 \bar{s} \right) + \mathcal{O}(\varepsilon^3) \quad (1.5)$$

and

$$\theta = \theta_0 + \varepsilon (\gamma - 1) \frac{\theta_0}{\rho_0} \bar{\rho} + \mathcal{O}(\varepsilon^2). \quad (1.6)$$

By substituting these expansions in the system (1.3) retaining up to $O(\varepsilon^2)$ terms, we obtain the simplified system of (1.1).

The first mathematical results on the radiating gas model are obtained in [13]. They show that the radiating gas model admits traveling waves and retains monotonicity like viscous Burgers equation, while it does not smooth out initial discontinuities. Later in [6], it is shown that the first order derivative of the solution blows up in finite time if the initial gradient is smaller than a certain negative constant even for smooth initial data.

From (1.1b), we can rewrite (1.1) in the Fourier multiplier form as

$$u_t + uu_x + \mathcal{F}^{-1} \left[\frac{\xi^2}{1 + \xi^2} \mathcal{F}[u](\xi) \right] = 0. \quad (1.7)$$

This expression is derived also in [12] by regularizing the Chapman-Enskog expansion. Comparing (1.7) with the viscous Burgers equation, we speculate that the radiating gas model has an intermediate property between the inviscid Burgers equation and the viscous Burgers equation.

In view of these properties of the radiating gas model, a generalized notion of solutions, called admissible solutions, are introduced in [6] following ideas by Kruzkov in [8]. Precisely, we define an admissible solution as follows.

Definition 1.1. We define an admissible solution $(u, q)(t, x)$ to (1.1) and (1.2) in the weak sense by a set of functions $(u, q) \in L^\infty([0, T] \times \mathbb{R})$ which satisfies

$$\int_0^T \int_{-\infty}^{+\infty} |u - k| \phi_t + \text{sign}(u - k) \left(\frac{1}{2} u^2 - \frac{1}{2} k^2 \right) \phi_x - \text{sign}(u - k)(u - Ku)\phi \, dxdt \geq 0 \quad (1.8)$$

for an arbitrary nonnegative function $\phi \in C_0^\infty((0, T) \times \mathbb{R})$, and an arbitrary constant $k \in \mathbb{R}$,

$$\int_{-\infty}^{+\infty} -q\psi_{xx} + q\psi - u\psi_x \, dx = 0 \quad (1.9)$$

for an arbitrary $\psi \in \mathcal{S}(\mathbb{R})$, and the initial condition

$$u(0, x) = u_0(x) \text{ almost every } x \in \mathbb{R}. \quad (1.10)$$

Our main focus is the traveling wave solution to (1.1), which is expressed in the form of

$$(u, q)(t, x) = (U, Q)(\eta), \quad \eta = x - st \quad (1.11)$$

for a certain constant s . By substituting (1.11) in (1.1), we obtain

$$-sU' + UU' + Q' = 0, \quad (1.12a)$$

$$-Q'' + Q + U' = 0. \quad (1.12b)$$

The conditions for the existence of traveling waves are obtained in [5] as follows.

Proposition 1.2. Assume traveling waves are piecewise smooth with the first kind of discontinuities. Set $\delta_S := u_- - u_+$.

(i) Suppose there exists a traveling wave solution $(U, Q)(\eta)$ to (1.1) which satisfies

$$U(\eta) \rightarrow u_\pm \text{ as } \eta \rightarrow \pm\infty. \quad (1.13)$$

Then we have

$$u_- > u_+, \quad s = (u_- + u_+)/2 \quad (1.14)$$

and

$$Q(\eta) \rightarrow 0 \text{ as } \eta \rightarrow \pm\infty. \quad (1.15)$$

(ii) Conversely, we suppose that (1.14) holds. Then there exists a traveling wave $(U, Q)(\eta)$ satisfying (1.13) and (1.15). This traveling wave is unique up to a shift. Moreover, the differentiability of the traveling wave solution depends on the shock strength:

(a) If $\delta_S \leq \sqrt{2}$, then $U(\eta) \in B^1(\mathbb{R})$ and $Q(\eta) \in B^2(\mathbb{R})$.

