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Long time asymptotic problems for stochastic optimal control and related variational problems

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Abstract

In this note we present some recent results on the large time behavior of solutions to viscous Hamilton-Jacobi equations arising in stochastic control. Our equations possess superlinear nonlinearity in gradients, and solutions are unbounded on the whole Euclidean space. We prove that, as the time tends to infinity, the solution approaches to a steady state in a suitable sense. We also establish a variational representation formula for the limit.

1 Introduction

Let us consider semilinear parabolic equations of the form

$$\partial_t u-\frac{1}{2}\operatorname{tr}(a(x)D^2u)+H(x,Du)=0 \quad \text{in} \quad (0,\infty)\times\mathbb{R}^N, \tag{1.1}$$

where $\partial_t u=\partial u/\partial t$, $D^2u=(\partial^2u/\partial x_i\partial x_j)$, and $Du=(\partial u/\partial x_i)$. We are concerned with the large time behavior of solutions of (1.1). It turns out under suitable assumptions on $a=(a_{ij}(x))$, $H=H(x,p)$, and initial datum $u(0,\cdot)$, that the solution $u=u(t,x)$ of (1.1) approaches as $t\to\infty$ to a function of the form $\lambda t+\phi(x)+c$ for some real constants λ, c, and function $\phi=\phi(x)$ on $\mathbb{R}^N$ with $\phi(0)=0$. More precisely, one can prove the following convergence:

$$u(t,x)-(\lambda t+\phi(x)+c)\longrightarrow 0 \qquad \text{in} \ \ C(\mathbb{R}^N) \ \ \text{as} \ \ T\to\infty. \tag{1.2}$$

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Here, convergence "in $C(\mathbb{R}^N)$" stands for locally uniform convergence in $\mathbb{R}^N$. We call the triplet (λ, ϕ, c) *asymptotic solution* if $\lambda t + \phi(x) + c$ solves (1.1). Any asymptotic solution should satisfy the stationary equation

$$\lambda - \frac{1}{2}\operatorname{tr}(a(x)D^2\phi) + H(x, D\phi) = 0 \quad \text{in } \mathbb{R}^N, \quad \phi(0) = 0. \tag{1.3}$$

Finding a pair (λ, ϕ) satisfying (1.3) is called *ergodic problem.* Remark that λ and ϕ in (1.2) are specified from the stationary equation (1.3), whereas the constant c needs to be determined from the evolutionary equation (1.1). Asymptotic problems of this type have been largely studied in various settings. We refer to [1, 2, 5, 12, 13] for recent results from the PDE viewpoint, and to [3, 4, 6, 8, 9, 10, 11] from the probabilistic viewpoint, especially, in connection with mathematical finance.

In this note, we concentrate on a more specific equation: we consider the Cauchy problem

$$\begin{cases} \partial_t u - \dfrac{1}{2}\Delta u + \dfrac{1}{m}|Du|^m = f & \text{in } (0, +\infty) \times \mathbb{R}^N, \\ u\big|_{t=0} = g & \text{on } \{0\} \times \mathbb{R}^N, \end{cases} \tag{CP}$$

where m, f, and g are assumed to satisfy the following conditions:

(H1) $m > 1$.

(H2) $f \in C^2(\mathbb{R}^N)$, and there exist some $C > 0$ and $\beta > 0$ such that

$$C^{-1}|x|^\beta - C \le f(x) \le C(|x|^\beta + 1), \quad |Df(x)| \le C(|x|^{\beta-1} + 1), \qquad x \in \mathbb{R}^N.$$

(H3) $g \in C(\mathbb{R}^N)$ is bounded below on $\mathbb{R}^N$.

In the first half of this note, we discuss, according to [9], the large time behavior of solutions to (CP). It holds convergence (1.2) for some (λ, ϕ, c) under (H1)-(H3). In the second half, we study a variational representation formula for the limit c, which seems to be new to the best of our knowledge.

