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ON THE SECOND COEFFICIENT OF THE TWISTED ALEXANDER POLYNOMIAL

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1. INTRODUCTION

Let K be a knot in the 3-sphere S^3 and $G(K)$ its knot group. Namely it is the fundamental group of $N = S^3 \backslash K$. In this note we consider the twisted Alexander polynomial $\Delta_{K,\rho}(t)$ (see [6] and [11]) associated to a parabolic representation $\rho : G(K) \to SL(2, \mathbb{C})$, which sends the meridian μ of K to a parabolic element of $SL(2, \mathbb{C})$, i.e. $\operatorname{tr} \rho(\mu) = 2$ holds. For a hyperbolic knot K in S^3 (that is, N admits a complete hyperbolic metric of finte volume), a typical example is the holonomy representation $\rho_0 : G(K) \to SL(2, \mathbb{C})$, which is a lift of a discrete faithful representation $\bar{\rho}_0 : G(K) \to PSL(2, \mathbb{C}) \cong \mathrm{Isom}^+(\mathbb{H}^3)$ so that $\mathbb{H}^3/G(K) \cong N$ (see [10]). It is known that such a representation is determined uniquely up to conjugation.

In [1], Dunfield, Friedl and Jackson studied the twisted Alexander polynomial $\Delta_{K,\rho_0}(t) \in \mathbb{C}[t^{\pm 1}]$ associated to the holonomy representation ρ_0, which is called the hyperbolic torsion polynomial and denoted by $\mathcal{T}_K$. It is well known that the coefficient of the highest degree term of $\mathcal{T}_K$ has information on fiberedness of knots. Based on huge numerical calculations for hyperbolic knots of 15 or fewer crossings, they found an interesting pattern on the second coefficient of $\mathcal{T}_K$ and asked the following question as one of the open problems (see [1, Section 1.13]): *For fibered knots, why is the second coefficient of $\mathcal{T}_K$ so often real?* As explained in [1, Section 6.5], for fibered knots, the twisted homology of the universal cyclic cover can be identified with that of the fiber, hence the action of a generator of the deck transformation group on this homology of the cover can be thought of as the action of the monodromy of the bundle on the twisted homology of the fiber. Then the second coefficient of $\mathcal{T}_K$ is just the sum of the eigenvalues of this monodromy map, but it is unclear why this should often be a real number. In contrast, the second coefficient is real for only few nonfibered knots. Hence we would like to explain these phenomena reasonably, but it is still widely open.

The purpose of this note is to observe the above question and confirm the property of the second coefficient of the twisted Alexander polynomial $\Delta_{K,\rho}(t)$ associated to parabolic representations ρ (not only for the holonomy representation ρ_0) for well known two infinite families of knots. One is a torus knot $T_{2,q}$ and the other is a twist knot $J(2, 2n)$. Every torus knot is non-hyperbolic fibered and we can see that the second coefficient of $\Delta_{K,\rho}(t)$ is actually real (precisely zero) for any irreducible representation $\rho : G(T_{2,q}) \to SL(2, \mathbb{C})$. For parabolic representations of twist knots, we can show that the second coefficient is an integer for fibered case and not rational for nonfibered knots except for $J(2, 4)$.

In Section 2.1, we review the representation space of irreducible representations into $SL(2, \mathbb{C})$ (see [3]) and the formula of the twisted Alexander polynomial (see [5]) of the torus knot $T_{p,q}$. In Section 2.2, we review the Riley polynomial (see [9]) and the formula

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of $\Delta_{K,\rho}(t)$ for $K = J(2, 2n)$ (see [7]). In particular, we calculate the second coefficient of the twisted Alexander polynomial for the twist knot.

2. Observation

In this section, we observe the second coefficient of the twisted Alexander polynomial associated to a parabolic $SL(2, \mathbb{C})$-representation of the knot group.

