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Global bifurcation diagrams and exact multiplicity of positive solutions for a one-dimensional prescribed mean curvature problem arising in MEMS

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1. Introduction

In this paper we study global bifurcation diagrams and exact multiplicity of positive solutions $u \in C^2(-L, L) \cap C[-L, L]$ for the one-dimensional prescribed mean curvature problem arising in electrostatic MEMS

$$\begin{cases} -\left(\dfrac{u'(x)}{\sqrt{1+(u'(x))^2}}\right)' = \dfrac{\lambda}{(1-u)^p}, \quad u<1, \quad -L<x<L, \\ u(-L)=u(L)=0, \end{cases} \tag{1.1}$$

where $\lambda > 0$ is a *bifurcation* parameter, and $p, L > 0$ are two *evolution* parameters. The singular nonlinearity

$$f(u) \equiv \frac{1}{(1-u)^p}, \ p>0$$

satisfies

$$f(0)=1, \ \lim_{u\to 1^-} f(u)=\infty, \ \text{and } f'(u), f''(u)>0 \text{ on } [0,1). \tag{1.2}$$

Notice that the improper integral of f over $[0,1)$ satisfies

$$\int_0^1 f(u)du = \begin{cases} \infty & \text{if } p \geq 1, \\ \frac{1}{1-p} < \infty & \text{if } 0<p<1. \end{cases}$$

The prescribed mean curvature problem

$$\begin{cases} -\left(\dfrac{u'(x)}{\sqrt{1+(u'(x))^2}}\right)' = \lambda \tilde{f}(u), \quad -L<x<L, \\ u(-L)=u(L)=0, \end{cases} \tag{1.3}$$

and n-dimensional problem of it, with general nonlinearity $\tilde{f}(u)$ or with many different types nonlinearities, like $\exp(u)$, $(1+u)^p$ $(p>0)$, $\exp(u)-1$, u^p $(p>0)$, a^u $(a>0)$, $u-u^3$, and $u^p + u^q$ ($0 \le p < q < \infty$) have been recently investigated by many authors, see e.g. [1, 2, 3, 4, 5, 6]. The methods for (1.3) they used are based on a detailed analysis of time maps. Note that, in geometry, a solution $u(x)$ of (1.3) is also called a graph of prescribed (mean) curvature $\lambda\tilde{f}(u)$.

A solution $u \in C^2(-L,L) \cap C[-L,L]$ of (1.3) with $u' \in C([-L,L],[-\infty,\infty])$ is called *classical* if $|u'(\pm L)| < \infty$, and it is called *non-classical* if $u'(-L) = \infty$ or $u'(L) = -\infty$, see [5]. In this paper we always allow that solutions $u \in C^2(-L,L) \cap C[-L,L]$ satisfy $u' \in C([-L,L],[-\infty,\infty])$. Notice that it can be shown that (see [5]):

(i) Any non-trivial solution $u \in C^2(-L,L) \cap C[-L,L]$ of (1.1) is *concave* and *positive* on $(-L,L)$.

(ii) A positive solution $u \in C^2(-L,L) \cap C[-L,L]$ of (1.1) must be *symmetric* on $[-L,L]$. Thus $u'(-L) = -u'(L)$.

(iii) A classical solution $u \in C^2(-L,L) \cap C[-L,L]$ of (1.1) belongs to $C^2[-L,L]$.

(iv) A non-classical solution $u \in C^2(-L,L) \cap C[-L,L]$ of (1.1) satisfies $|u'(\pm L)| = \infty$.

For any fixed $p, L > 0$, we define the bifurcation diagram $C_{p,L}$ of (1.1) by

$$C_{p,L} \equiv \{(\lambda, \|u_\lambda\|_\infty) : \lambda > 0 \text{ and } u_\lambda \text{ is a positive solution of (1.1)}\}.$$

We say the bifurcation diagram $C_{p,L}$ is $\supset$-shaped (see e.g. Fig. 1(i) depicted below) if there exists $\lambda^* > 0$ such that $C_{p,L}$ consists of a continuous curve with exactly one turning point at some point $(\lambda^*, \|u_{\lambda^*}\|_\infty)$ where the bifurcation diagram $C_{p,L}$ turns to the left.

This research is motivated by very recent papers of Pan and Xing [6] and Brubaker and Pelesko [1, 7]. Brubaker and Pelesko [7] studied existence and multiplicity of positive solutions of the prescribed mean curvature problem

$$\begin{cases} -\text{div}\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}} = \dfrac{\lambda}{(1-u)^2}, \quad u < 1, \quad \mathbf{x} \in \Omega_L, \\ u = 0, \quad \mathbf{x} \in \partial\Omega_L, \end{cases} \tag{1.4}$$

where $\lambda > 0$ is a bifurcation parameter and $\Omega_L \subset \mathbb{R}^n$ ($n \ge 1$) is a smooth bounded domain depending on some parameter $L > 0$. Problem (1.4) with an inverse square type nonlinearity $f(u) = (1-u)^{-p}$, $p = 2$ is a derived variant of a *canonical* model used in the modeling of electrostatic Micro-Electro Mechanical Systems (MEMS) device obeying the electrostatic Coulomb law with the Coulomb force satisfies the *inverse square law* with respect to the distance of the two charged objects, which is a function of the deformation variable (cf. [8, p. 1324].) The modeling of electrostatic MEMS device consists of a thin dielectric elastic membrane with boundary supported at 0 below a rigid plate located at +1. In (1.4), u is the unknown profile of the deflecting MEMS membrane, λ is the drop voltage between the ground plate and the deflecting membrane, and the term $|\nabla u|^2$ is called a fringing field (cf. [7]). When a voltage λ is applied, the membrane deflects towards the ceiling plate and a snap-through may occur when it exceeds a certain critical value λ^*, referred to as the "pull-in voltage". (So if voltage λ exceeds pull-in voltage λ^*, an equilibrium defection is no longer attainable and the lower surface will touch up on the upper plate.) This creates a so-called "pull-in instability" which greatly affects the design of many devices. Also, in the actual design of a MEMS device, typically, one of the primary device design goals is to achieve the maximum possible stable steady-state deflection (that is, $\|u_{\lambda^*}\|_\infty$ (< 1), cf. Theorems 1–2 and Figs. 1–2 below), referred to as the "pull-in distance", with a relatively small applied voltage. We refer to [7] and the book [9] for detailed discussions on MEMS devices modeling. We also refer to the book [10] for mathematical analysis of electrostatic MEMS problem (1.4). Notice that the physically relevant dimensions are $n = 1$ (In this case Ω_L is a rectangular strip with two opposite edges at $x = \pm L$ fixed ($2L$ is the length of the strip) and the remaining two edges free, the deflection $u = u(x, y)$ may be assumed a function of x only.) and $n = 2$ (Ω_L is a planar bounded domain with smooth boundary, and L is the characteristic length (diameter) of the domain. In particular, Ω_L is a circular disk of radius L.)

With general $p > 0$, (1.1) is a generalized MEMS problem under the assumption that the Coulomb force satisfies the *inverse p-th power law* with respect to the distance of the two charged objects, where $p > 0$ characterizes the *force strength.* See [11, 12, 13, 14] for related references in which the Coulomb force satisfies *inverse p-th power law with various positive numbers* $p \neq 2$.

Pan and Xing [6] and Brubaker and Pelesko [1] studied global bifurcation diagrams and exact multiplicity of positive solutions for the one dimensional problem of (1.4),

$$\begin{cases} -\left(\dfrac{u'(x)}{\sqrt{1+(u'(x))^2}}\right)' = \dfrac{\lambda}{(1-u)^2}, \quad u < 1, \quad -L < x < L, \\ u(-L) = u(L) = 0. \end{cases} \tag{1.5}$$

(Notice that, problem (1.1) reduces to problem (1.5) when $p = 2$.) Pan and Xing [6, Theorem 1.1] and Brubaker and Pelesko [1, Theorem 1.1] independently proved that there exists $L^* > 0$ such that, on the $(\lambda, \|u\|_\infty)$-plane, the bifurcation diagram $C_{2,L}$ of (1.5) consists of a (continuous) $\supset$-shaped curve when $L \geq L^*$, and as L transitions from greater than or equal to L^* to less than L^* the upper branch of the bifurcation diagram $C_{2,L}$ of (1.5) splits into two parts. See Fig. 1 and see [6, Theorem 1.1] and [1, Theorem 1.1] for details. Note that Brubaker and Pelesko [1, Theorem 1.1] showed that $L^* \approx 0.3499676$ and they also gave some computational results, see [1, Fig. 2].