(b) If $\delta_S < 2\sqrt{2n}/(n+1)$ for $n \in \mathbb{N}$, then $U(\eta) \in B^n(\mathbb{R})$ and $Q(\eta) \in B^{n+1}(\mathbb{R})$.

Furthermore, the following estimates hold:

$$\left| \frac{d^n}{d\eta^n} U(\eta) \right| \leq C\delta_S^{n+1} \text{ for } n \in \mathbb{N}, \quad (1.16)$$

where C is a positive constant depending only on n , and specifically,

$$|U(\eta) - u_{\pm}| \leq \frac{1}{2} \delta_S e^{-\sigma|\eta|}, \quad -\frac{1}{4} \delta_S^2 \leq U'(\eta) < 0 \quad \text{for } n \in \mathbb{N}, \quad (1.17)$$

where σ is a positive constant depending only on δ_S .

Remark 1.3. *In the case $\delta_S > \sqrt{2}$, by extending the definition of traveling waves (see [5] for the definition of the admissible traveling waves), there is still a traveling wave solution to the system in which U is continuous except for one point, while Q is Lipschitz continuous.*

Here we review known results about the asymptotic stability of the traveling wave. The paper [5] proves the asymptotic stability of the traveling wave assuming that $\delta_S < \sqrt{6}/2$ (the case with $n = 3$ in Proposition 1.2 (ii)(c)), the initial perturbation is in $L^1 \cap H^2(\mathbb{R})$ so that it does not contain discontinuity, and its anti-derivative is small in the Sobolev space $H^3(\mathbb{R})$. In [9], the asymptotic stability of the traveling wave with $\delta_S \leq 1/2$ is studied in the case that the initial perturbation has a piecewise B^1 regularity except a discontinuity at $x = 0$, as an extension of the stability results obtained in [6] for the Riemann initial data with $\delta_S \leq 1/2$. In these papers, the convergence rate of $t^{-1/4}$ is obtained additionally assuming in [5, 9] that the anti-derivative of the initial perturbation belongs to L^1 . The authors in [10] improve the results of [6] by showing that if the initial data is the Riemann data with $\delta_S \leq 1/2$, the solution uniformly converges to the corresponding traveling wave exponentially fast. The proof uses the property (1.17), i.e., the exponential convergence of the traveling wave towards asymptotic values. They also show that if the initial data is smooth and the perturbation from the traveling wave with algebraic weight belongs to a suitable Sobolev space, then the perturbation decays algebraically fast. In [15], assuming the pointwise algebraic decay of the initial data, the pointwise algebraic decay in time is derived. For researches with other initial conditions, readers are referred to [7, 14]. The extension of this problem to multidimensional problems are seen in [1, 2] for example.

The main purpose of the present paper is to improve or generalize results in [9, 10]. Namely, using the weighted energy method, we obtain a convergence rate subject to the spatial decay rate of the initial perturbation. Hence in many cases, our results yield better decay rates than [9]. Our results apply not only to the Riemann data, but also to general initial data which have a discontinuity at one point. Moreover, our results admit the traveling wave which satisfies $\delta_S < \sqrt{6}/2$, which is the maximum value to assure $U \in B^3$. On the other hand, [9, 10] assume $\delta_S \leq 1/2$ to make use of the maximum principle for u_x from below, which makes it easy to obtain a global solution. Hence our results have a wider applicability than the previous results. Instead, extra care is needed in constructing local solutions with weight functions and in obtaining a-priori estimates of discontinuous data for which we no longer have an a-priori lower bound of u_x .

To state our main result precisely, we define some functions and quantities. Hereafter $u_- > u_+$ is always assumed and u_0 is assumed to be discontinuous only at $x = 0$. We suppose

$$u_0 - u_S \in L^1, \quad (1.18)$$

where u_S is a step function defined by

$$u_S(x) = \begin{cases} u_- & \text{for } x < 0 \\ u_+ & \text{for } x > 0. \end{cases} \quad (1.19)$$

Setting the shift of the traveling wave solution so that $U(0) = (u_- + u_+)/2$ holds, we define 'the center of mass' by

$$x_0 := \frac{1}{u_- - u_+} \int_{-\infty}^{\infty} (u_0(x) - U(x)) dx, \quad (1.20)$$

and the initial perturbation and its anti-derivative by

$$\phi_0(x) := u_0(x) - U(x - x_0), \quad (1.21)$$

$$\Phi_0(x) := \int_{-\infty}^x \phi_0(y) dy. \quad (1.22)$$

Note

$$\int_{-\infty}^{\infty} \phi_0(x) dx = \int_{-\infty}^{\infty} (u_0(x) - U(x - x_0)) dx = 0 \quad (1.23)$$

holds by the definition of x_0 in (1.20).

