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ON VOLUME PRESERVING MOVES ON GRAPHS WITH PARABOLIC MERIDIANS

HIDETOSHI MASAI

1. INTRODUCTION

A spatial graph G in a closed 3-manifold M is a topological embedding of a graph into M. By abusing notation, we also regard G as the image of the embedding. We consider hyperbolic structures of $M \setminus G$ with totally geodesic boundary and of finite volume. A hyperbolic structure is determined once we fix whether the meridian of each edge corresponds to a hyperbolic or a parabolic isometry. Note that every choice dose not necessarily give a hyperbolic structure, however, the hyperbolic structure, if any, is unique i.e. rigidity theorem still holds (see [5], [3], and [10]). Heard has developed a computer program named Orb [4], which computes hyperbolic structures and many hyperbolic invariants of spatial graph complements. Orb is based on Weeks' computer program SnapPea. By using Orb, in [5], Heard, Hodgson, Martelli and Petronio enumerated many hyperbolic graphs with parabolic meridians i.e. the meridian of each edge corresponds to a parabolic isometry. However unlike hyperbolic knots and links, not so many works have been done about hyperbolic graphs. In this paper, we consider the hyperbolic graphs with parabolic meridians and discuss a relation between hyperbolic planar graphs and fully augmented links (see section 3 for the definition). From now on, by the word "hyperbolic graph", we always mean a hyperbolic graph with parabolic meridians. In section 2, we introduce moves which preserve volume. These moves correspond to cutting or gluing along a thrice punctured sphere. By this move, we can obtain a hyperbolic link from a hyperbolic graph of the same volume. In particular, we can get a fully augmented link if we apply those moves suitably on *planar* graphs. Then in section 3, we discuss Lackenby's volume estimates [7] in terms of the twist number. Lackenby (also an improvement by Agol-Thurston, see appendix of [7]) has used fully augmented links to get an upper bound of the volume of a given hyperbolic link. By the above moves, we have a corresponding planar trivalent simple graph for a given fully augmented link. There are only finitely many such graphs with n vertices and hence the best possible upper bounds B_n is attained by one of such graphs.

In the latter half of section 3 we compute approximated value of the best possible upper bounds for twist number $n = 2, 3, \ldots, 8$. First we see that a given planar graph is a hyperbolic graph if and only if the graph is cyclically m-connected for some $m \geq 3$ (see section 3 for the definition of the cyclically connectedness). Then to compute approximated value, we observe that if a planar graph is cyclically 3-connected, then the computation of the volume can be reduced to the case of fewer vertices. Hence we only need to investigate cyclically m-connected graphs with $m \geq 4$. We used plantri [2] to enumerate graphs with given number of vertices and with certain connectedness and Orb [4] to compute the approximated value of the hyperbolic volume.

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2. Volume preserving moves on hyperbolic graphs

From now on, we consider spatial graphs in S^3. We first recall a topological definition of a hyperbolic graph i.e. a hyperbolic graph with parabolic meridians (see also [5], [9]). Let G be a trivalent spatial finite graph in S^3 and $V \subset S^3$ the set of vertices of G. We define N_G as $S^3 \setminus G \setminus (\bigcup_{v \in V} \mathcal{N}(v))$ where $\mathcal{N}(v)$ is an open regular neighborhood of v. N_G is a 3-manifold with thrice punctured sphere boundary components, one corresponds to each vertex of G. G is said to be hyperbolic if N_G admits a hyperbolic metric of finite volume with totally geodesic boundary.

The following lemma relates hyperbolic graphs with hyperbolic links. The same proof can be found in [6], but we include this for the completeness.