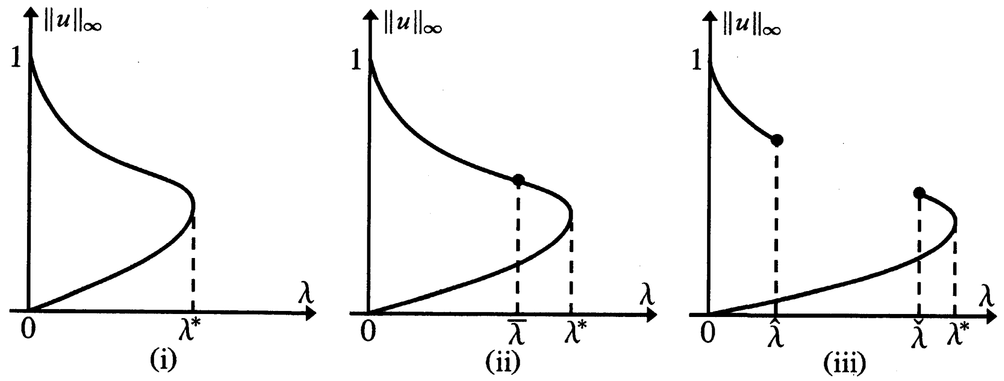

Fig. 1. Global bifurcation diagrams $C_{p,L}$ with $p \geq 1$.
(i) $L > L^*$. (ii) $L = L^*$. (iii) $0 < L < L^*$.

In this paper we extend and improve the results of Pan and Xing [6, Theorem 1.1] and Brubaker and Pelesko [1, Theorem 1.1] by generalizing the nonlinearity $f(u) = (1 - u)^{-2}$ in (1.5) to $f(u) = (1 - u)^{-p}$ with general $p \in [1, \infty)$, see Theorem 2.1 stated below. Our results (Theorems 2.1 and 2.2) also answer an open question raised by Brubaker and Pelesko

[1, section 4] on the (possible) extension of (global) bifurcation diagram results of generalized MEMS problem (1.1) under the assumption that the Coulomb force satisfies the *inverse p-th power law* with respect to the distance of the two charged objects, where $p > 0$ characterizes the *force strength.* To this open question, we find and prove that global bifurcation diagrams $C_{p,L}$ for $0 < p < 1$ are different to and more complicated than those for $p \geq 1$; compare Fig. 2 depicted below with Fig. 1. Thus p is also a bifurcation parameter to prescribed mean curvature problem (1.1). This result is of particular interest since p is *not* a bifurcation parameter to the corresponding *semilinear* problem of *quasilinear* problem (1.1),

$$\begin{cases} -u''(x) = \dfrac{\lambda}{(1-u)^p}, \quad u < 1, \quad -L < x < L, \\ u(-L) = u(L) = 0. \end{cases} \tag{1.6}$$

For (1.6) with any $p > 0$ and $L > 0$, by applying (1.2) and Laetsch [15, Theorems 2.5, 2.9 and 3.2], we obtain that, on the $(\lambda, \|u\|_\infty)$-plane, the bifurcation diagram of positive solutions consists of a (continuous) $\supset$-shaped curve which starts from the origin and ends at $(0, 1)$, cf. Fig. 1(i).

The paper is organized as follows. Section 2 contains statements of main results. Section 3 contains several lemmas needed to prove the main results. Section 4 contains the proofs of the main results.

2. Main results

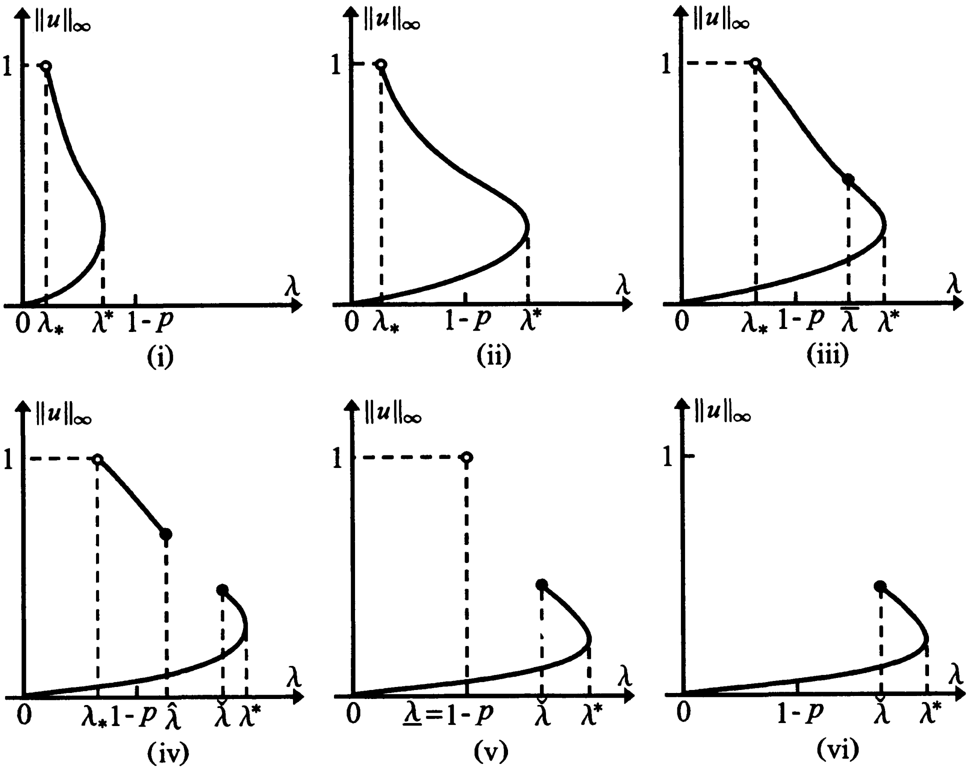

Fig. 2. Global bifurcation diagrams $C_{p,L}$ with $0 < p < 1$.
(i)–(ii) $L > L^*$. (iii) $L = L^*$. (iv) $L_* < L < L^*$. (v) $L = L_*$. (vi) $0 < L < L_*$.

The main results in this paper are next Theorems 2.1 and 2.2 for (1.1).

Theorem 2.1 (See Fig. 1). *Consider* (1.1) *with* $p \geq 1$. *There exists* $L^* = L^*(p) > 0$ *such that the following assertions (i)–(iii) hold:*

(i) *(See Fig. 1(i).) If* $L > L^*$, *then there exists* $\lambda^* > 0$ *such that* (1.1) *has exactly two positive solutions* u_λ, v_λ *with* $\|u_\lambda\|_\infty < \|v_\lambda\|_\infty$ *for* $0 < \lambda < \lambda^*$, *exactly one positive solution* u_λ *for* $\lambda = \lambda^*$, *and no positive solution for* $\lambda > \lambda^*$.

(ii) *(See Fig. 1(ii).) If* $L = L^*$, *then there exist* $0 < \bar{\lambda}$ $(= \bar{\lambda}(p)) < \lambda^*$ *such that* (1.1) *has exactly two positive solutions* u_λ, v_λ *with* $\|u_\lambda\|_\infty < \|v_\lambda\|_\infty$ *for* $0 < \lambda < \lambda^*$, *exactly one positive solution* u_λ *for* $\lambda = \lambda^*$, *and no positive solution for* $\lambda > \lambda^*$.

(iii) *(See Fig. 1(iii).) If* $0 < L < L^*$, *then there exist* $0 < \hat{\lambda} < \check{\lambda} < \lambda^*$ *such that* (1.1) *has exactly two positive solutions* u_λ, v_λ *with* $\|u_\lambda\|_\infty < \|v_\lambda\|_\infty$ *for* $0 < \lambda \leq \hat{\lambda}$ *and* $\check{\lambda} \leq \lambda < \lambda^*$, *exactly one positive solution* u_λ *for* $\hat{\lambda} < \lambda < \check{\lambda}$ *and* $\lambda = \lambda^*$, *and no positive solution for* $\lambda > \lambda^*$.

Theorem 2.2 (See Fig. 2). *Consider* (1.1) *with* $0 < p < 1$. *There exist* $0 < L_*$ $(= L_*(p))$ $< L^*$ $(= L^*(p))$ *such that the following assertions (i)–(iv) hold:*

(i) *(See Fig. 2(i)–(ii).) If* $L > L^*$, *then there exist* $0 < \lambda_* < \lambda^*$ *such that* (1.1) *has exactly two positive solutions* u_λ, v_λ *with* $\|u_\lambda\|_\infty < \|v_\lambda\|_\infty$ *for* $\lambda_* < \lambda < \lambda^*$, *exactly one positive solution* u_λ *for* $0 < \lambda \leq \lambda_*$ *and* $\lambda = \lambda^*$, *and no positive solution for* $\lambda > \lambda^*$.

(ii) *(See Fig. 2(iii).) If* $L = L^*$, *then there exist* $0 < \lambda_* < \bar{\lambda}$ $(= \bar{\lambda}(p)) < \lambda^*$ *satisfying* $\lambda_* < 1 - p < \bar{\lambda}$ *such that* (1.1) *has exactly two positive solutions* u_λ, v_λ *with* $\|u_\lambda\|_\infty < \|v_\lambda\|_\infty$ *for* $\lambda_* < \lambda < \lambda^*$, *exactly one positive solution* u_λ *for* $0 < \lambda \leq \lambda_*$ *and* $\lambda = \lambda^*$, *and no positive solution for* $\lambda > \lambda^*$.