Our main focus is to obtain the convergence rate subject to the spatial decay rate of the initial perturbation. Here and hereafter we assume $u_- + u_+ = 0$ and thus $s = 0$ without loss of generality. The discontinuities of the solution are known ([9]) to be located on a C^1 -curve $\{x = d(t)\}$ with $d(0) = 0$ in the (t, x) -space and it moves at a speed of

$$\dot{d}(t) = \frac{d}{dt}d(t) = \frac{1}{2} \left(\lim_{x \rightarrow d(t)-0} u(t, x) + \lim_{x \rightarrow d(t)+0} u(t, x) \right). \quad (1.24)$$

Then we define the perturbation as

$$\phi(\tau, \xi) := u(\tau, \xi + d(\tau)) - U(\xi + d(\tau) - x_0), \quad (1.25a)$$

$$\psi(\tau, \xi) := q(\tau, \xi + d(\tau)) - Q(\xi + d(\tau) - x_0), \quad (1.25b)$$

where $\tau \geq 0$ and $\xi \in \mathbb{R}_0$. By these steps, we see ϕ always has a discontinuity only on $\xi = 0$. The governing equations for ϕ and ψ are obtained from (1.1) and (1.12) as

$$\phi_\tau + (U - \dot{d}(\tau)) \phi_\xi + U' \phi + \phi \phi_\xi + \psi_\xi = 0, \quad (1.26a)$$

$$-\psi_{\xi\xi} + \psi + \phi_\xi = 0. \quad (1.26b)$$

By the definition of ϕ_0 in (1.21) and $d(0) = 0$, the initial data is

$$\phi(0, \xi) = \phi_0(\xi). \tag{1.27}$$

In the case $\phi(\tau, \cdot) \in L^1(\mathbb{R}_0)$, we define the anti-derivative of ϕ by

$$\Phi(\tau, \xi) := \int_{-\infty}^{\xi} \phi(\tau, \eta) d\eta. \tag{1.28}$$

From (1.24) and (1.26a), we see that $\int_{-\infty}^{\infty} \phi(\tau, \xi) d\xi = 0$ and $\Phi(\tau, \xi) = \int_{\infty}^{\xi} \phi(\tau, \eta) d\eta$. By integrating (1.26a) over $(-\infty, \xi]$, we have

$$\Phi_{\tau} + (U - \dot{d}(\tau))\Phi_{\xi} + \frac{1}{2}\Phi_{\xi}^2 + \psi = 0. \tag{1.29}$$

Now we are ready to state our main results.

Theorem 1.4. ([11])

Assume $\phi_0 \in L^1(\mathbb{R}_0)$, $u_0(0-0) > u_0(0+0)$ and $0 < u_- - u_+ < \sqrt{6}/2$ hold.

(i) Suppose $e^{\lambda|\xi|/2}\Phi_0(\xi) \in H^3(\mathbb{R}_0)$ holds for $\lambda \in (0, \lambda_0)$, where λ_0 is a certain positive constant less than $\sqrt{2}$. Then there exists a positive constant ε such that if $\|e^{\lambda|\xi|/2}\Phi_0(\xi)\|_{H^3} \leq \varepsilon$, the initial value problem (1.26) and (1.27) has a unique global admissible solution as $e^{\lambda|\xi|/2}\Phi(t, \xi) \in \mathfrak{X}_2^1([0, \infty), \mathbb{R}_0)$, and $e^{\lambda|\xi|/2}\psi(t, \xi) \in \mathfrak{X}_2^1([0, \infty), \mathbb{R}_0)$, which satisfies

$$\inf_{t \geq 0, x \neq d(t)} u_x(t, x) > -1. \tag{1.30}$$

The discontinuity approaches the center of mass:

$$d(t) \rightarrow x_0 \text{ as } t \rightarrow \infty. \tag{1.31}$$

The solution also verifies the decay estimate

$$\begin{aligned} & \|\Phi(t, \xi)\|_{H^3}^2 + \|\psi(t, \xi)\|_{H^3}^2 \\ & \leq C \{ \|e^{\lambda|\xi|/2}\Phi_0(\xi)\|_{H^3}^2 + |[\phi_0]|(1 + \|u_0\|_{L^\infty} + \|U\|_{L^\infty}) \} e^{-\gamma t} \end{aligned} \tag{1.32}$$

for certain positive constants C and γ independent of t .

(ii) Suppose $\langle \xi \rangle^{\alpha/2}\Phi_0(\xi) \in H^3(\mathbb{R}_0)$ holds for a certain positive constant α . Then there exists a positive constant ε such that if $\|\langle \xi \rangle^{\alpha/2}\Phi_0(\xi)\|_{H^3} \leq \varepsilon$, the initial value problem (1.26) and (1.27) has a unique global admissible solution as $\langle \xi \rangle^{\alpha/2}\Phi(t, \xi) \in \mathfrak{X}_2^1([0, \infty), \mathbb{R}_0)$, and $\langle \xi \rangle^{\alpha/2}\psi(t, \xi) \in \mathfrak{X}_2^1([0, \infty), \mathbb{R}_0)$, which satisfies (1.30) and (1.31). Moreover, the solution verifies the decay estimate

$$\begin{aligned} & \|\Phi(t, \xi)\|_{H^3}^2 + \|\psi(t, \xi)\|_{H^3}^2 \\ & \leq C \{ \|\langle \xi \rangle^{\alpha/2}\Phi_0(\xi)\|_{H^3}^2 + |[\phi_0]|(1 + \|u_0\|_{L^\infty} + \|U\|_{L^\infty}) \} (1+t)^{-\alpha} \end{aligned} \tag{1.33}$$

for a certain positive constant C independent of t .

Notation: We define $\mathbb{R}_0, \mathbb{R}_-$ and \mathbb{R}_+ , subsets of \mathbb{R} , as follows; $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$, $\mathbb{R}_- := \{x \in \mathbb{R}; x < 0\}$, $\mathbb{R}_+ := \{x \in \mathbb{R}; x > 0\}$. For a nonnegative integer $l \geq 0$ and Ω , a domain of \mathbb{R} , $H^l(\Omega)$ denotes the l -th order Sobolev space in the L^2 sense, equipped with the norm $\|\cdot\|_{H^l}$, and $C^k([0, T]; H^l(\Omega))$ denotes the space of k -times continuously differentiable functions on the interval $[0, T]$ with values in $H^l(\Omega)$. The function space \mathfrak{X}_i^j is defined by

$$\mathfrak{X}_i^j([0, T], \Omega) := \bigcap_{k=0}^i C^k([0, T]; H^{j+i-k}(\Omega)), \quad i, j \in \mathbb{Z}, \quad i, j \geq 0.$$

We define algebraically weighted norm by $|f|_{\alpha, i} := \left(\sum_{j=0}^i \|\langle x \rangle^{\alpha/2} \partial_x^j f(x)\|_{L^2}^2 \right)^{1/2}$, where $\langle x \rangle := (1 + x^2)^{1/2}$. We often omit the last subscript if $i = 0$: $|f|_{\alpha} = |f|_{\alpha, 0}$. For a function f over \mathbb{R}_0 , $[f]$ denotes the jump quantity of a function f at 0, i.e., $[f] := f(0 + 0) - f(0 - 0)$. Finally, c and C denote generic positive constants and C_{ε} denotes a generic positive constant depending on a positive constant ε .

Remark 1.5. *The definitions of weight functions in Theorem 1.4(i)(ii) are both based on the distance from the location of discontinuity $|\xi|$. If we define the weight functions based on the distance from the center of mass, we deduce essentially the same conclusions as Theorem 1.4. This is well understood by notifying that the discontinuity approaches the center of mass as shown in (1.31). This definition has a benefit because it can be applicable to the stability problem of traveling waves which do not have discontinuity or have multiple discontinuities.*

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