Equation (CP) naturally appears in the stochastic control theory. Let us consider the following minimizing problem

$$\begin{aligned} &\text{Minimize} && J(T, x; \xi) := E\Big[\int_0^T \Big(\frac{1}{m^*}|\xi_t|^{m^*} + f(X_t^\xi)\Big)\, dt + g(X_T^\xi)\Big], \\ &\text{subject to} && X_t^\xi = x - \int_0^t \xi_s\, ds + W_t, \quad t \ge 0, \end{aligned}$$

where $m^* := m/(m-1) > 1$, and $W = (W_t)$ denotes an N-dimensional Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, P; (\mathcal{F}_t))$. The control process

$\xi = (\xi_t)$ is taken from the admissible class $\mathcal{A}_T$ which is defined as the collection of all $(\mathcal{F}_t)$-progressively measurable processes $\xi = (\xi_t)$ in $\mathbb{R}^N$ such that

$$E^x\Big[\int_0^T (|\xi_t|^{m^*} + |X_t^\xi|^\beta)\,dt\Big] < \infty, \quad x \in \mathbb{R}^N, \tag{1.4}$$

where β is the constant in (H2). Then, we see that the value function

$$u(T,x) := \inf_{\xi \in \mathcal{A}_T} J(T,x;\xi) \tag{1.5}$$

is a classical solution of (CP).

This note is organized as follows. In the next section, we survey some results obtained in [9]. In Section 3, we discuss a variational representation formula for the constant c in (1.2).

2 Convergence of solutions

We begin with the solvability of (CP).

Theorem 2.1. Let (H1)-(H3) hold. Then, u defined by (1.5) is the minimal solution of (CP) in the class

$$\Phi := \{u \in C^{1,2}((0,\infty)\times\mathbb{R}^N)\times C([0,\infty)\times\mathbb{R}^N) \mid \inf_{[0,T]\times\mathbb{R}^N} u > -\infty,\ T > 0\}.$$

Proof. The proof is based on the verification theorem. See [9, Theorem 2.1] for details. □

As the limiting equation of (CP), we derive the ergodic problem

$$\lambda - \frac{1}{2}\Delta\phi + \frac{1}{m}|D\phi|^m = f \quad \text{in } \mathbb{R}^N, \quad \phi(0) = 0. \tag{EP}$$

Recall that the unknown is $(\lambda,\phi) \in \mathbb{R}\times C^2(\mathbb{R}^N)$. Equation (EP) has a unique solution in the following sense.

Theorem 2.2. Let (H1)-(H3) hold. Then, there exists a unique solution $(\lambda^*,\varphi) \in \mathbb{R}\times C^2(\mathbb{R}^N)$ of (EP) such that $\inf_{\mathbb{R}^N}\varphi > -\infty$. Moreover, there exists some $C > 0$ such that the solution φ satisfies the following estimate:

$$C^{-1}|x|^{(\beta/m)+1} - C \le \varphi(x) \le C(|x|^{(\beta/m)+1} + 1), \quad x \in \mathbb{R}^N.$$

Proof. See [9, Theorem 2.2]. □

Remark 2.3. The condition $\inf_{\mathbb{R}^N} \varphi > -\infty$ is necessary to derive the uniqueness of solution. Indeed, there exist infinitely may pairs (λ, ϕ) satisfying (EP) if we do not put this condition.

Let (λ^*, φ) be the unique solution of (EP) given in Theorem 2.2. Then, we see that the solution u of (CP) converges to an asymptotic solution (λ^*, φ, c) for some $c \in \mathbb{R}$.

Theorem 2.4. Let (H1)-(H3) hold. Let u and (λ^*, φ) be the solutions of (CP) and (EP), respectively. Assume that $\beta \geq m^*$. Then, there exists a constant c such that

$$u(T, \cdot\,) - (\lambda^* T + \varphi(\,\cdot\,) + c) \longrightarrow 0 \quad \text{in} \quad C(\mathbb{R}^N) \quad \text{as} \quad T \to \infty. \tag{2.1}$$

Remark 2.5. Under (H1)-(H3), we can prove that

$$\frac{u(T, \cdot\,)}{T} \longrightarrow -\lambda^* \quad \text{in} \quad C(\mathbb{R}^N) \quad \text{as} \quad T \to \infty.$$

However, we do not know, in general, if (2.1) is valid without assuming $\beta \geq m^*$.

