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A Report on Studies of Relative Randomness

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Abstract

We report some results of our recent studies. Let Γ be a set of (Turing) oracles. A set Z is called Γ-random if Z is ML-random relative to A for all $A \in \Gamma$. We use $\mathbb{L}$ and $\mathbb{G}$ to denote the set of low sets and the set of 1-generic sets, respectively. In [7], Yu proved that $\mathbb{L}$-randomness is equivalent to $\emptyset'$-Schnorr randomness, where $\emptyset'$ denotes the halting problem. We show that $(\mathbb{L} \cap \mathbb{G})$-randomness is still equivalent to $\emptyset'$-Schnorr randomness. We also proved that $(\mathbb{L} \cap \mathbb{MLR})$-randomness is equivalent to $\emptyset'$-Schnorr randomness.

1 Introduction

For a definition of random sequences, many approaches have been made until a definition was proposed by Martin-Löf [3] in 1966, which for the first time included all standard statistical properties of random sequences. The relativized randomness was first studied by Gaifman and Snir. We say that a set is n-random if it is ML-random relative to $\emptyset^{(n-1)}$. So it is 1-random if it is ML-random. 2-random if it is ML-random relative to $\emptyset'$. 2-randomness was first studied by Kurtz [6]. He also considered weak 2-randomness, an interesting notion lying strictly between Martin-Löf randomness and 2-randomness. In this report, we will introduce other randomness notions which between Martin-Löf randomness and 2-randomness.

Γ-randomness was first studied in [9], and is strongly connected with Yu's research [7]. The Γ-randomess notion could sometimes produce alternative proofs of existing results. For instance, some properties of $\emptyset'$-Schnorr randomness are proved more easily by the characterization due to $\mathbb{L}$-randomness than the usual methods. In section 3, we will report some new characterizations of $\mathbb{L}$-randomness. The detail proof of these results will be published in the future literature.

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2 Preliminaries

The collection of binary strings is denoted by $2^{<\mathbb{N}}$, i.e. the set of all functions from $\{0,\dots,n\}$ to $\{0,1\}$ for some $n \in \mathbb{N}$. We use $\sigma, \tau, \cdots$ to denote the elements of $2^{<\mathbb{N}}$. Let $2^{\mathbb{N}}$ denote the set of infinite binary sequences. Subsets of $\mathbb{N}$ can be identified with element of $2^{\mathbb{N}}$. These are also called *reals*. For sets A, B, Let $A \oplus B = \{2x : x \in A\} \cup \{2x+1 : x \in B\}$, namely the set which is A on the even bit positions and B on the odd positions.

For $\sigma \in 2^{<\mathbb{N}}$, we write $|\sigma|$ for the length of σ. Equivalently, $|\sigma| = \#\mathrm{dom}(\sigma)$. Here the cardinality of a set A is denoted by $\#A$. The empty string is denoted by λ. For strings σ and τ, let $\sigma \preceq \tau$ denotes that σ is a prefix of τ, i.e., $\mathrm{dom}(\sigma) \subseteq \mathrm{dom}(\tau)$ and $\sigma(m) = \tau(m)$ holds for each $m \in \mathrm{dom}(\sigma)$. The concatenation of two strings σ and τ is denoted by $\sigma\tau$. For a set A, $A \upharpoonright n$ is the prefix of A of length n. A topology of $2^{\mathbb{N}}$ is induced by basic open sets $[\sigma] = \{X \in 2^{\mathbb{N}} : X \succeq \sigma\}$ for all strings $\sigma \in 2^{<\mathbb{N}}$. So each open set of $2^{\mathbb{N}}$ is generated by a subset of $2^{<\mathbb{N}}$, that is $[S]^{\prec} = \{X \in 2^{\mathbb{N}} : \exists \sigma \in S\ \sigma \preceq X\}$. With this topology, $2^{\mathbb{N}}$ is called *the Cantor space.*

The *Lebesgue measure* on $2^{\mathbb{N}}$ is induced by giving each basic open set $[\sigma]$ measure $\mu([\sigma]) := 2^{-|\sigma|}$. for each string σ. If a class $G \subseteq 2^{\mathbb{N}}$ is open then $\mu(G) = \sum_{\sigma \in B} 2^{-|\sigma|}$ where B is a prefix-free set of strings such that $G = \bigcup_{\sigma \in B} [\sigma]$. A class $\mathcal{C} \subseteq 2^{\mathbb{N}}$ is called *null* if $\mu(\mathcal{C}) = 0$. If $2^{\mathbb{N}} - \mathcal{C}$ is null we say that $\mathcal{C}$ is *conull.*

3 Γ-randomness

ML-randomness is a central notion of algorithmic randomness for subsets of $\mathbb{N}$, which defined in the following way.

Definition 1 (Martin-Löf [3]). (i) A *Martin-Löf* test, or ML-test for short, is a uniformly c.e. sequence $(G_m)_{m \in \mathbb{N}}$ of open sets such that $\forall m \in \mathbb{N}\ \mu(G_m) \leq 2^{-m}$.

(ii) A set $Z \subseteq \mathbb{N}$ *fails* the test if $Z \in \bigcap_m G_m$, otherwise Z *passes* the test.

(iii) Z is *ML-random* if Z passes each ML-test. Let *MLR* denote the class of ML-random sets. Let *non-MLR* denote its complement in $2^{\mathbb{N}}$.

Following Schnorr [10], we will look at other natural notion of randomness, which refine the notion of Martin-Löf randomness.