Lemma 2.1. *For hyperbolic graphs, the moves from the left (or right) to center (cutting), and center to left (or right) (glueing) of Figure 1 are volume preserving.*

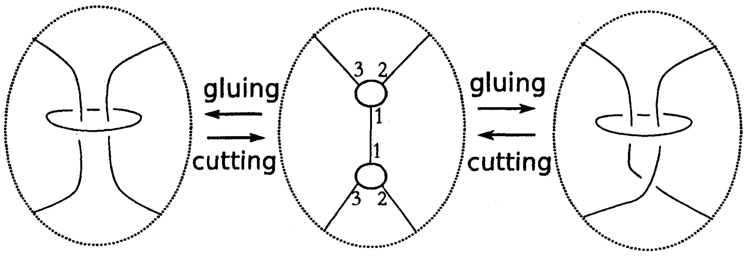

FIGURE 1. Volume preserving moves on hyperbolic graphs with parabolic meridians

Proof. Note that each vertex in graphs corresponds to a totally geodesic thrice punctured sphere and each edge corresponds to an annulus cusp. Moreover, the hyperbolic structure on a thrice punctured sphere is unique and hence any orientation preserving homeomorphism between hyperbolic thrice punctured spheres is isotopic to an isometry. Since homeomorphisms between thrice punctured spheres are uniquely determined by its action on cusps, we may denote homeomorphisms as elements of S_3, the symmetry group of degree 3. Fix labels of the cusps of thrice punctured sphere as in Figure 1. We glue the thrice punctured boundary components via the homeomorphism corresponding to $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$. Then we get the tangle in the left of Figure 1. When we glue the thrice punctured boundary components via the homeomorphism corresponding to $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$, we get the tangle in the right of Figure 1. To see this, we regard $S^3 = B^3 \cup B^3$, and reverse the inside and the outside with respect to one of the boundary components. After gluing, we get a link or graph in $S^2 \times S^1$. The component coming from the cusp labelled by 1 is going to be a loop which corresponds to a generator of the fundamental group of $S^2 \times S^1$. Therefore its complement is homeomorphic to a solid torus. Figure 2 depicts the argument. Thus we get a tangle in the left or right of Figure 1. The inverse move (left or right to middle) is verified by the fact that any essential thrice punctured sphere (or, 2-punctured disk) in hyperbolic manifolds is totally geodesic ([1]). Since we are dealing with hyperbolic graphs, the 2-punctured disk in the left or right of Figure 1 is essential (see [8] Lemma 2.1). □

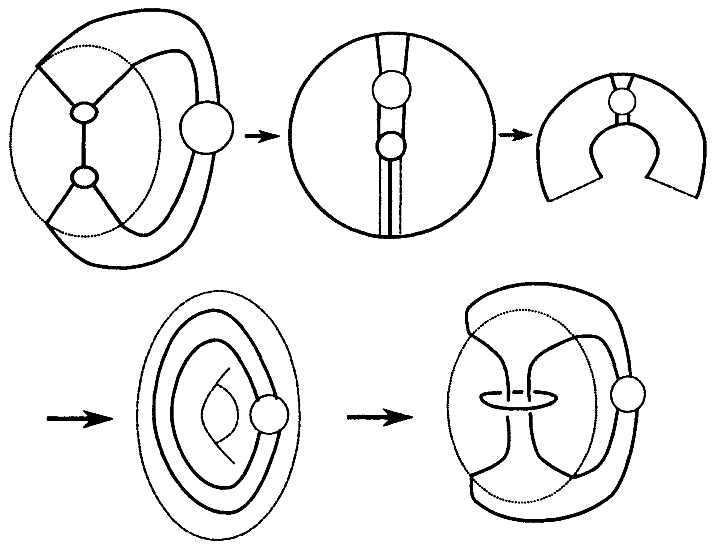

FIGURE 2. Glueing by a automorphism

Remark 2.2. In [9], van der Veen has demonstrated above moves for planar diagrams.

3. Some applications of volume preserving moves

In this section, we apply Lemma 2.1 to *fully augmented links*. First we recall the definition of a fully augmented link (see also [8]). Let D be a diagram of a link in S^3.

Definition 3.1. A *twist* of D is either a connected collection of bigon regions arranged in a row, which is maximal in the sense that it is not part of a longer row of bigons, or a single crossing adjacent to no bigon regions. The *twist number* of a diagram D is its number of twists and is denoted by $\mathrm{tw}(D)$.

Definition 3.2. A *fully augmented link* is a link obtained by encircling each twist by a single unknoted component and removing full-twists (see Figure 3). We call each added unknotted component a *crossing circle*. A component which is not a crossing circle is called a *knot component*.