In the rest of this section, we give a sketch of the proof for Theorem 2.4. We refer to [9, Section 5.2] for a complete proof. Let u be the solution of (CP) defined by (1.5), and let (λ^*, φ) be the solution of (EP) such that $\inf_{\mathbb{R}^N} \varphi > -\infty$. We set $w(T, x) := u(T, x) - (\varphi(x) + \lambda^* T)$ for $(T, x) \in (0, \infty) \times \mathbb{R}^N$. The goal is to prove that $w(T, \cdot\,)$ converges in $C(\mathbb{R}^N)$ to a constant as $T \to \infty$. Observe that w is a solution of

$$\partial_t w - A^\varphi w + H_\varphi(x, Dw) = 0 \quad \text{in} \quad (0, \infty) \times \mathbb{R}^N \tag{2.2}$$

with $w(0, \cdot\,) = g - \varphi$ in $\mathbb{R}^N$, where A^φ is the second order differential operator given by

$$A^\varphi := \frac{1}{2}\Delta - |D\varphi(x)|^{m-2} D\varphi(x) \cdot D,$$

and $H_\varphi(x, p)$ is defined by

$$H_\varphi(x, p) := \frac{1}{m}|p + D\varphi(x)|^m - \frac{1}{m}|D\varphi(x)|^m - |D\varphi(x)|^{m-2} D\varphi(x) \cdot p. \tag{2.3}$$

Notice that $H_\varphi \geq 0$ in $\mathbb{R}^{2N}$ since the mapping $p \mapsto (1/m)|p|^m$ is convex.

Let $X^\varphi = (X_t^\varphi)_{t \geq 0}$ be the A^φ-diffusion, that is, the solution of the stochastic differential equation

$$dX_t^\varphi = -|D\varphi(X_t^\varphi)|^{m-2} D\varphi(X_t^\varphi)\, dt + dW_t, \quad t \geq 0.$$

Note that X^φ is ergodic with an invariant probability measure $\mu = \mu(dx)$ such that $\int_{\mathbb{R}^N} |x|^l \mu(dx) < \infty$ for all $l > 0$ (see [9, Proposition 4.13]).

Lemma 2.6. Let (λ^*, φ) be the unique solution of (EP) given in Theorem 2.2, and let $X^\varphi = (X_t^\varphi)$ be the A^φ-diffusion. Then,

$$w(T+S,x) \le E^x[w(T, X_S^\varphi)], \quad T, S \ge 0, \quad x \in \mathbb{R}^N.$$

Proof. In view of Ito's formula to $w(T+S-t, X_t^\varphi)$ and equation (2.2), we see that

$$\begin{aligned}
& w(T+S-S\wedge\tau_R, X^\varphi_{S\wedge\tau_R}) - w(T+S, X_0^\varphi) \\
&= \int_0^{S\wedge\tau_R} (-\partial_t w + A^\varphi w)(T+S-t, X_t^\varphi)\,dt + \int_0^{S\wedge\tau_R} Dw(T+S-t, X_t^\varphi)\,dW_t \\
&\ge \int_0^{S\wedge\tau_R} Dw(T+S-t, X_t^\varphi)\,dW_t,
\end{aligned}$$

where $\tau_R := \inf\{t>0 \,|\, |X_t^\varphi| \ge R\}$. Taking expectation, we have

$$w(T+S,x) \le E^x[w(T+S-S\wedge\tau_R, X^\varphi_{S\wedge\tau_R})].$$

Since $|w(t,x)| \le C(1+|x|^q)$ in $[0,T+S]\times\mathbb{R}^N$ for some $C, q>1$, and $\{|X^\varphi_{S\wedge\tau_R}|^q\,;\, R>1\}$ is uniformly integrable, we obtain the desired estimate after sending $R\to\infty$. $\square$

Proposition 2.7. The family $\{w(T,\cdot\,) \,|\, T>1\}$ is uniformly bounded from above on $\overline{B}_R := \{x\in\mathbb{R}^N \,|\, |x|\le R\}$ for any $R>0$. Moreover, if $\beta\ge m^*$, then it is also uniformly bounded from below on $\overline{B}_R$.