2.1. **Torus knots.** Let (p, q) be a pair of coprime natural numbers and $T_{p,q}$ the (p, q)-torus knot. We take and fix the following presentation of $G_{p,q} = G(T_{p,q})$:

$$G_{p,q} = \langle x, y \mid x^p y^{-q} \rangle.$$

Here we note that this is not a Wirtinger presentation.

Let R be the set of irreducible $SL(2, \mathbb{C})$-representations of $G_{p,q}$ and $\hat{R}$ the quotient space of R by conjugate action of $SL(2, \mathbb{C})$. We denote the conjugacy class, which contains a representation ρ, by $\hat{\rho}$. In general $\hat{R}$ has several components. Choosing a pair (r, s) of natural numbers satisfying $ps - qr = 1$, then $\mu = x^{-r}y^s \in G_{p,q}$ represents a meridian of $T_{p,q}$. For $\hat{\rho} \in \hat{R}$, let $\alpha^{\pm 1} = \exp(\pm\sqrt{-1}\pi a/p)$ and $\beta^{\pm 1} = \exp(\pm\sqrt{-1}\pi b/q)$ be the eigenvalues of $\rho(x)$ and $\rho(y)$ respectively, where we can assume that $0 < a < p$ and $0 < b < q$.

Under the notations above, the space of conjugacy classes of irreducible $SL(2, \mathbb{C})$-representations of the torus knot group $G_{p,q}$ is described as follows:

Proposition 2.1 (Johnson [3]). *The representation space $\hat{R}$ consists of $(p-1)(q-1)/2$ components, denoted by $\hat{R}_{a,b}$, which are determined by the following data:*

1. $0 < a < p$ *and* $0 < b < q$.
2. $a \equiv b \mod 2$.
3. *For every $\hat{\rho} \in \hat{R}_{a,b}$, we have* $\operatorname{tr}\rho(x) = 2\cos(\pi a/p)$, $\operatorname{tr}\rho(y) = 2\cos(\pi b/q)$ *and* $\rho(x^p) = \rho(y^q) = (-I)^a$.
4. $\operatorname{tr}\rho(\mu) \neq 2\cos\pi(ra/p \pm sb/q)$.

In particular, $\hat{R}_{a,b}$ is parametrized by $\operatorname{tr}\rho(\mu)$ *and hence* $\dim_{\mathbb{C}}(\hat{R}_{a,b}) = 1$.

Remark 2.2. An easy argument shows that there is just one conjugacy class of parabolic representations on each component $\hat{R}_{a,b}$ (see [5, Proposition 4.4]).

Let $K = T_{p,q}$ and $\hat{\rho} \in \hat{R}_{a,b}$. Then the twisted Alexander polynomial is given by

$$\Delta_{K,\rho}(t) = \frac{(1 - (-1)^a t^{pq})^2}{(1 - \alpha t^q)(1 - \alpha^{-1}t^q)(1 - \beta t^p)(1 - \beta^{-1}t^p)},$$

and moreover we see that each coefficient of $\Delta_{K,\rho}(t)$ is a locally constant function on $\hat{R}$. Namely every coefficient of the twisted Alexander polynomial is a constant function on each component $\hat{R}_{a,b}$ (see [5]).

Example 2.3. Let $K = T(2, 3)$, the trefoil knot. In this case, $a = b = 1$, $\alpha^{\pm 1} = \pm\sqrt{-1}$ and $\beta^{\pm 1} = (1 \pm \sqrt{-3})/2$. Hence there is just one component $\hat{R}_{1,1}$ and we see that

$$\Delta_{K,\rho}(t) = \frac{(1 + t^6)^2}{(1 + t^6)(1 - t^2 + t^4)} = 1 + t^2$$

holds for any ρ with $\hat{\rho} \in \hat{R}_{1,1}$.