The fully augmented link obtained from a diagram D has as many crossing circles as the twist number $\mathrm{tw}(D)$.

In [7] Lackenby (improved by Agol-Thurston) has proved the following theorem

Theorem 3.3 ([7]). *Let L be a hyperbolic link in S^3, D a diagram for L. Then,*

$$\mathrm{Vol}(S^3 \setminus L) < 10v_3(\mathrm{tw}(D) - 1)$$

where $\mathrm{tw}(D)$ is the twist number of the diagram D.

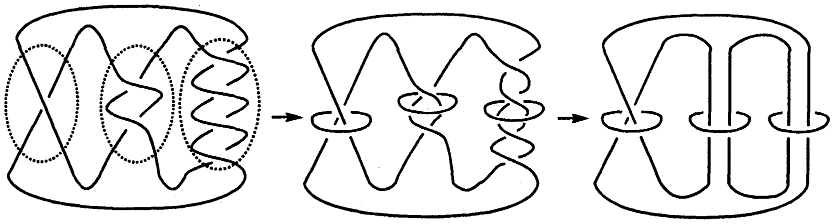

FIGURE 3. (left) a link diagram and its twists. (center) the augmented link diagram. (right) A diagram of a fully augmented link

Thanks to Thurston's Dehn surgery theorem (see [10]), we see that this inequality can be obtained by estimating the volume of the fully augmented link which we get from D. From now on, we will give more precise estimates of such volume by using graphs.

By lemma 2.1, we get a trivalent graph from a fully augmented link. Moreover,

Proposition 3.4. *Let L be a fully augmented link, and let G_L denote the spatial trivalent graph which we get by applying the cutting move in Lemma 2.1 to each 2-punctured disk bounded by a crossing circle. Then G_L*

(1) *is trivalent with $2c$ vertices,*
(2) *is simple, and*
(3) *has a planar diagram.*

Proof. By the construction, the graph G_L is trivalent and has a planar diagram. If G_L is not simple then it has self-loop or multi-arcs. By lemma 2.1, if it has self-loop then the L is splittable and if it has multi-arcs, then the complement of L has an incompressible annulus. These contradict the assumption that L is hyperbolic. □

Since for given $n \in \mathbb{N}$, there are only finitely many trivalent simple planar graphs with n vertices, we can enumerate all of them.

Example 3.5. There are only one simple planar trivalent graph with 4 or 6 vertices. Their volumes are $2v_8$ and $4v_8$ respectively. Where v_8 is the volume of regular ideal octahedron. Therefore, if a link has a diagram with 2 (resp. 3) twist regions, its volume is less than $2v_8$ (resp. $4v_8$).

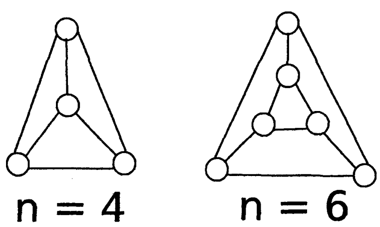

FIGURE 4. Simple planar trivalent graph with 4 or 6 vertices.

From now on, we only consider planar graphs and regard each planar graph as a diagram of a spatial graph in S^3. In order to compute the upper bounds, we do not have to enumerate all planar graphs. The cyclically connectivity of graphs allows us to reduce the target of enumeration.

Definition 3.6. A *t-cut* of a graph is a collection of t edges whose removal is disconnected. A t-cut is *nontrivial* if each component of its removal graph contains a cycle. A graph is cyclically k-connected if it has no non-trivial t-cuts for $0 \leq t \leq k-1$.

If a planar graph has a nontrivial 1 or 2-cut, then its complement (as a spatial graph with a planar diagram in S^3) has a disk which compresses the meridian of the cut edge or an essential annulus respectively. Therefore those graphs can not be hyperbolic and hence, we only need to enumerate cyclically 3-connected graphs. Moreover, by Thurston's uniformization theorem, it turns out that a trivalent planar graph is hyperbolic if and only if it is simple and cyclically 3-connected (see [5], Theorem 2.4).

Let P be a planar graph with n vertices such that N_P admits a hyperbolic metric of finite volume. Let $\omega(P) = \mathrm{Vol}(N_P)/n$ and $U_m = \max\{\omega(Q) \mid Q$ trivalent planar graph with m vertices$\}$. We call $\omega(P)$ the *normalized volume* of P.