Proof. Let $X^\varphi = (X_t^\varphi)_{t\ge 0}$ be the A^φ diffusion. Then, in view of Lemma 2.6, we see that

$$w(T,x) \le E^x[(g-\varphi)(X_T^\varphi)] \longrightarrow \int_{\mathbb{R}^N} (g-\varphi)(y)\mu(dy) < \infty \quad \text{as} \quad T\to\infty.$$

Since the convergence above is uniform in $\overline{B}_R$, we see that $w(T,\cdot\,)$ is bounded above on $\overline{B}_R$ uniformly in $T>1$.

To get a lower bound, we assume $\beta\ge m^*$. Set $v(T,x) := (1-e^{-\delta T})\varphi(x)+\lambda T+q(T)$ for some $\delta>0$ and $q\in C^1([0,\infty))$ that will be determined later. Then, noting $\varphi(x)\le K(1+|x|^{(\beta/m)+1})$ in $\mathbb{R}^N$ for some $K>0$ and observing $\beta\ge(\beta/m)+1$ in view of $\beta\ge m^*$, we have

$$\begin{aligned}
\partial_t v + F[v] &\le e^{-\delta T}\delta\varphi + \lambda + q' + (1-e^{-\delta T})F[\varphi] + e^{-\delta T}F[0] \\
&\le e^{-\delta T}(\delta K - c_1)|x|^\beta + q' + e^{-\delta T}(2\delta K + |\lambda| + C_1)
\end{aligned}$$

for some $c_1, C_1>0$. We now choose $\delta := c_1/K$ and $q(T) := \inf_{\mathbb{R}^N} g - \delta^{-1}(2\delta K+|\lambda|+C_1)(1-e^{-\delta T})$. Then, $\partial_t v+F[v]\le 0$ in $(0,\infty)\times\mathbb{R}^N$ and $v(0,\cdot\,)\le g$ in $\mathbb{R}^N$. In particular,

v is a subsolution of (CP). Applying the comparison principle ([9, Proposition 3.6]), we obtain $v \le u$ in $(0,\infty)\times\mathbb{R}^N$. This infers that $-e^{-\delta T}\varphi(x)+q(T)\le w(T,x)$ for all $(T,x)\in(0,\infty)\times\mathbb{R}^N$. Since $\inf_{T>1} q(T) > -\infty$, we conclude that $w(T,\,\cdot\,)$ is bounded below on $\overline{B}_R$ uniformly in $T>1$. Hence, we have completed the proof. $\square$

Let Γ be the totality of all ω-limits of $\{w(T,\,\cdot\,)\,|\,T>1\}$ in $C(\mathbb{R}^N)$, namely,

$$\Gamma := \{w_\infty \in C(\mathbb{R}^N)\,|\, \lim_{j\to\infty} w(T_j,\,\cdot\,) = w_\infty \text{ in } C(\mathbb{R}^N) \text{ for some } \lim_{j\to\infty} T_j = \infty\}.$$

In view of Proposition 2.7 and the standard gradient estimate for w, we see that the family $\{w(T,\,\cdot\,)\,|\,T>1\}$ is pre-compact in $C(\mathbb{R}^N)$. In particular, $\Gamma\neq\emptyset$.

We are now in position to complete the proof of Theorem 2.4.