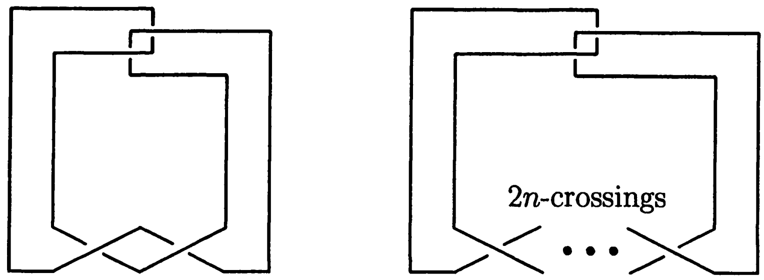

FIGURE 1. The figure eight knot $J(2,-2)$ and $J(2,2n)$

The above example shows the second coefficient of $\Delta_{K,\rho}(t)$ is zero for every irreducible representation of the trefoil knot. This is not an accident and in fact we have the following:

Claim 2.4. *For any $\hat{\rho} \in \hat{R}$ of $K = T_{2,q}$, the second coefficient of $\Delta_{K,\rho}(t)$ is zero. In particular, it is zero for every parabolic representation.*

Proof. When $p = 2$, the twisted Alexander polynomial of $K = T_{2,q}$ is given by

$$\Delta_{K,\rho}(t) = (t^2+1) \prod_{0<j<q,\ j:\mathrm{odd},\ j\neq b} (t^2 - \xi_j)\,(t^2 - \bar{\xi}_j)$$

for $\hat{\rho} \in \hat{R}_{1,b}$, where $\xi_j = \exp(\sqrt{-1}\pi j/q)$. Namely $\Delta_{K,\rho}(t)$ is a polynomial in t^2 and hence $\Delta_{K,\rho}(t)$ has only even degree terms. Therefore the second coefficient of $\Delta_{K,\rho}(t)$ is zero for any irreducible $SL(2,\mathbb{C})$-representation. □

A representation $\rho : G(K) \to SL(2,\mathbb{C})$ is called metabelian if the commutator subgroup of $G(K)$ is sent to an abelian subgroup in $SL(2,\mathbb{C})$ by ρ. As shown in [12], the twisted Alexander polynomial associated to an irreducible metabelian representation ρ satisfies $\Delta_{K,\rho}(-t) = \Delta_{K,\rho}(t)$ and hence it has only even degree terms. Since $T_{2,q}$ is a 2-bridge knot, every component $\hat{R}_{1,b}$ ($0 < b < q$ and b is odd) contains an irreducible metabelian representation (see [4, Lemma 4.7]). Moreover each coefficient of $\Delta_{K,\rho}(t)$ is a locally constant function on $\hat{R}$, so that Claim 2.4 also follows from these facts.

Remark 2.5. If a component $\hat{R}_{a,b}$ contains an irreducible metabelian representation, then by Remark 2.2, there is a parabolic representation $\hat{\rho} \in \hat{R}_{a,b}$ so that the second coefficient of $\Delta_{K,\rho}(t)$ is zero. However, in general, some component may contain no irreducible metabelian representation.

2.2. **Twist knots.** Let $K = J(\pm 2, k)$ be the twist knot ($k \in \mathbb{Z}$). It is known that $J(\pm 2, 2n+1)$ is equivalent to $J(\mp 2, 2n)$ and $J(\pm 2, k)$ is the mirror image of $J(\mp 2, -k)$. Hence we only consider the case where $K = J(2,2n)$ for $n \in \mathbb{Z}$ (see Figure 1). The knot $J(2,0)$ presents the trivial knot, so that we always assume $n \neq 0$. The typical examples are the trefoil knot $J(2,2)$ and the figure eight knot $J(2,-2)$.

The Alexander polynomial of $K = J(2,2n)$ is given by $\Delta_K(t) = n - (2n-1)t + nt^2$. Since the twist knots are 2-bridge knots, $J(2,2n)$ is fibered if and only if $|n| = 1$. It is also known that $J(2,2n)$ is hyperbolic if $n \notin \{0,1\}$.