Proposition 3.7. *If P has a non-trivial 3-cut, then for some $4 \leq k \leq n-2$, $\omega(P) \leq U_k$.*

Proof. By [1], any essential thrice punctured sphere in N_P is isotopic to totally geodesic one. We claim that the thrice punctured sphere E determined by the non-trivial 3-cut is essential. If it has a boundary compressing disk F, then we may assume that the boundary ∂F encircles exactly 1 puncture. Therefore F gives a disk which compresses the meridian of the edge corresponds to the encircled puncture. This contradicts the hyperbolicity of N_P. Hence F is a totally geodesic thrice punctured sphere and we may cut along F to get two graphs with planar diagram whose number of vertices are less than or equal to $n-2$. $\square$

Example 3.8. The graph in the left of Figure 5 has a non-trivial 3-cut. The dotted line depicts a thrice punctured sphere that may cut the complement into two pieces. Each piece is homeomorphic to the complement of a graph. The graphs in the middle and the right in Figure 5 are the corresponding graphs.

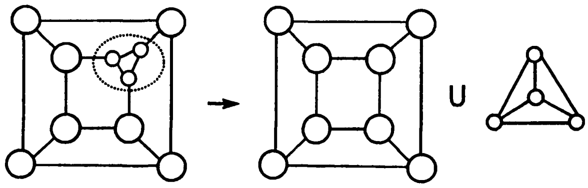

FIGURE 5. Graph with nontrivial 3-cut.

Thus, putting all the discussion above together, we get

Theorem 3.9. *Let B_n denote the best possible upper bound of the volumes of hyperbolic links that have diagrams with n twists. Then we have $B_n = 2nU_{2n}$. Further, B_n is attained by some cyclically 4-connected graph P_{2n} if we have $\omega(P_{2n}) \geq U_{2k}$ for all $2 \leq k \leq n-1$.*

Remark 3.10. It seems quite likely that B_n is attained by some 4-connected graph for all n.

In Table 1, we collect the number of cyclically 3 or 4 connected graph.

By plantri [2], we enumerate all planar cyclically 4-connected trivalent graphs and estimate the volume by Orb[4]. We computed an approximate value of U_{2n} for $n \leq 8$ (Figure 6).

$\lvert V \rvert$	4	6	8	10	12	14	16	18
3-connected	1	1	2	5	14	50	233	1249
4-connected	0	0	1	1	2	4	10	25

TABLE 1. The number of cyclically 3 or 4 -connected planar graphs.

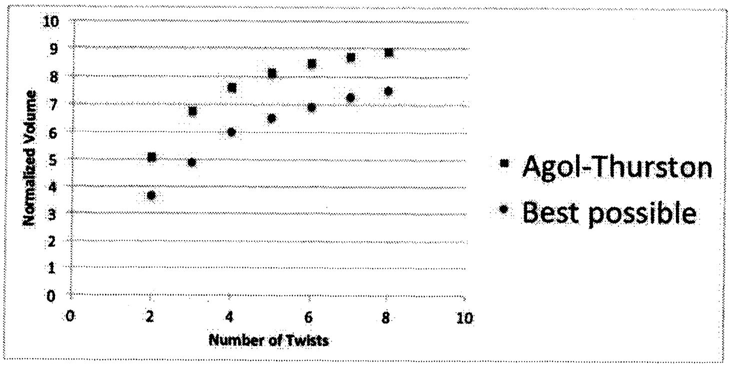

FIGURE 6. The values of $10v_3(n-1)/n$ (the upper bounds by Agol-Thurston) and U_{2n}.

So far, U_n is monotonically increasing as a function of n. It is interesting to compute exact values of U_n and observe the difference between $10v_3(n-1)/n$ and U_n for general n.

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DEPARTMENT OF MATHEMATICAL AND COMPUTING SCIENCES, TOKYO INSTITUTE OF TECHNOLOGY, O-OKAYAMA, MEGURO-KU, TOKYO 152-8552 JAPAN

E-mail address: `masai9 at is.titech.ac.jp`