(iii) *(See Fig. 2(iv).) If* $L_* < L < L^*$, *then there exist* $0 < \lambda_* < \hat{\lambda} < \check{\lambda} < \lambda^*$ *satisfying* $\lambda_* < 1 - p < \hat{\lambda}$ *such that* (1.1) *has exactly two positive solutions* u_λ, v_λ *with* $\|u_\lambda\|_\infty < \|v_\lambda\|_\infty$ *for* $\lambda_* < \lambda \leq \hat{\lambda}$ *and* $\check{\lambda} \leq \lambda < \lambda^*$, *exactly one positive solution* u_λ *for* $0 < \lambda \leq \lambda_*$, $\hat{\lambda} < \lambda < \check{\lambda}$ *and* $\lambda = \lambda^*$, *and no positive solution for* $\lambda > \lambda^*$.

(iv) *(See Fig. 2(v)–(vi).) If* $0 < L \leq L_*$, *then there exist* $0 < \check{\lambda} < \lambda^*$ *satisfying* $1 - p < \check{\lambda}$ *such that* (1.1) *has exactly two positive solutions* u_λ, v_λ *with* $\|u_\lambda\|_\infty < \|v_\lambda\|_\infty$ *for* $\check{\lambda} \leq \lambda < \lambda^*$, *exactly one positive solution* u_λ *for* $0 < \lambda < \check{\lambda}$ *and* $\lambda = \lambda^*$, *and no positive solution for* $\lambda > \lambda^*$.

3. Lemmas

In this section, in the next Lemmas 3.1–3.8, we develop some time-map techniques to prove Theorems 2.1–2.4. First, we introduce the time-map method used in [4, 5]. Let $F(u) \equiv \int_0^u f(t)dt$. We have that:

(I) If $p \geq 1$, $F : [0, 1) \to [0, \infty)$ and hence F^{-1} is well defined on $[0, \infty)$. Then for any $\lambda > 0$, the time map formula for (1.1) takes the form as follows:

$$T_\lambda(r) = \int_0^r \frac{1 + \lambda F(u) - \lambda F(r)}{\sqrt{1 - [1 + \lambda F(u) - \lambda F(r)]^2}} du, \quad r = \|u\|_\infty \in (0, F^{-1}(\frac{1}{\lambda})], \tag{3.1}$$

where
$$F(u)=\int_0^u f(t)dt=\begin{cases} -\log(1-u) & \text{if } p=1,\\ \frac{-1+(1-u)^{1-p}}{p-1} & \text{if } p\in(1,\infty).\end{cases} \tag{3.2}$$
Notice that it can be proved that $T_\lambda(r)\in C^2((0,F^{-1}(\frac{1}{\lambda})])$, see [2, Lemma 3.1].

(II) If $0<p<1$, $F:[0,1]\to[0,\frac{1}{1-p}]$ and hence F^{-1} is only defined on $[0,\frac{1}{1-p}]$. Then the time map formula for (1.1) takes the form as follows:
$$T_\lambda(r)=\int_0^r\frac{1+\lambda F(u)-\lambda F(r)}{\sqrt{1-[1+\lambda F(u)-\lambda F(r)]^2}}du,\quad r=\|u\|_\infty\in\begin{cases}(0,F^{-1}(\frac{1}{\lambda})] & \text{if } \lambda>1-p,\\ (0,1) & \text{if } 0<\lambda\le 1-p,\end{cases} \tag{3.3}$$
where
$$F(u)=\int_0^u f(t)dt=\frac{1-(1-u)^{1-p}}{1-p}. \tag{3.4}$$
Note that the time map formula $T_\lambda(r)$ in (3.3) with $0<p<1$ is the same as that in (3.1) with $p\ge 1$. But the domain of $T_\lambda(r)$ in (3.3) with $0<p<1$ is different from that in (3.1) with $p\ge1$, since $\lim_{u\to 1^-}f(u)=\infty$ and $F^{-1}:[0,\frac{1}{1-p})\to[0,1)$ when $0<p<1$. Notice that it also can be proved that $T_\lambda(r)\in C^2((0,F^{-1}(\frac{1}{\lambda})])$ if $\lambda>1-p$ and $T_\lambda(r)\in C^2((0,1))$ if $0<\lambda\le 1-p$.

Observe that positive solutions u_λ for (1.1) correspond to
$$\|u_\lambda\|_\infty=r \text{ and } T_\lambda(r)=L. \tag{3.5}$$
Thus, studying of the exact number of positive solutions of (1.1) for any fixed $\lambda>0$ is equivalent to studying the shape of the time map $T_\lambda(r)$ on its domain.

First, we determine the limit behaviors of $T_\lambda(r)$ and $T'_\lambda(r)$ in the following lemma.

Lemma 3.1. *Consider $T_\lambda(r)$. Then*

(i) For fixed $p>0$, $\lim_{r\to0^+}T_\lambda(r)=0$ and $\lim_{r\to0^+}T'_\lambda(r)=\infty$ for any $\lambda>0$.

(ii) For fixed $p\ge1$, $T'_\lambda(F^{-1}(\frac{1}{\lambda}))<0$ for any $\lambda>0$.

(iii) For fixed $p\in(0,1)$, $T'_\lambda(F^{-1}(\frac{1}{\lambda}))<0$ for any $\lambda>1-p$ and $\lim_{r\to1^-}T'_\lambda(r)=-\infty$ for any $0<\lambda\le1-p$.

Proof of Lemma 3.1. First, the results in parts (i)–(ii) follow from [4, Propositions 2.6, 2.7, 2.10] since $f(0)=1>0$ and $f'(u)=p(1-u)^{-p-1}>0$ on $[0,1)$.

Finally, for part (iii), for fixed $p\in(0,1)$, the result $T'_\lambda(F^{-1}(1/\lambda))<0$ if $\lambda>1-p$ follows from [4, Propositions 2.10]. The remaining part of the proof of part (iii) is to prove $\lim_{r\to1^-}T'_\lambda(r)=-\infty$ for $0<\lambda\le1-p$.

Let $u=rs$, then (3.3) becomes
$$T_\lambda(r)=r\int_0^1\frac{1+\lambda F(rs)-\lambda F(r)}{\sqrt{1-[1+\lambda F(rs)-\lambda F(r)]^2}}ds,\quad r\in(0,1).$$
We compute that
$$T'_\lambda(r)=I_1(r)+I_2(r),\quad r\in(0,1), \tag{3.6}$$

where

$$I_1(r) \equiv \int_0^1 \frac{1+\lambda F(rs)-\lambda F(r)}{\sqrt{1-[1+\lambda F(rs)-\lambda F(r)]^2}}ds,$$

and

$$I_2(r) \equiv \int_0^1 \frac{\lambda r\,[f(rs)s-f(r)]}{\left\{1-[1+\lambda F(rs)-\lambda F(r)]^2\right\}^{3/2}}ds.$$

We compute that

$$\begin{aligned}
\lim_{r\to 1^-} I_1(r) &= \lim_{r\to 1^-}\int_0^1 \frac{1+\lambda F(rs)-\lambda F(r)}{\sqrt{1-[1+\lambda F(rs)-\lambda F(r)]^2}}ds \\
&= \int_0^1 \lim_{r\to 1^-} \frac{1+\lambda F(rs)-\lambda F(r)}{\sqrt{1-[1+\lambda F(rs)-\lambda F(r)]^2}}ds \\
&= \int_0^1 \frac{1-\lambda\frac{(1-s)^{1-p}}{1-p}}{\sqrt{1-\left[1-\lambda\frac{(1-s)^{1-p}}{1-p}\right]^2}}ds \\
&= \int_{1-\frac{\lambda}{1-p}}^1 \frac{y}{\sqrt{1-y^2}}\frac{\left[\frac{(1-y)(1-p)}{\lambda}\right]^{\frac{p}{1-p}}}{\lambda}dy, \quad \left(\text{set } y = 1-\lambda\frac{(1-s)^{1-p}}{1-p}\right) \\
&= \frac{(1-p)^{\frac{p}{1-p}}}{\lambda^{\frac{1}{1-p}}}\int_{1-\frac{\lambda}{1-p}}^1 \frac{y\,(1-y)^{\frac{p}{1-p}}}{\sqrt{1-y^2}}dy \\
&< \infty
\end{aligned} \tag{3.7}$$

by simple analysis of the last integral for y near 1^-.

On the other hand, we show that $\lim_{r\to 1^-} I_2(r) = -\infty$. For any fixed $r \in (0,1)$, since both F and f are increasing function on $(0,1)$, we obtain that $f(rs)s - f(r) < 0$ for $s \in (0,1)$ and $\left\{1-[1+\lambda F(rs)-\lambda F(r)]^2\right\}^{3/2}$ is strictly decreasing in $s \in (0,1)$. Hence we compute that

$$\begin{aligned}
I_2(r) &= \int_0^1 \frac{\lambda r\,[f(rs)s-f(r)]}{\left\{1-[1+\lambda F(rs)-\lambda F(r)]^2\right\}^{3/2}}ds \\
&\le \int_0^1 \frac{\lambda r\,[f(rs)s-f(r)]}{\left\{1-[1-\lambda F(r)]^2\right\}^{3/2}}ds \\
&= \frac{\lambda r}{\left\{1-[1-\lambda F(r)]^2\right\}^{3/2}}\int_0^1 [f(rs)s-f(r)]\,ds \\
&= \frac{\lambda r}{\left\{1-[1-\lambda F(r)]^2\right\}^{3/2}}\left[\frac{(1-r)^p+pr-1-r^2(1-p)^2}{(1-p)\,(2-p)\,(1-r)^p r^2}\right].
\end{aligned}$$

This implies that

$$\lim_{r\to 1^-} I_2(r) \le \lim_{r\to 1^-}\frac{1}{\left\{1-[1-\lambda F(r)]^2\right\}^{3/2}}\left[\frac{(1-r)^p+pr-1-r^2(1-p)^2}{(1-p)\,(2-p)\,(1-r)^p r^2}\right] = -\infty. \tag{3.8}$$

Combining (3.6)–(3.8), we obtain that

$$\lim_{r\to 1^-} T'_\lambda(r) = \lim_{r\to 1^-} I_1(r) + \lim_{r\to 1^-} I_2(r) = -\infty \quad \text{for } 0<\lambda\le 1-p.$$

This completes the proof of Lemma 3.1. ∎

In the next lemma, we then prove that $T_\lambda(r)$ has exactly one critical point, a local maximum, on its domain.