The knot group $G(J(2,2n))$ has the presentation (see [2]):

$$G(J(2,2n)) = \langle x, y \mid w^n x = y w^n \rangle,$$

where $w = [y, x^{-1}]$. Suppose that $\rho : G(J(2,2n)) \to SL(2,\mathbb{C})$ is a parabolic representation. After conjugating, if necessary, we may assume that for a complex number u

$$\rho(x) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \rho(y) = \begin{pmatrix} 1 & 0 \\ -u & 1 \end{pmatrix}.$$

It is known that ρ defines a representation when u satisfies $\phi_n(u) = 0$ (see [9, Theorem 2]), where

$$\phi_n(u) = (1-u)\frac{\lambda_+^n - \lambda_-^n}{\lambda_+ - \lambda_-} - \frac{\lambda_+^{n-1} - \lambda_-^{n-1}}{\lambda_+ - \lambda_-}, \quad \text{and} \quad \lambda_\pm(u) = \frac{u^2 + 2 \pm \sqrt{u^4 + 4u^2}}{2}$$

denote the eigenvalues of the matrix $\rho(w)$. We call $\phi_n(u)$ the Riley polynomial of the twist knot $J(2,2n)$. See also [7, Proposition 3.1] for the above formula of $\phi_n(u)$. Of course, the holonomy representation ρ_0 corresponds to one of the roots of $\phi_n(u) = 0$ if $n \notin \{0,1\}$.

Example 2.6. We can easily check that $\phi_1(u) = 1 - u$, $\phi_{-1}(u) = 1 + u + u^2$, $\phi_2(u) = 1 - 2u + u^2 - u^3$, $\phi_{-2}(u) = 1 + 2u + 3u^2 + u^3 + u^4$ and $\phi_3(u) = 1 - 3u + 3u^2 - 4u^3 + u^4 - u^5$.

In general the Riley polynomial $\phi_n(u)$ satisfies the following:

(a) $\phi_n(u) \in \mathbb{Z}[u]$ is irreducible and its highest coefficient is ± 1.
(b) $\deg \phi_n(u) = 2|n| - \max\{\mathrm{sign}(n), 0\}$.

For a parabolic representation $\rho : G(K) \to SL(2,\mathbb{C})$ of the twist knot $K = J(2,2n)$, the twisted Alexander polynomial $\Delta_{K,\rho}(t)$ is given by

$$\Delta_{K,\rho}(t) \;=\; \alpha\beta + \left\{\alpha + \beta - 2\alpha\beta + \frac{\lambda_+ - \lambda_-}{2 + \lambda_+ + \lambda_-}(\alpha - \beta)\right\} t + \alpha\beta t^2,$$

where $\alpha = 1 + \lambda_+ + \lambda_+^2 + \cdots + \lambda_+^{n-1}$ and $\beta = 1 + \lambda_- + \lambda_-^2 + \cdots + \lambda_-^{n-1}$ (see [7, Theorem 4.1]).

Example 2.7. For the figure eight knot $K = J(2,-2)$, we can easily check that there are two parabolic representations corresponding to $u = (-1 \pm \sqrt{-3})/2$ up to conjugation. We then obtain $\Delta_{K,\rho}(t) = 1 - 4t + t^2$ for both of them.

Let $\delta_n(u)$ be the second coefficient of $\Delta_{K,\rho}(t)$. As we saw in Examples 2.3 and 2.7, $\delta_{\pm 1}(u)$ are integers for the fibered twist knots $J(2,\pm 2)$.

Next we consider the nonfibered case. As stated in [8, Remark 3.3], $\delta_n(u) \in \mathbb{Z}[u]$ and $\deg \delta_n(u) = 2n - 4$ for $n \geq 2$. First we check the degree of $\delta_n(u)$. Let $\tau_n(u) = \mathrm{tr}\,\rho(w^n) = \lambda_+^n + \lambda_-^n$. Easy calculations show that