Definition 2 (Schnorr [10]). A *Schnorr test* is a ML-test $(G_m)_{m \in \mathbb{N}}$ such that μG_m is computable uniformly in m. A set $Z \subseteq \mathbb{N}$ *fails* the test if $Z \in \bigcap_m G_m$, otherwise Z *passes* the test. Z is Schnorr random if Z passes each Schnorr test.

We recall some definitions in [9].

Definition 3. Let $\Gamma \subset \omega^{\omega}$. A set Z is Γ*-random* if Z is ML-random relative to f for all $f \in \Gamma$. Any ML-test relative to $f \in \Gamma$ is called a Γ-test.

For $f \in \omega^\omega$, we say f-random and f-test instead of $\{f\}$-random and $\{f\}$-test, respectively. Recall that a set A is low if $A' \leq_T \emptyset'$. In particular, Γ-randomness is called $\mathbb{L}$-*randomness* if Γ is the set of low sets.

Since a ML-test is a uniformly c.e. sequence $(G_m)_{m\in\mathbb{N}}$ of open sets such that $\forall m \in \mathbb{N}\ \mu G_m \leq 2^{-m}$. Thus, we can define an $\mathbb{L}$-test to be a sequence $(G_m)_{m\in\mathbb{N}}$ of open sets, which is uniformly c.e in some low set, such that $\forall m \in \mathbb{N}\ \ \mu G_m \leq 2^{-m}$.

The randomness notions between ML-randomness and 2-randomness have been extensively investigated in the literature by many researchers. In 2012, Yu [7] show that $\mathbb{L}$-randomness lying strictly between Martin-Löf randomness and 2-randomness.

Theorem 1 (Yu [7]). *$\mathbb{L}$-randomness is equivalent to $\emptyset'$-Schnorr randomness.*

In [8], we also give another characterization of $\mathbb{L}$-randomness. Let PA denote the set of all functions of PA degrees.

Proposition 1 (Peng, Higuchi, Yamazaki and Tanaka [8]). *$\mathbb{L}$-randomness is equivalent to $\mathbb{L} \cap$ PA-randomness.*

Let $\mathbb{G}$ denote the set of all 1-generic elements of 2^ω. Here, recall that an element Z of 2^ω is *1-generic* if for any c.e. subset W of $2^{<\omega}$, there exists $\sigma \prec Z$ such that either $\sigma \in W$ or $[\sigma] \cap W = \emptyset$ holds. It is well-known that any 1-generic element Z of 2^ω is generalized low, i.e., $Z \oplus \emptyset'$ computes Z'. Thus a 1-generic element of 2^ω is computable relative to $\emptyset'$ if and only if it is low.

Now we have the following theorem.

Theorem 2. *$(\mathbb{L} \cap \mathbb{G})$-randomness is equivalent to $\emptyset'$-Schnorr randomness.*

The following answer a question in [8].

Theorem 3. *$(\mathbb{L} \cap \mathbb{MLR})$-randomness is equivalent to $\emptyset'$-Schnorr randomness.*

A natural of Turing reducibility from the point of view of ML-randomness is the LR-reducibility which was introduced in [5].

Definition 4 (Nies [5]). For any $A, B \subseteq \mathbb{N}$, we say that A is *LR-reducible* to B, abbreviated $A \leq_{LR} B$, if

$$\forall X(X\ is\ B{-}random\ \Rightarrow X\ is\ A{-}random)$$

Intuitively this means that if oracle A can identify some patterns on some real x, oracle B can also find patterns on x. In other words, B is at least as good as A for this purpose.

In 2012, Diamondstone [2] show a surprising divergence between the LR degrees and the Turing degrees.

Theorem 4 (David, [2]). *For any low real X, Y, there exists a low c.e. real Z such that $X, Y \leq_{\mathrm{LR}} Z$.*

We also show some similar results as follows.

Theorem 5. *For any low real X, Y, there exists a low 1-generic real Z such that $X, Y \leq_{\mathrm{LR}} Z$.*

The above can be shown from theorem 2.

Theorem 6. *For any low real X, Y, there exists a low Martin-Löf random real Z such that $X, Y \leq_{\mathrm{LR}} Z$.*

This follows from theorem 3.

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References

[1] Rod G. Downey, Denis R. Hirshfeldt: Algorithmic Randomness and Complexity. *Springer-Verlag, Berlin. November, 2010.*

[2] David Diamondstone: Low upper bounds in the LR degrees. *Annals of Pure and Applied Logic, vol.* ***163****, no.3, pp. 314-320, 2012.*

[3] Per. Martin-Löf: The definition of random sequences. *Information and Control, vol.* ***9****, no. 6, pp. 602-619, 1966.*

[4] André Nies: Computability and Randomness. *Oxford University Press, 2009.*

[5] Andre. Nies: Lowness properties and randomness. *Advances in Mathematics, vol.* ***197****, pp. 274-305, 2005.*

[6] Stuart A. Kurtz: Randomness and genericity in the degrees of unsolvability. *In: Ph.D. Dissertation, University of Illinois, Urbana, 1981.*

[7] Liang Yu: Characterizing strong randomness via Martin-Löf randomness. *Annals of Pure and Applied Logic, vol.****163****, no. 3, pp. 214-224, 2012.*

[8] NingNing Peng, Kojiro Higuchi, Takeshi Yamazaki and Kazuyuki Tanaka Relative Randomness for Martin-Löf random Sets. *Lecture Notes in Computer Science, vol.****7318****, pp. 581-588, 2012.*

[9] NingNing Peng: The notions between Martin Löf randomness and 2-randomness. *RIMS Kôkyûroku, No.* ***1792****, pp. 117-122, 2010.*

[10] Claus Peter Schnorr: A unified approach to the definition of a random sequence, *Mathematical Systems Theory,* ***5****, 246-258, 1971.*