Lemma 3.2. *Consider $T_\lambda(r)$. Then*

(i) For fixed $p \geq 1$, $T_\lambda(r)$ has exactly one critical point, a local maximum, on $(0, F^{-1}(1/\lambda))$ for any $\lambda > 0$.

(ii) For fixed $p \in (0,1)$, $T_\lambda(r)$ has exactly one critical point, a local maximum, on $(0, F^{-1}(1/\lambda))$ for any $\lambda > 1-p$.

(iii) For fixed $p \in (0,1)$, $T_\lambda(r)$ has exactly one critical point, a local maximum, on $(0,1)$ for any $0 < \lambda \leq 1-p$.

Proof of Lemma 3.2. For part (i) with $p \geq 1$ be fixed. Since $f(0) = 1 > 0$, $f'(u) = p(1-u)^{-p-1} > 0$ on $[0,1)$, and $f''(u) = p(p+1)(1-u)^{-p-2} > 0$ on $[0,1)$, (1.1) has at most two positive solutions for any $\lambda, L > 0$ by [3, Theorem 3.4]. Suppose that, on the contrary, part (i) does not hold. Then by Lemma 3.1(i)–(ii), $T_\lambda(r)$ has at least two critical points, a local maximum and a local minimum, on $(0, F^{-1}(1/\lambda))$. So by (3.5), (1.1) has at least three positive solutions for some $\lambda, L > 0$, which contradicts to the fact that (1.1) has at most two positive solutions. So part (i) follows.

The proofs of parts (ii) and (iii) are similar to that of part (i), so we omit them.

The proof of Lemma 3.2 is complete. ∎

For any $p \geq 1$, let

$$h_p(\lambda) \equiv \sup\left\{T_\lambda(r) : r \in (0, F^{-1}(\frac{1}{\lambda})]\right\}, \ \lambda > 0. \tag{3.9}$$

For any $0 < p < 1$, let

$$h_p(\lambda) \equiv \begin{cases} \sup\left\{T_\lambda(r) : r \in (0, F^{-1}(\frac{1}{\lambda})]\right\} & \text{if } \lambda > 1-p, \\ \sup\left\{T_\lambda(r) : r \in (0,1)\right\} & \text{if } 0 < \lambda \leq 1-p. \end{cases} \tag{3.10}$$

We mainly determine some basic properties of $h_p(\lambda)$ in the following lemma.

Lemma 3.3. *Consider $T_\lambda(r)$ and $h_p(\lambda)$ with fixed $p > 0$. Then*

(i) For fixed $r \in (0,1)$, $T_\lambda(r)$ is a continuous, strictly decreasing function of $\lambda > 0$. Moreover, $\lim_{\lambda \to 0^+} T_\lambda(r) = \infty$.

(ii) $h_p(\lambda)$ is a continuous, strictly decreasing function of $\lambda > 0$. Moreover, $\lim_{\lambda \to 0^+} h_p(\lambda) = \infty$ and $\lim_{\lambda \to \infty} h_p(\lambda) = 0$.

Proof of Lemma 3.3. Let $p > 0$ be fixed.

(i) First, for fixed $r \in (0,1)$, it can be proved that $T_\lambda(r)$ is a continuous function of $\lambda > 0$. The proof is easy but tedious and we omit it. For any fixed $r \in (0,1)$ and $0 < u < r$, $\frac{1+\lambda F(u) - \lambda F(r)}{\sqrt{1-[1+\lambda F(u)-\lambda F(r)]^2}}$ is strictly decreasing in λ since $0 < F(r) - F(u) < 1/\lambda$. So $T_\lambda(r)$ is a strictly decreasing function of $\lambda > 0$. Moreover, $\lim_{\lambda \to 0^+} T_\lambda(r) = \infty$ follows directly from the time map formula (3.1). So part (i) follows.

(ii) By Lemma 3.2 and part (i), we obtain that $h_p(\lambda)$ is a continuous, strictly decreasing function of $\lambda > 0$, and $\lim_{\lambda\to 0^+} h_p(\lambda) = \infty$. On the other hand, since $\lim_{\lambda\to\infty} F^{-1}(\frac{1}{\lambda}) = 0$ and $0 < r \le F^{-1}(\frac{1}{\lambda})$ for large λ, we have $\lim_{\lambda\to\infty} h_p(\lambda) = 0$. So part (ii) follows.

The proof of Lemma 3.3 is complete. ■

For any $p \ge 1$, let

$$g_p(\lambda) \equiv T_\lambda(F^{-1}(\frac{1}{\lambda})),\ \lambda > 0. \tag{3.11}$$

For any $0 < p < 1$, let

$$g_p(\lambda) \equiv \begin{cases} T_\lambda(F^{-1}(\frac{1}{\lambda})) & \text{if } \lambda > 1-p, \\ \lim_{r\to 1^-} T_\lambda(r) & \text{if } 0 < \lambda \le 1-p. \end{cases} \tag{3.12}$$

Let $\alpha = F^{-1}(\frac{1}{\lambda})$ and $u = \alpha s$, then by (3.1),

$$\begin{aligned} T_\lambda(F^{-1}(\frac{1}{\lambda})) &= \int_0^{F^{-1}(\frac{1}{\lambda})} \frac{\lambda F(u)}{\sqrt{1-[\lambda F(u)]^2}}du \\ &= \alpha\int_0^1 \frac{\lambda F(\alpha s)}{\sqrt{1-[\lambda F(\alpha s)]^2}}ds \\ &= \int_0^1 \frac{1}{\sqrt{1-t^2}}\frac{\frac{t}{\lambda}}{f(F^{-1}(\frac{t}{\lambda}))}dt \end{aligned}$$

by change of variable $t = \lambda F(\alpha s)$. So for $p \ge 1$, (3.11) implies

$$g_p(\lambda) = T_\lambda(F^{-1}(\frac{1}{\lambda})) = \int_0^1 \frac{1}{\sqrt{1-t^2}}\frac{\frac{t}{\lambda}}{f(F^{-1}(\frac{t}{\lambda}))}dt,\ \lambda > 0. \tag{3.13}$$

For $0 < p < 1$, by (3.3) and (3.4),

$$\begin{aligned} \lim_{r\to 1^-} T_\lambda(r) &= \int_0^1 \frac{1+\lambda F(u) - \lambda F(1)}{\sqrt{1-[1+\lambda F(u)-\lambda F(1)]^2}}du \\ &= \int_0^1 \frac{(1-p)-\lambda(1-u)^{1-p}}{\sqrt{2\lambda(1-p)(1-u)^{1-p} - \lambda^2(1-u)^{2-2p}}}du. \end{aligned}$$

So for $0 < p < 1$, (3.12) implies

$$g_p(\lambda) = \begin{cases} \int_0^1 \frac{1}{\sqrt{1-t^2}}\frac{\frac{t}{\lambda}}{f(F^{-1}(\frac{t}{\lambda}))}dt & \text{if } \lambda > 1-p, \\ \int_0^1 \frac{(1-p)-\lambda(1-u)^{1-p}}{\sqrt{2\lambda(1-p)(1-u)^{1-p}-\lambda^2(1-u)^{2-2p}}}du & \text{if } 0 < \lambda \le 1-p. \end{cases} \tag{3.14}$$

We first determine some basic properties of $g_p(\lambda)$ in the following lemma.

Lemma 3.4. *Consider $g_p(\lambda)$. Then*

(i) For fixed $p > 0$, $g_p(\lambda)$ is a continuous function of $\lambda > 0$.

(ii) For fixed $p \ge 1$, $\lim_{\lambda\to 0^+} g_p(\lambda) = \lim_{\lambda\to\infty} g_p(\lambda) = 0$.

(iii) For fixed $p \in (0,1)$, $\lim_{\lambda\to 0^+} g_p(\lambda) = \infty$ and $\lim_{\lambda\to\infty} g_p(\lambda) = 0$.