$$\begin{aligned}
\alpha + \beta &= (1 + \lambda_+ + \lambda_+^2 + \cdots + \lambda_+^{n-1}) + (1 + \lambda_- + \lambda_-^2 + \cdots + \lambda_-^{n-1}) \\
&= 2 + \tau_1 + \tau_2 + \cdots + \tau_{n-2} + \tau_{n-1}, \\
\alpha - \beta &= (1 + \lambda_+ + \lambda_+^2 + \cdots + \lambda_+^{n-1}) - (1 + \lambda_- + \lambda_-^2 + \cdots + \lambda_-^{n-1}) \\
&= (\lambda_+ - \lambda_-)\left\{1 + (\lambda_+ + \lambda_-) + (\lambda_+^2 + 1 + \lambda_-^2) + (\lambda_+^3 + \lambda_+ + \lambda_- + \lambda_-^3) + \cdots\right\} \\
&= (\lambda_+ - \lambda_-)\left\{1 + \tau_1 + (\tau_2 + 1) + (\tau_3 + \tau_1) + \cdots + (\tau_{n-2} + \tau_{n-4} + \cdots)\right\}, \\
\alpha\beta &= (1 + \lambda_+ + \lambda_+^2 + \cdots + \lambda_+^{n-1})(1 + \lambda_- + \lambda_-^2 + \cdots + \lambda_-^{n-1}) \\
&= \tau_{n-1} + 2\tau_{n-2} + (\text{some polynomial in } \tau_1, \ldots, \tau_{n-3}).
\end{aligned}$$

Since $\lambda_+ - \lambda_- = \sqrt{u^4+4u^2}$ and $2+\lambda_+ + \lambda_- = u^2+4$, we have

$$\begin{aligned}
\delta_n(u) &= (2+\tau_1+\cdots+\tau_{n-1}) - 2\left\{\tau_{n-1}+2\tau_{n-2}+(\text{some polynomial in } \tau_1,\ldots,\tau_{n-3})\right\}\\
&\quad + u^2\left\{\tau_{n-2}+(\text{some polynomial in } \tau_1,\ldots,\tau_{n-3})\right\}\\
&= -\tau_{n-1}-(3-u^2)\tau_{n-2}+(\text{some polynomial in } \tau_1,\ldots,\tau_{n-3}).
\end{aligned}$$

Now as shown in [7, Corollary 4.3], τ_n is given by

$$\begin{aligned}
\tau_n &= \frac{1}{2^{|n|-1}} \sum_{0\le j\le |n|,\ |n|-j:\text{even}} \binom{|n|}{j} {\tau_1}^j \left({\tau_1}^2-4\right)^{\frac{|n|-j}{2}}\\
&= {\tau_1}^n + (\text{some polynomial in } {\tau_1}^{n-2}, {\tau_1}^{n-4},\ldots)
\end{aligned}$$

and $\tau_n(u)=\tau_{-n}(u)\in\mathbb{Z}[u]$ is a monic polynomial of degree $2|n|$. Since ${\tau_1}^n=(u^2+2)^n = u^{2n}+2nu^{2n-2}+\cdots+2^n$, we have

$$\begin{aligned}
-\tau_{n-1}-(3-u^2)\tau_{n-2} &= -\left(u^{2(n-1)}+2(n-1)u^{2(n-1)-2}+\cdots\right)\\
&\quad -(3-u^2)\left(u^{2(n-2)}+2(n-2)u^{2(n-2)-2}+\cdots\right)\\
&= -5u^{2n-4}+(\text{lower degree terms})
\end{aligned}$$

and hence $\deg\delta_n(u)=2n-4$ $(n\ge 2)$.

If $\delta_n(u)=\frac{l}{m}\in\mathbb{Q}$ for a parabolic representation corresponding to a root of $\phi_n(u)=0$, then $\phi_n(u)$ divides $m\delta_n(u)-l$, because of the property (a) of the Riley polynomial $\phi_n(u)$. However this contradicts the fact that

$$\deg\phi_n(u)=2|n|-\max\{\operatorname{sign}(n),0\}>2n-4=\deg\delta_n(u).$$

Therefore we can conclude that $\delta_n(u)$ with $n\ge 3$ is not a rational number for every parabolic representation.