Proof of Theorem 2.4. It suffices to prove that $\Gamma=\{c\}$ for some $c\in\mathbb{R}$. We first show that any element of Γ is constant. Let $w_\infty\in\Gamma$, i.e., $w(T_j,\,\cdot\,)\to w_\infty$ in $C(\mathbb{R}^N)$ as $j\to\infty$ for some diverging sequence $\{T_j\}$. By Lemma 2.6, we see that

$$w(T+S,x)\le E^x[w(T,X_S^\varphi)],\quad T,S\ge 0,\quad x\in\mathbb{R}^N. \tag{2.4}$$

Take $S:=T_j-T$ and send $j\to\infty$. Then, we have

$$w_\infty(x)\le\int w(T,y)\mu(dy).$$

Choosing $T:=T_j$ and letting $j\to\infty$,

$$w_\infty(x)\le\int w_\infty(y)\mu(dy).$$

Taking the sup over $x\in\mathbb{R}^N$, we obtain

$$0\le\int(w_\infty(y)-\sup_{\mathbb{R}^N} w_\infty)\mu(dy)\le 0.$$

From the last estimate and the fact that $\operatorname{supp}\mu=\mathbb{R}^N$, we conclude that $w_\infty=\sup_{\mathbb{R}^N} w_\infty$ in $\mathbb{R}^N$. Hence, w_∞ is constant in $\mathbb{R}^N$.

We next show that Γ consists of a single element. Suppose that there exist two diverging sequences $\{T_j\}$ and $\{S_j\}$ such that $w(T_j,\,\cdot\,)\to c_1$ and $w(S_j,\,\cdot\,)\to c_2$ in $C(\mathbb{R}^N)$ as $j\to\infty$ for some $c_1,c_2\in\mathbb{R}$. We choose $S:=S_j-T$ and $T:=T_k$ in (2.4), and let $j\to\infty$ and $k\to\infty$ in this order. Then,

$$c_2\le\lim_{k\to\infty}\int w(T_k,y)\mu(dy)=\int c_1\mu(dy)=c_1.$$

Thus, $c_2\le c_1$. Changing the role of $\{T_j\}$ and $\{S_j\}$, we also have $c_1\le c_2$. Hence, $c_1=c_2$, and therefore $\Gamma=\{c\}$ for some $c\in\mathbb{R}$. $\square$

3 A representation formula

In this section, we discuss the dependence of c in (2.1) with respect to the initial function g. Let u and (λ^*, φ) be the solutions of (CP) and (EP), respectively. As in the previous section, we set

$$w(T,x) := u(T,x) - (\lambda^* T + \varphi(x)), \quad T \geq 0, \quad x \in \mathbb{R}^N. \tag{3.1}$$

Then, w satisfies (2.2) with $w(0,\,\cdot\,) = g - \varphi$. In the rest of this section, we set $\eta := w(0,\,\cdot\,)$, which is viewed as a small perturbation of stationary state φ. In view of Theorem 2.4, we can prove the following theorem.

Theorem 3.1. For any $\eta \in C_b(\mathbb{R}^N)$, there exists a real constant $c = c(\eta)$ such that $w(t,\,\cdot\,) \to c$ in $C(\mathbb{R}^N)$ as $t \to \infty$. Moreover, let $\mu = \mu(dx)$ be the invariant probability measure for the A^φ-diffusion. Then, the function

$$t \mapsto \langle w(t,\,\cdot\,), \mu \rangle := \int_{\mathbb{R}^N} w(t,x)\mu(dx)$$

is non-increasing. In particular,

$$c(\eta) = \inf_{t>0} \langle w(t,\,\cdot\,), \mu \rangle = \lim_{t\to\infty} \langle w(t,\,\cdot\,), \mu \rangle.$$

In what follows, we assume $\eta \in C_b(\mathbb{R}^N)$ and regard $c = c(\eta)$ as a functional of η taken from the Banach space $(C_b(\mathbb{R}^N), \|\cdot\|_\infty)$, where $\|\eta\|_\infty := \sup_{\mathbb{R}^N} |\eta|$.