Proof of Lemma 3.4. (i) Since the map $\lambda \mapsto \frac{\frac{t}{\lambda}}{f(F^{-1}(\frac{t}{\lambda}))}$ is a composition of $y \mapsto \frac{F(y)}{f(y)}$ and $y = F^{-1}\left(\frac{t}{\lambda}\right)$. For fixed $p \geq 1$, $g_p(\lambda)$ is a continuous function of $\lambda > 0$ by (3.13). For fixed $p \in (0,1)$, $g_p(\lambda)$ is a continuous function of $\lambda \in (0, 1-p)$ by (3.13) and (3.14). In addition, $g_p(\lambda)$ is a continuous function of $\lambda < 1-p$ by (3.14). Moreover, since

$$\lim_{\lambda \to (1-p)^-} g_p(\lambda) = \lim_{\lambda \to (1-p)^+} g_p(\lambda) = g_p(1-p) = \lim_{r \to 1^-} T_{1-p}(r) = \int_0^1 \frac{1-u^{1-p}}{\sqrt{2u^{1-p} - u^{2-2p}}} du, \quad (3.15)$$

$g_p(\lambda)$ is a continuous at $\lambda = 1-p$. So part (i) follows.

(ii) For fixed $p \geq 1$, by (3.2), we first obtain that

$$\lim_{y \to 0^+} \frac{F(y)}{f(y)} = \lim_{y \to 0^+} \frac{\frac{1-(1-y)^{p-1}}{(p-1)(1-y)^{p-1}}}{\frac{1}{(1-y)^p}} = \lim_{y \to 0^+} \frac{(1-y)-(1-y)^p}{p-1} = 0 \quad \text{for } p > 1, \quad (3.16)$$

and

$$\lim_{y \to 0^+} \frac{F(y)}{f(y)} = \lim_{y \to 0^+} \frac{-\log(1-y)}{\frac{1}{1-y}} = 0 \quad \text{for } p = 1. \quad (3.17)$$

We change variables in (3.13) by writing $y = F^{-1}(\frac{t}{\lambda})$, then

$$\lim_{\lambda \to \infty} g_p(\lambda) = \int_0^1 \frac{1}{\sqrt{1-t^2}} \lim_{\lambda \to \infty} \frac{\frac{t}{\lambda}}{f(F^{-1}(\frac{t}{\lambda}))} dt = \int_0^1 \frac{1}{\sqrt{1-t^2}} \lim_{y \to 0^+} \frac{F(y)}{f(y)} dt = 0$$

by (3.16) and (3.17). On the other hand,

$$\lim_{y \to 1^-} \frac{F(y)}{f(y)} = \lim_{y \to 1^-} \frac{\frac{1-(1-y)^{p-1}}{(p-1)(1-y)^{p-1}}}{\frac{1}{(1-y)^p}} = \lim_{y \to 1^-} \frac{(1-y)-(1-y)^p}{p-1} = 0 \quad \text{for } p > 1, \quad (3.18)$$

and

$$\lim_{y \to 1^-} \frac{F(y)}{f(y)} = \lim_{y \to 1^-} \frac{-\log(1-y)}{\frac{1}{1-y}} = -\lim_{y \to 1^-} (1-y)\log(1-y) = 0 \quad \text{for } p = 1. \quad (3.19)$$

We change variables in (3.13) by writing $y = F^{-1}(\frac{t}{\lambda})$, then for fixed $p \geq 1$,

$$\lim_{\lambda \to 0^+} g_p(\lambda) = \int_0^1 \frac{1}{\sqrt{1-t^2}} \lim_{\lambda \to 0^+} \frac{\frac{t}{\lambda}}{f(F^{-1}(\frac{t}{\lambda}))} dt = \int_0^1 \frac{1}{\sqrt{1-t^2}} \lim_{y \to 1^-} \frac{F(y)}{f(y)} dt = 0.$$

So part (ii) follows.

(iii) For fixed $p \in (0,1)$, by (3.14),

$$\begin{aligned}
\lim_{\lambda \to 0^+} g_p(\lambda) &= \lim_{\lambda \to 0^+} \int_0^1 \frac{(1-p) - \lambda(1-u)^{1-p}}{\sqrt{2\lambda(1-p)(1-u)^{1-p} - \lambda^2(1-u)^{2-2p}}} du \\
&= \int_0^1 \lim_{\lambda \to 0^+} \frac{(1-p) - \lambda(1-u)^{1-p}}{\sqrt{2\lambda(1-p)(1-u)^{1-p} - \lambda^2(1-u)^{2-2p}}} du \\
&= \infty.
\end{aligned}$$

In addition, the proof of $\lim_{\lambda \to \infty} g_p(\lambda) = 0$ is similar to that of part (ii), and hence we omit it. So part (iii) follows.

The proof of Lemma 3.4 is complete. ■

In the following Lemmas 3.5–3.7, for $p \geq 1$, we mainly prove that $g_p(\lambda)$ has exactly one critical point, a local maximum, on $(0, \infty)$.

Lemma 3.5. *Consider $g_p(\lambda)$ with fixed $p \geq 2$. Then*

(i) $g_p'(\lambda) < 0$ on $[1, \infty)$.

(ii) $g_p''(\lambda) < 0$ on $(0, 1)$.

(iii) $g_p(\lambda)$ has exactly one critical point, a local maximum, at some $\bar{\lambda}$ ($\in (0,1)$) on $(0, \infty)$.

Proof of Lemma 3.5. Let $p \geq 2$ be fixed.

(i) We change variables in (3.13) by writing $y = F^{-1}(\frac{t}{\lambda})$, then

$$g_p(\lambda) = \int_0^1 \frac{1}{\sqrt{1-t^2}} \frac{\frac{t}{\lambda}}{f(F^{-1}(\frac{t}{\lambda}))} dt = \int_0^1 \frac{1}{\sqrt{1-t^2}} \frac{F(y)}{f(y)} dt, \ \lambda > 0.$$

Since $F(u) = \frac{-1+(1-u)^{1-p}}{p-1}$ is a differential, strictly increasing function, by the Inverse Function Theorem, we compute that

$$\begin{aligned} g_p'(\lambda) &= \int_0^1 \frac{1}{\sqrt{1-t^2}} \frac{f^2(y) - f'(y)F(y)}{f^2(y)} \frac{1}{f(F^{-1}(\frac{t}{\lambda}))} \frac{-t}{\lambda^2} dt \\ &= \frac{1}{\lambda^2} \int_0^1 \frac{t}{\sqrt{1-t^2}} \frac{f'(y)F(y) - f^2(y)}{f^3(y)} dt \\ &= \frac{1}{(p-1)\,\lambda^2} \int_0^1 \frac{t}{\sqrt{1-t^2}} \left[(1-y)^p - p(1-y)^{2p-1}\right] dt. \end{aligned} \tag{3.20}$$

Since $F^{-1}(u) = 1 - [1-(1-p)u]^{\frac{1}{1-p}}$ and $y = F^{-1}(\frac{t}{\lambda})$,

$$g_p'(\lambda) = \frac{1}{\lambda^3} \int_0^1 \frac{t\,(t-\lambda)}{\sqrt{1-t^2}} \left[1 + (p-1)\frac{t}{\lambda}\right]^{-\frac{2p-1}{p-1}} dt < 0 \tag{3.21}$$

for all $\lambda \geq 1$. So part (i) follows.

(ii) Since $F(u) = \frac{-1+(1-u)^{1-p}}{p-1}$ is a differential, strictly increasing function, by (3.20) and the Inverse Function Theorem, we compute that

$$\begin{aligned} &g_p''(\lambda) \\ &= \frac{1}{\lambda^3} \int_0^1 \frac{t}{\sqrt{1-t^2}} \frac{3f'^2(y)F^2(y) - f(y)f''(y)F^2(y) - 4f^2(y)f'(y)F(y) + 2f^4(y)}{f^5(y)} dt \\ &= \frac{1}{(p-1)^2\lambda^3} \int_0^1 \frac{t}{\sqrt{1-t^2}} \left\{ \frac{2p^2-p}{[1+(p-1)t/\lambda]^{\frac{3p-2}{p-1}}} - \frac{2p}{[1+(p-1)t/\lambda]^{\frac{2p-1}{p-1}}} + \frac{2-p}{[1+(p-1)t/\lambda]^{\frac{p}{p-1}}} \right\} dt. \qquad (3.22) \\ &= \frac{1}{(p-1)^4\lambda} \int_1^{1+(p-1)\frac{1}{\lambda}} \frac{w-1}{\sqrt{1-\left(\frac{\lambda}{p-1}\right)^2 (w-1)^2}} w^{\frac{3p-2}{1-p}} \left[(2-p)w^2 - 2pw + (2p^2-p)\right] dw, \end{aligned} \tag{3.23}$$

where $w \equiv 1 + (p-1)\frac{t}{\lambda}$. Then:

(1) For $p > 2$, we define $\eta_0 \equiv \frac{-(p-1)\sqrt{2p}-p}{p-2}$ and $\eta_1 \equiv \frac{(p-1)\sqrt{2p}-p}{p-2}$ be the two zeros of the quadratic polynomial $(2-p)w^2 - 2pw + (2p^2-p)$ such that

$$(2-p)w^2 - 2pw + (2p^2-p) \begin{cases} > 0 \text{ on } (\eta_0, \eta_1), \\ < 0 \text{ on } (-\infty, \eta_0) \cup (\eta_1, \infty). \end{cases}$$

Observe that $\eta_0 = \frac{-(p-1)\sqrt{2p}-p}{p-2} < 0 < 1 < \frac{(p-1)\sqrt{2p}-p}{p-2} = \eta_1 < p < 1 + (p-1)\frac{1}{\lambda}$ for $p > 2$ and $0 < \lambda < 1$.