A similar argument can be applied to the case of $J(2,2n)$ with $n\le -2$. In this case $\alpha+\beta$ and $\alpha\beta$ are the same as that of $n\ge 2$, but $\alpha-\beta$ changes to the opposite sign. Hence we obtain

$$\begin{aligned}
\delta_n(u) &= (2+\tau_1+\cdots+\tau_{n-1}) - 2\left\{\tau_{n-1}+2\tau_{n-2}+(\text{some polynomial in } \tau_1,\ldots,\tau_{n-3})\right\}\\
&\quad - u^2\left\{\tau_{n-2}+(\text{some polynomial in } \tau_1,\ldots,\tau_{n-3})\right\}\\
&= -\tau_{n-1}-(3+u^2)\tau_{n-2}+(\text{some polynomial in } \tau_1,\ldots,\tau_{n-3})\\
&= -2u^{2|n|-2}+(\text{lower degree terms})
\end{aligned}$$

and $\deg\delta_n(u)=2|n|-2$. Therefore $\delta_n(u)$ is not a rational number for $n\le -2$.

Finally we consider the nonfibered hyperbolic knot $K=J(2,4)$. In this case we can easily check that $\alpha+\beta=u^2+4$, $\alpha-\beta=\sqrt{u^4+4u^2}$ and $\alpha\beta=u^2+4$. Hence we have

$$\delta_2(u)=(u^2+4)-2(u^2+4)+\frac{\sqrt{u^4+4u^2}}{u^2+4}\cdot\sqrt{u^4+4u^2}=-4$$

and $\Delta_{K,\rho}(t)=(u^2+4)-4t+(u^2+4)t^2$ for every parabolic representation. In particular, we see that the second coefficient $\delta_2(u)$ is an integer for the holonomy representation $\rho_0 : G(J(2,4))\to SL(2,\mathbb{C})$.

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REFERENCES

[1] N. Dunfield, S. Friedl and N. Jackson, *Twisted Alexander polynomials of hyperbolic knots*, Experiment. Math. **21** (2012), 329–352.

[2] J. Hoste and P. D. Shanahan, *Trace fields of twist knots*, J. Knot Theory Ramifications **10** (2001), 625–639.

[3] D. Johnson, *A geometric form of Casson's invariant and its connection to Reidemeister torsion*, unpublished lecture notes.

[4] T. Kim and T. Morifuji, *Twisted Alexander polynomials and character varieties of 2-bridge knot groups*, Internat. J. Math. **23** (2012), 1250022, 24 pp.

[5] T. Kitano and T. Morifuji, *Twisted Alexander polynomials for irreducible $SL(2,\mathbb{C})$-representations of torus knots*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **11** (2012), 395–406.

[6] X.-S. Lin, *Representations of knot groups and twisted Alexander polynomials*, Acta Math. Sin. (Engl. Ser.) **17** (2001), 361–380.

[7] T. Morifuji, *Twisted Alexander polynomials of twist knots for nonabelian representations*, Bull. Sci. Math. **132** (2008), 439–453.

[8] T. Morifuji, *On a conjecture of Dunfield, Friedl and Jackson*, C. R. Acad. Sci. Paris, Ser I **350** (2012), 921–924.

[9] R. Riley, *Parabolic representations of knot groups, I*, Proc. London Math. Soc. (3) **24** (1972), 217–242.

[10] W. Thurston, *Three-dimensional geometry and topology*, Vol. 1, volume 35 of Princeton Mathematical Series, Princeton University Press, Princeton, NJ, 1997. Edited by Silvio Levy.

[11] M. Wada, *Twisted Alexander polynomial for finitely presentable groups*, Topology **33** (1994), 241–256.

[12] Y. Yamaguchi, *On the twisted Alexander polynomial for metabelian represetations into* $\mathrm{SL}_2(\mathbb{C})$, arXiv:1101.3989.

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