Proposition 3.2. Let $c = c(\eta)$ be the constant given in Theorem 3.1. Then, $c(\eta)$ satisfies the following properties:
(a) $c(0) = 0$ and $c(\eta + a) = c(\eta) + a$ for any $\eta \in C_b(\mathbb{R}^N)$ and $a \in \mathbb{R}$.
(b) $\eta_1 \leq \eta_2$ in $\mathbb{R}^N$ implies $c(\eta_1) \leq c(\eta_2)$.
(c) $|c(\eta_1) - c(\eta_2)| \leq \|\eta_1 - \eta_2\|_\infty$ for all $\eta_1, \eta_2 \in C_b(\mathbb{R}^N)$.
(d) c is concave, i.e., $c(\delta\eta_1 + (1-\delta)\eta_2) \geq \delta c(\eta_1) + (1-\delta)c(\eta_2)$ for all $\eta_1, \eta_2 \in C_b(\mathbb{R}^N)$ and $\delta \in [0,1]$.

Proof. (a). Let $(T_t)_{t\geq 0}$ be the nonlinear semigroup associated with (2.2), that is, for each $\eta \in C_b(\mathbb{R}^N)$, we set $T_t\eta := w(t,\,\cdot\,) \in C_b(\mathbb{R}^N)$, where w denotes the unique solution of (2.2) with $w(0,\,\cdot\,) = \eta$. Then, by the uniqueness of solution, it is easy to see that $T_t 0 \equiv 0$ and $T_t(\eta + a) = T_t\eta + a$. In particular, $c(0) = 0$ and $c(\eta + a) = c(\eta) + a$.
(b). Since $T_t(\eta_1) \leq T_t(\eta_2)$ in view of comparison, we obtain $c(\eta_1) \leq c(\eta_2)$ after sending $t \to \infty$.

(c). Set $\eta := \eta_2 + \|\eta_1 - \eta_2\|_\infty$. Note that $\eta_1 \le \eta$ in $\mathbb{R}^N$. Taking into acount (a) and (b), we see that

$$T_t\eta_1 \le T_t\eta = T_t(\eta_2 + \|\eta_1 - \eta_2\|_\infty) = T_t\eta_2 + \|\eta_1 - \eta_2\|_\infty.$$

Letting $t \to \infty$, we obtain $c(\eta_1) \le c(\eta_2) + \|\eta_1 - \eta_2\|_\infty$. Changing the role of η_1 and η_2, we obtain the desired inequality.

(d). In view of the convexity of $H_\varphi(x,p)$ in p, we see that $\delta T_t(\eta_1) + (1-\delta)T_t(\eta_2)$ is a subsolution of (2.2) with $w(0,\cdot) := \delta\eta_1 + (1-\delta)\eta_2$. By the comparison theorem, we have $\delta T_t(\eta_1) + (1-\delta)T_t(\eta_2) \le T_t(\delta\eta_1 + (1-\delta)\eta_2)$. Letting $t \to \infty$, we obtain the concavity of c. □

We now derive a variational formula for $c(\eta)$. Let $(\Omega, \mathcal{F}, P; (\mathcal{F}_t))$ be a given filtered probability space on which is defined an N-dimensional standard $(\mathcal{F}_t)$-Brownian motion $W = (W_t)_{t\ge 0}$. Let $\mathcal{A}_T$ denote the totality of $(\mathcal{F}_t)$-progressively measurable processes $\xi = (\xi_t)_{0\le t\le T}$ with values in $\mathbb{R}^N$. For each $T > 0$, $\xi \in \mathcal{A}_T$, and a given initial law, we define the stochastic process $X^\xi = X^\xi$ as the solution to the stochastic differential equation

$$dX_t^\xi = -\xi_t\,dt - |D\varphi(X_t^\xi)|^{m-2}D\varphi(X_t^\xi)\,dt + dW_t, \quad 0 \le t \le T. \tag{3.2}$$

Let $H_\varphi = H_\varphi(x,p)$ be the function defined by (2.3), and set

$$L(x,p) := \sup_{p\in\mathbb{R}^N} (\xi\cdot p - H_\varphi(x,p)), \quad (x,p) \in \mathbb{R}^{2N}.$$

Note that L satisfies the following properties:

(L1) $L \in C^2(\mathbb{R}^N \times (\mathbb{R}^N - \{0\}))$.
(L2) $\min\{L(x,\xi) \mid \xi \in \mathbb{R}^N\} = 0$ for all $x \in \mathbb{R}^N$.
(L3) $L(x,\xi)$ is strictly convex and superlinear with respect to ξ for all $x \in \mathbb{R}^N$.