(2) For $p=2$, we define $\eta_1 \equiv 3/2$ such that

$$(2-p)w^2 - 2pw + (2p^2-p) = -4w + 6 \begin{cases} > 0 \text{ on } (1, \eta_1), \\ < 0 \text{ on } (\eta_1, \infty). \end{cases}$$

Then for $p \geq 2$ and $0 < \lambda < 1$, by (3.23), we compute that

$$\begin{aligned}
& (p-1)^4 \lambda g_p''(\lambda) \\
= & \int_1^{\eta_1} \frac{w-1}{\sqrt{1-\left(\frac{\lambda}{p-1}\right)^2 (w-1)^2}} w^{\frac{3p-2}{1-p}} \left[(2-p)w^2 - 2pw + (2p^2-p)\right] dw \\
& + \int_{\eta_1}^{1+(p-1)\frac{1}{\lambda}} \frac{w-1}{\sqrt{1-\left(\frac{\lambda}{p-1}\right)^2 (w-1)^2}} w^{\frac{3p-2}{1-p}} \left[(2-p)w^2 - 2pw + (2p^2-p)\right] dw \\
< & \int_1^{\eta_1} \frac{w-1}{\sqrt{1-\left(\frac{\lambda}{p-1}\right)^2 (\eta_1-1)^2}} w^{\frac{3p-2}{1-p}} \left[(2-p)w^2 - 2pw + (2p^2-p)\right] dw \\
& + \int_{\eta_1}^{1+(p-1)\frac{1}{\lambda}} \frac{w-1}{\sqrt{1-\left(\frac{\lambda}{p-1}\right)^2 (\eta_1-1)^2}} w^{\frac{3p-2}{1-p}} \left[(2-p)w^2 - 2pw + (2p^2-p)\right] dw \\
= & \frac{1}{\sqrt{1-\left(\frac{\lambda}{p-1}\right)^2 (\eta_1-1)^2}} \int_1^{1+(p-1)\frac{1}{\lambda}} (w-1)\, w^{\frac{3p-2}{1-p}} \left[(2-p)w^2 - 2pw + (2p^2-p)\right] dw \\
= & \frac{1}{\sqrt{1-\left(\frac{\lambda}{p-1}\right)^2 (\eta_1-1)^2}} \frac{(p-1)^4(\lambda-1)}{\lambda^3} \left(\frac{\lambda+p-1}{\lambda}\right)^{\frac{2p-1}{p-1}} \\
< & \ 0.
\end{aligned}$$

By the above analyses, we obtain that $g_p''(\lambda) < 0$ for $p \geq 2$ and $0 < \lambda < 1$. So part (ii) follows.

(iii) Part (iii) follows from parts (i)–(ii) and Lemma 3.4(i)–(ii).

The proof of Lemma 3.5 is complete. ■

Lemma 3.6. *Consider $g_p(\lambda)$ with fixed $p \in (1,2)$. Then*

(i) $g_p'(\lambda) < 0$ on $[1, \infty)$.

(ii) $g_p'(\lambda) > 0$ on $(0, \frac{4}{3\pi}]$.

(iii) $g_p''(\lambda) < 0$ whenever $g_p'(\lambda) = 0$ for $\lambda \in (\frac{4}{3\pi}, 1)$.

(iv) $g_p(\lambda)$ has exactly one critical point, a local maximum, at some $\bar{\lambda}$ $(\in (\frac{4}{3\pi}, 1))$ on $(0, \infty)$.

Proof of Lemma 3.6. Let $p \in (1,2)$ be fixed.

(i) The proof of part (i) is the same as that of Lemma 3.5(i), and hence we omit it.

(ii) For any given $\lambda \in (0,1)$, (3.21) implies that

$$\begin{aligned} g_p'(\lambda) &= \frac{1}{\lambda^3}\int_0^\lambda \frac{t(t-\lambda)}{\sqrt{1-t^2}}\frac{1}{\left[1+(p-1)\frac{t}{\lambda}\right]^{\frac{2p-1}{p-1}}}dt + \frac{1}{\lambda^3}\int_\lambda^1 \frac{t(t-\lambda)}{\sqrt{1-t^2}}\frac{1}{\left[1+(p-1)\frac{t}{\lambda}\right]^{\frac{2p-1}{p-1}}}dt \\ &> \frac{1}{\lambda^3}\int_0^\lambda \frac{t(t-\lambda)}{\sqrt{1-\lambda^2}}\frac{1}{\left[1+(p-1)\frac{t}{\lambda}\right]^{\frac{2p-1}{p-1}}}dt + \frac{1}{\lambda^3}\int_\lambda^1 \frac{t(t-\lambda)}{\sqrt{1-\lambda^2}}\frac{1}{\left[1+(p-1)\frac{t}{\lambda}\right]^{\frac{2p-1}{p-1}}}dt \\ &= \frac{1}{\lambda^3\sqrt{1-\lambda^2}}\int_0^1 \frac{t(t-\lambda)}{\left[1+(p-1)\frac{t}{\lambda}\right]^{\frac{2p-1}{p-1}}}dt \\ &= \frac{\Psi_p(\lambda)}{\left(\frac{\lambda+p-1}{\lambda}\right)^{\frac{p}{p-1}}(2-p)\,\lambda^2\sqrt{1-\lambda^2}}, \end{aligned} \tag{3.24}$$

where $\Psi_p(\lambda) \equiv \lambda^{\frac{p-2}{p-1}}(\lambda+p-1)^{\frac{p}{p-1}} - \lambda(\lambda+p) - 1$. We compute that

$$\Psi_p'(\lambda) = (p+2\lambda-2)\left(\frac{\lambda+p-1}{\lambda}\right)^{\frac{1}{p-1}} - (p+2\lambda)$$

and

$$\lambda(\lambda+p-1)\,\Psi_p'(\lambda) - (p+2\lambda-2)\,\Psi_p(\lambda) = -(1-\lambda)(2-p) < 0$$

since $1 < p < 2$ and $0 < \lambda < 1$. This implies that $\Psi_p(\lambda)$ has at most one zero in $(0,1)$ for all $p \in (1,2)$. Moreover, since $\lim_{\lambda\to 0^+}\Psi_p(\lambda) = \infty$ and

$$\Psi_p\left(\frac{4}{3\pi}\right) = \left(\frac{4}{3\pi}\right)^{\frac{p-2}{p-1}}\left(\frac{4}{3\pi}+p-1\right)^{\frac{p}{p-1}} - \frac{4}{3\pi}\left(\frac{4}{3\pi}+p\right) - 1 \geq 0 \quad \text{for all } p \in (1,2),$$

we find $\Psi_p(\lambda) \geq 0$ for all $p \in (1,2)$ and $\lambda \in (0, \frac{4}{3\pi}]$. By (3.24), $g_p'(\lambda) > 0$ on $(0, \frac{4}{3\pi}]$ for $p \in (1,2)$. So part (ii) follows.

(iii) By (3.21) and (3.22), we find

$$\lambda^3 g_p''(\lambda) + 2\lambda^2 g_p'(\lambda) = \int_0^1 \frac{t}{\sqrt{1-t^2}}\frac{pt(t-2\lambda)}{\lambda^2\left[1-(1-p)t/\lambda\right]^{\frac{3p-2}{p-1}}}dt. \tag{3.25}$$

(1) If $\lambda \in [\frac{1}{2}, 1)$, $\lambda^3 g_p''(\lambda) + 2\lambda^2 g_p'(\lambda) < 0$ by (3.25). Hence, we find that $g_p''(\lambda) < 0$ whenever $g_p'(\lambda) = 0$ for $\lambda \in [\frac{1}{2}, 1)$.

(2) If $\lambda \in (\frac{4}{3\pi}, \frac{1}{2})$,

$$\begin{aligned} &\lambda^3 g_p''(\lambda) + 2\lambda^2 g_p'(\lambda) \\ &= \int_0^{2\lambda} \frac{t}{\sqrt{1-t^2}}\frac{pt(t-2\lambda)}{\lambda^2\left[1-(1-p)t/\lambda\right]^{\frac{3p-2}{p-1}}}dt + \int_{2\lambda}^1 \frac{t}{\sqrt{1-t^2}}\frac{pt(t-2\lambda)}{\lambda^2\left[1-(1-p)t/\lambda\right]^{\frac{3p-2}{p-1}}}dt \\ &< \int_0^{2\lambda} \frac{t}{\sqrt{1-t^2}}\frac{pt(t-2\lambda)}{\lambda^2\left[1-2(1-p)\right]^{\frac{3p-2}{p-1}}}dt + \int_{2\lambda}^1 \frac{t}{\sqrt{1-t^2}}\frac{pt(t-2\lambda)}{\lambda^2\left[1-2(1-p)\right]^{\frac{3p-2}{p-1}}}dt \\ &= \frac{p}{\lambda^2(2p-1)^{\frac{3p-2}{p-1}}}\int_0^1 \frac{t^2(t-2\lambda)}{\sqrt{1-t^2}}dt \\ &= \frac{p(4-3\pi\lambda)}{6\lambda^2(2p-1)^{\frac{3p-2}{p-1}}} \\ &\leq 0. \end{aligned} \tag{3.26}$$

Hence, we find that $g_p''(\lambda) < 0$ whenever $g_p'(\lambda) = 0$ by (3.26) for $\lambda \in (\frac{4}{3\pi}, \frac{1}{2})$.

By the above analyses, we obtain that $g_p''(\lambda) < 0$ whenever $g_p'(\lambda) = 0$ for $p \in (1,2)$ and $\lambda \in (\frac{4}{3\pi}, 1)$. So part (iii) follows.

(iv) Part (iv) follows from parts (i)–(iii) and Lemma 3.4(i)–(ii).

The proof of Lemma 3.6 is complete. ■

Lemma 3.7. *Consider* $g_p(\lambda)$ *with* $p = 1$. *Then*

(i) $g_p'(\lambda) < 0$ on $[1, \infty)$.

(ii) $g_p'(\lambda) > 0$ on $(0, \frac{1}{2}]$.

(iii) $g_p''(\lambda) < 0$ whenever $g_p'(\lambda) = 0$ for $\lambda \in (\frac{1}{2}, 1)$.

(iv) $g_p(\lambda)$ has exactly one critical point, a local maximum, at some $\bar{\lambda}$ $(\in (\frac{1}{2}, 1))$ on $(0, \infty)$.

Proof of Lemma 3.7. (i) Consider $f(u) = (1-u)^{-1}$. Then $f'(u) = (1-u)^{-2}$, $F(u) = -\log(1-u)$, and $F^{-1}(u) = 1 - e^{-u}$. Hence, by (3.11), we find that

$$g_p(\lambda) = \frac{1}{\lambda} \int_0^1 \frac{t}{e^{t/\lambda}\sqrt{1-t^2}} dt$$

and hence

$$g_p'(\lambda) = \frac{1}{\lambda^3} \int_0^1 \frac{t(t-\lambda)}{e^{t/\lambda}\sqrt{1-t^2}} dt < 0 \quad \text{for } \lambda \geq 1.$$

So part (i) follows.

(ii) For $\lambda \in (0, \frac{1}{2}]$, we have

$$\begin{aligned}
\lambda^3 g_p'(\lambda) &= \int_0^\lambda \frac{t(t-\lambda)}{e^{t/\lambda}\sqrt{1-t^2}} dt + \int_\lambda^1 \frac{t(t-\lambda)}{e^{t/\lambda}\sqrt{1-t^2}} dt \\
&> \int_0^\lambda \frac{t(t-\lambda)}{e^{t/\lambda}\sqrt{1-\lambda^2}} dt + \int_\lambda^1 \frac{t(t-\lambda)}{e^{t/\lambda}\sqrt{1-\lambda^2}} dt \\
&= \frac{1}{\sqrt{1-\lambda^2}} \int_0^1 \frac{t(t-\lambda)}{e^{t/\lambda}} dt \\
&= \frac{\lambda}{e^{\frac{1}{\lambda}}\sqrt{1-\lambda^2}} \left[\lambda^2 e^{\frac{1}{\lambda}} - (1 + \lambda + \lambda^2)\right].
\end{aligned}$$

Since

$$\lambda^2 e^{\frac{1}{\lambda}} - (1 + \lambda + \lambda^2) > 0 \quad \text{for } \lambda \in (0, \frac{1}{2}].$$

We obtain that $g_p'(\lambda) > 0$ for $p = 1$ and $\lambda \in (0, \frac{1}{2}]$. So part (ii) follows.

(iii) By parts (i)–(ii), $g_p(\lambda)$ has critical points in $(\frac{1}{2}, 1)$. If $\lambda \in (\frac{1}{2}, 1)$,

$$g_p''(\lambda) + \frac{2}{\lambda} g_p'(\lambda) = \frac{1}{\lambda^5} \int_0^1 \frac{t^2(t-2\lambda)}{e^{t/\lambda}\sqrt{1-t^2}} dt < 0.$$

Hence, $g_p''(\lambda) < 0$ whenever $g_p'(\lambda) = 0$ for $p = 1$ and $\lambda \in (\frac{1}{2}, 1)$. So part (iii) follows.

(iv) Part (iv) follows from parts (i)–(iii) and Lemma 3.4(i)–(ii).

The proof of Lemma 3.7 is complete. ■

In the final lemma of this section, for $0 < p < 1$, we mainly prove that $g_p(\lambda)$ has exactly one local minimum and exactly one local maximum on $(0, \infty)$.

Lemma 3.8. *Consider $g_p(\lambda)$ with fixed $p \in (0,1)$. Then*

(i) $g_p'(\lambda) < 0$ on $[1, \infty)$.

(ii) $g_p'(\lambda) < 0$ on $(0, 1-p)$ and $g_p'((1-p)^-) < 0$.

(iii) $g_p'((1-p)^+) > 0$. In particular, for $1/2 < p < 1$, $g_p'(\lambda) > 0$ on $(1-p, \frac{1}{2})$.

(iv) $g_p''(\lambda) < 0$ whenever $g_p'(\lambda) = 0$ for $\lambda \in (1-p, 1)$.

(v) $g_p(\lambda)$ has exactly two critical points, one local minimum at $\underline{\lambda} = 1-p$ and one local maximum at $\bar{\lambda} \in (\underline{\lambda}, 1)$, on $(0, \infty)$.

Proof of Lemma 3.8. Let $p \in (0,1)$ be fixed.

(i) For $\lambda \geq 1-p$ and similar argument as (3.21), we find that

$$g_p'(\lambda) = \frac{1}{\lambda^3} \int_0^1 \frac{t\,(t-\lambda)}{\sqrt{1-t^2}} \frac{1}{\left[1-(1-p)\frac{t}{\lambda}\right]^{\frac{2p-1}{p-1}}} dt < 0 \quad \text{for all } \lambda \geq 1. \tag{3.27}$$

So part (i) follows.

(ii) Recall that

$$g_p(\lambda) = \int_0^1 \frac{(1-p) - (1-s)^{1-p}\lambda}{\sqrt{2\,(1-p)\,(1-s)^{1-p}\lambda - (1-s)^{2-2p}\lambda^2}} ds \quad \text{if } \lambda < 1-p.$$

Then we compute that

$$\begin{aligned}
g_p'(\lambda) &= \int_0^1 \frac{-2\,(1-p)^2\,(1-s)^{1-p} + 2\,(1-p)\,(1-s)^{3-3p}\lambda - (1-s)^{3-3p}\lambda^2}{\left[2\,(1-p)\,(1-s)^{1-p}\lambda - (1-s)^{2-2p}\lambda^2\right]^{3/2}} ds \\
&= \int_\lambda^0 \frac{-2\,(1-p)^2\,y/\lambda + 2\,(1-p)\,y^3/\lambda^2 - y^3/\lambda}{\left[2\,(1-p)\,y - y^2\right]^{3/2}} \frac{(y/\lambda)^{\frac{p}{1-p}}}{-(1-p)\lambda} dy \;\; (\text{set } y = (1-s)^{1-p}\lambda) \\
&= \int_0^\lambda \left[(2-\lambda-2p)\,y^2 - 2\lambda\,(p-1)^2\right] \frac{y^{\frac{3p-1}{2(1-p)}}}{(1-p)\left[2\,(1-p) - y\right]^{3/2} \lambda^{\frac{3-2p}{1-p}}} dy \\
&< 0 \quad \text{for } \lambda \in (0, 1-p),
\end{aligned}$$

since $y = (1-s)^{1-p}\lambda < \lambda$ and

$$\begin{aligned}
(2-\lambda-2p)\,y^2 - 2\lambda\,(p-1)^2 &< (2-\lambda-2p)\,\lambda^2 - 2\lambda\,(p-1)^2 \\
&= \left\{-\left[\lambda-(1-p)\right]^2 - (p-1)^2\right\}\lambda \\
&< 0 \quad \text{for } \lambda \in (0, 1-p).
\end{aligned}$$

So part (ii) follows.