For given $\mu, \nu \in \mathcal{M}_1$, where $\mathcal{M}_1 = \mathcal{M}_1(\mathbb{R}^N)$ is the set of Borel probability measures on $\mathbb{R}^N$, we consider the minimization problem

$$\begin{aligned}&\text{Minimize} \quad && J_T(\mu,\nu;\xi) := E\left[\int_0^T L(X_t^\xi, \xi_t)\,dt\right]\\ &\text{subject to} \quad && P(X_0^\xi)^{-1} = \mu, \quad P(X_T^\xi)^{-1} = \nu, \qquad \xi \in \mathcal{A}_T.\end{aligned}$$

Recall that $X^\xi = (X_t^\xi)$ is governed by (3.2). Furthermore, for each $T > 0$ and $\mu, \nu \in \mathcal{M}_1$, we set

$$\begin{aligned}\mathcal{A}_T(\mu,\nu) &:= \{\xi \in \mathcal{A}_T \mid P(X_0^\xi)^{-1} = \mu,\ P(X_T^\xi)^{-1} = \nu\},\\ V_T(\mu,\nu) &:= \inf\{J(\mu,\nu;\xi) \mid \xi \in \mathcal{A}_T(\mu,\nu)\},\\ V(\mu,\nu) &:= \inf\{V_T(\mu,\nu) \mid T > 0\}.\end{aligned}$$

We set $V_T(\mu,\nu) := +\infty$ if $\mathcal{A}_T(\mu,\nu) = \emptyset$. Under this notation, function w defined by (3.1) can be written as

$$w(T,x) = \inf\{V_T(\delta_x,\nu) + \langle \eta,\nu\rangle \mid \nu \in \mathcal{M}_1\},$$

where δ_x stands for the unit distribution concentrated on $x \in \mathbb{R}^N$.

Theorem 3.3. Let $c = c(\eta)$ be the constant given in Theorem 3.1. Then, for any $\eta \in C_b(\mathbb{R}^N)$, one has

$$c(\eta) = \inf\{V(\mu,\nu) + \langle \eta,\nu\rangle \mid \nu \in \mathcal{M}_1\},$$

where μ denotes the invariant probability measure for the A^φ-diffusion.

Proof. Fix any $\eta \in C_b(\mathbb{R}^N)$ and $\nu \in \mathcal{M}_1$. Then, for any $\varepsilon > 0$, there exists a $T > 0$ such that $V_T(\mu,\nu) < V(\mu,\nu) + \varepsilon$. By the definition of V_T, we can find a $\xi \in \mathcal{A}_T(\mu,\nu)$ such that

$$E\left[\int_0^T L(X_t^\xi,\xi_t)\,dt\right] < V_T(\mu,\nu) + \varepsilon.$$

In view of Theorem 3.1, we have

$$c(\eta) \le \langle w(T,\,\cdot\,),\mu\rangle \le E\left[\int_0^T L(X_t^\xi,\xi_t)\,dt + \eta(X_T^\xi)\right] < V(\mu,\nu) + \langle \eta,\nu\rangle + 2\varepsilon.$$

Letting $\varepsilon \to 0$ and then taking the inf over $\nu \in \mathcal{M}_1$, we obtain $c(\eta) \le \inf\{V(\mu,\nu) + \langle \eta,\nu\rangle \mid \nu \in \mathcal{M}_1\}$.

We next prove the opposite inequality. Fix an arbitrary $T > 0$. We consider the feedback control $\xi_T(t,x) := D_pH(x, Dw(T-t,x))$ and define the diffusion process $X = X^T$ by

$$dX_t = -\xi_T(t,X_t)\,dt - |D\varphi(X_t)|^{m-2}D\varphi(X_t)\,dt + dW_t, \qquad 0 \le t \le T,$$

with $P(X_0)^{-1} = \mu$. Then, by Ito's formula and the definition of H_φ, we easily see that

$$\langle w(T,\,\cdot\,),\mu\rangle = E\left[\int_0^T L(X_t,\xi_T(t,X_t))\,dt + \eta(X_T)\right].$$

Setting $\nu^T := P(X_T)^{-1}$, we obtain

$$\langle w(T,\,\cdot\,),\mu\rangle \ge V_T(\mu,\nu^T) + \langle \eta,\nu^T\rangle \ge \inf\{V(\mu,\nu) + \langle \eta,\nu\rangle \mid \nu \in \mathcal{M}_1\}.$$

Since $T > 0$ is arbitrary, we have the opposite inequality. Hence, we have completed the proof. $\square$

Now, we consider the case where $m = 2$. In this case, we have $H_\varphi(x,p) = (1/2)|p|^2$. In particular, w satisfies the equation

$$\partial_t w - \frac{1}{2}\Delta w + D\varphi \cdot Dw + \frac{1}{2}|Dw|^2 = 0 \quad \text{in} \quad (0,\infty) \times \mathbb{R}^N.$$

We set $v := e^{-w}$. Then, v is a solution of the linear equation

$$\partial_t v - \frac{1}{2}\Delta v + D\varphi \cdot Dv = 0 \quad \text{in} \quad (0,\infty) \times \mathbb{R}^N$$

with $v(0,\,\cdot\,) = e^{-\eta}$ in $\mathbb{R}^N$. Note that v is written as $v(T,x) = E_x[e^{-\eta(X_T)}]$, where X is governed by

$$dX_t = -D\varphi(X_t)\,dt + dW_t.$$

Since X is ergodic with invariant probability measure $\mu(dx) := e^{-2\varphi(x)}dx$, we have

$$v(T,x) = E_x[e^{-\eta(X_T)}] \longrightarrow \int_{\mathbb{R}^N} e^{-\eta(x)}\mu(dx) \quad \text{as} \quad T \to \infty.$$

Thus, $c(\eta)$ in Theorem 3.1 can be written as

$$c(\eta) = -\log \int_{\mathbb{R}^N} e^{-\eta(y)}\,\mu(dy), \qquad \eta \in C_b(\mathbb{R}^N). \tag{3.3}$$

Taking into account this observation, we have the following representation formula for $c(\eta)$.

Theorem 3.4. Assume that $m = 2$. Let $H(\nu|\mu)$ be the "relative entropy" defined by

$$H(\nu|\mu) := \int_{\mathbb{R}^N} \log \frac{d\nu}{d\mu}(x)\,\nu(dx), \qquad \nu \ll \mu,$$

where $H(\nu|\mu) := +\infty$ if ν is singular to μ. Then,

$$c(\eta) = \min\{\langle \eta, \nu\rangle + H(\nu|\mu) \mid \nu \in \mathcal{M}_1\}, \quad \mu := e^{-2\varphi}dx.$$

Moreover, the minimum is attained when $\nu = e^{-\eta}d\mu/\langle e^{-\eta}, \mu\rangle$.

Proof. Let $\nu \in \mathcal{M}_1$ be such that $\nu \ll \mu$, and set $p := d\nu/d\mu$. Then, for any $\eta \in C_b(\mathbb{R}^N)$, we have

$$c(\eta) - \langle \eta, \nu\rangle \le H(\nu|\mu).$$

Indeed, in view of (3.3), we see that the above inequality is equivalent to say that

$$\exp\left(\int_{\mathbb{R}^N} \{-\eta(x) - \log p(x)\}\nu(dx)\right) \le \int_{\mathbb{R}^N} e^{-\eta(x)}\,\mu(dx).$$

But this inequality is true in view of Jensen's inequality. Hence, we obtain

$$c(\eta) \le \langle \eta, \nu\rangle + H(\nu|\mu), \qquad \nu \in \mathcal{M}_1.$$

Note that the equality holds if and only if $-\eta(x) - \log p(x)$ is constant. This implies that p should be of the form $p(x) = e^{-\eta(x)}/\langle e^{-\eta}, \mu\rangle$. Hence, we have completed the proof. $\square$

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