(iii) By (3.27), we compute that

$$
\begin{aligned}
&(1-p)^3\, g_p'((1-p)^+)\\
= \;& \int_0^1 \frac{t\,(t-1+p)}{\sqrt{1-t^2}}\,(1-t)^{\frac{2p-1}{1-p}}\,dt\\
= \;& \int_0^{1-p} \frac{t\,(t-1+p)}{\sqrt{1-t^2}}\,(1-t)^{\frac{2p-1}{1-p}}\,dt + \int_{1-p}^1 \frac{t\,(t-1+p)}{\sqrt{1-t^2}}\,(1-t)^{\frac{2p-1}{1-p}}\,dt\\
> \;& \int_0^{1-p} \frac{t\,(t-1+p)}{\sqrt{1-(1-p)^2}}\,(1-t)^{\frac{2p-1}{1-p}}\,dt + \int_{1-p}^1 \frac{t\,(t-1+p)}{\sqrt{1-(1-p)^2}}\,(1-t)^{\frac{2p-1}{1-p}}\,dt\\
= \;& \frac{1}{\sqrt{1-(1-p)^2}} \int_0^1 t\,(t-1+p)\,(1-t)^{\frac{2p-1}{1-p}}\,dt\\
= \;& \frac{2\,(1-p)^3}{p\,(2-p)} + \frac{(p-1)\Gamma(\frac{p}{1-p})}{\Gamma(\frac{2-p}{1-p})} \quad (\Gamma(x) \equiv \int_0^1 t^{x-1}e^{-t}dt \text{ is the gamma function})\\
> \;& 0 \;\text{ for } 0<p<1.
\end{aligned}
$$

On the other hand, for $\lambda \in (0,1)$, (3.27) implies that

$$
\begin{aligned}
g_p'(\lambda) \;=\;& \frac{1}{\lambda^3}\int_0^\lambda \frac{t\,(t-\lambda)}{\sqrt{1-t^2}} \frac{1}{\left[1-(1-p)\frac{t}{\lambda}\right]^{\frac{2p-1}{p-1}}}dt + \frac{1}{\lambda^3}\int_\lambda^1 \frac{t\,(t-\lambda)}{\sqrt{1-t^2}} \frac{1}{\left[1-(1-p)\frac{t}{\lambda}\right]^{\frac{2p-1}{p-1}}}dt\\
>\;& \frac{1}{\lambda^3}\int_0^\lambda \frac{t\,(t-\lambda)}{\sqrt{1-\lambda^2}} \frac{1}{\left[1-(1-p)\frac{t}{\lambda}\right]^{\frac{2p-1}{p-1}}}dt + \frac{1}{\lambda^3}\int_\lambda^1 \frac{t\,(t-\lambda)}{\sqrt{1-\lambda^2}} \frac{1}{\left[1-(1-p)\frac{t}{\lambda}\right]^{\frac{2p-1}{p-1}}}dt\\
=\;& \frac{1}{\lambda^3\sqrt{1-\lambda^2}}\int_0^1 \frac{t\,(t-\lambda)}{\left[1-(1-p)\frac{t}{\lambda}\right]^{\frac{2p-1}{p-1}}}dt\\
=\;& \frac{\lambda^{\frac{p-2}{p-1}}\,(\lambda+p-1)^{\frac{p}{p-1}} - \lambda(\lambda+p)-1}{\left(\frac{\lambda+p-1}{\lambda}\right)^{\frac{p}{p-1}}\,(2-p)\,\lambda^2\sqrt{1-\lambda^2}}\\
=\;& \frac{\Psi_p(\lambda)}{\left(\frac{\lambda+p-1}{\lambda}\right)^{\frac{p}{p-1}}\,(2-p)\,\lambda^2\sqrt{1-\lambda^2}},
\end{aligned}
$$

where $\Psi_p(\lambda) \equiv \lambda^{\frac{p-2}{p-1}}\,(\lambda+p-1)^{\frac{p}{p-1}} - \lambda(\lambda+p)-1$. For $\lambda > 1-p$, we compute that

$$
\Psi_p'(\lambda) = (p+2\lambda-2)\left(\frac{\lambda+p-1}{\lambda}\right)^{\frac{1}{p-1}} - (p+2\lambda)
$$

and

$$
\begin{aligned}
\lambda\,(\lambda+p-1)\,\Psi_p'(\lambda) - (p+2\lambda-2)\,\Psi_p(\lambda) \;&=\; -(1-\lambda)\,(2-p)\\
&<\; 0 \;\text{ for all } p\in(0,1) \text{ and } \lambda \in (0,1).
\end{aligned}
$$

So $\Psi_p(\lambda)$ has at most one zero in $(0,1)$ for all $p \in (0,1)$. Moreover, since $\lim_{\lambda\to 1-p}\Psi_p(\lambda) = \infty$ and

$$
\Psi_p(\tfrac{1}{2}) = \frac{1}{4}\,(2p-1)^{\frac{p}{p-1}} - \frac{p}{2} - \frac{5}{4} > 0 \;\text{ if } 1-p < \frac{1}{2},
$$

we find that $\Psi_p(\lambda) \ge 0$ for all $p \in (\frac{1}{2},1)$ and $\lambda \in (1-p,\frac{1}{2})$. Hence, $g_p'(\lambda) > 0$ for $p \in (\frac{1}{2},1)$ and $\lambda \in (1-p,\frac{1}{2})$. So part (iii) follows.

(iv) By parts (ii)–(iii), we have that:
(a) if $p \in (0, \frac{1}{2}]$, then $g_p(\lambda)$ has critical points in $(1-p, 1)$,
(b) if $p \in (\frac{1}{2}, 1)$, then $g_p(\lambda)$ has critical points in $(\frac{1}{2}, 1) \subset (1-p, 1)$.
Moreover, we compute that

$$\begin{aligned} \lambda^3 g_p''(\lambda) + 2\lambda^2 g_p'(\lambda) &= \int_0^1 \frac{t}{\sqrt{1-t^2}} \frac{pt\,(t-2\lambda)}{\lambda^2\,[1-(1-p)t/\lambda]^{\frac{3p-2}{p-1}}} dt \\ &\begin{cases} < 0 & \text{for all } \lambda \in (1-p, 1) \subset (\frac{1}{2}, 1) \text{ if } p \in (0, \frac{1}{2}]. \\ < 0 & \text{for all } \lambda \in (\frac{1}{2}, 1) \text{ if } p \in (\frac{1}{2}, 1). \end{cases} \end{aligned}$$

By above analyses, $g_p''(\lambda) < 0$ whenever $g_p'(\lambda) = 0$ for $\lambda \in (1-p, 1)$. So part (iv) follows.
(v) Part (v) follows from parts (i)–(iv) and Lemma 3.4(i) and (iii).
The proof of Lemma 3.8 is complete. ∎

4. Proofs of main results

By (3.5), the positive solutions $u_\lambda \in C^2(-L, L) \cap C[-L, L]$ for (1.1) correspond to

$$\|u_\lambda\|_\infty = r \quad \text{and} \quad T_\lambda(r) = L.$$

Thus, we study the shape of the time map $T_\lambda(r)$ on its domain to find the exact number of positive solutions of (1.1) for any fixed $\lambda > 0$.
Proof of Theorem 2.1. Let $p \geq 1$ be fixed. By Lemmas 3.1–3.7, we have the following properties:

(1) $\lim_{r \to 0^+} T_\lambda(r) = 0$ for all $\lambda > 0$.

(2) $\lim_{r \to 0^+} T_\lambda'(r) = \infty$ for all $\lambda > 0$.

(3) $T_\lambda'(F^{-1}(1/\lambda)) < 0$ for all $\lambda > 0$.

(4) $T_\lambda(r)$ has exactly one critical point, a local maximum, on $(0, F^{-1}(1/\lambda))$.

(5) For fixed $r \in (0, 1)$, $T_\lambda(r)$ is a continuous, strictly decreasing function of $\lambda > 0$, and $\lim_{\lambda \to 0^+} T_\lambda(r) = \infty$.

(6) $h_p(\lambda)$ is a continuous, strictly decreasing function of λ, $\lim_{\lambda \to 0^+} h_p(\lambda) = \infty$ and $\lim_{\lambda \to \infty} h_p(\lambda) = 0$.

(7) $g_p(\lambda)$ has exactly one critical point, a local maximum, at $\bar{\lambda}$ $(\in (0, 1))$ on $(0, \infty)$ and $\lim_{\lambda \to 0^+} g_p(\lambda) = \lim_{\lambda \to \infty} g_p(\lambda) = 0$.

See, e.g., Fig. 3 for numerical computations of $T_\lambda(r)$ with $p = 1$ with varying $\lambda > 0$.

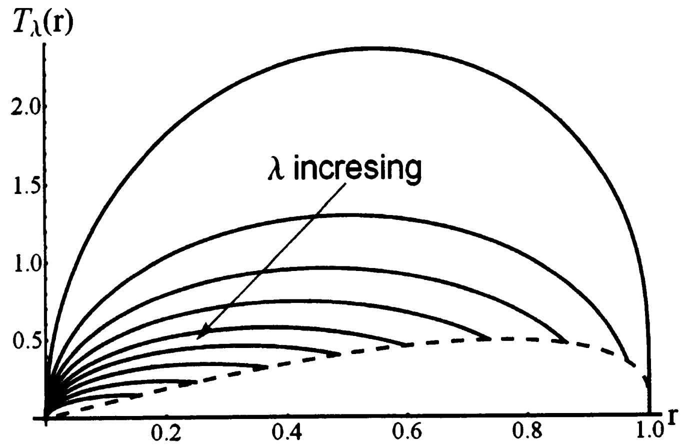

Fig. 3. Numerical computations of $T_\lambda(r)$ with $p = 1$.
$\lambda = 0.1, 0.3, 0.5, 0.75, 1.1, 1.5, 2.2, 3.5, 6$.

Let $L^* = T_{\bar{\lambda}}(F^{-1}(1/\bar{\lambda}))$ for $\lambda = \bar{\lambda}$. We obtain that:

(i) For $L > L^*$, there exists $\lambda^* > 0$ such that $h_p(\lambda^*) = L$. Thus part (i) follows immediately by properties (1)–(7) and (3.5).

(ii) For $L = L^*$, there exist positive numbers $\bar{\lambda} < \lambda^*$ such that $g_p(\bar{\lambda}) = L^*$ and $h_p(\lambda^*) = L^*$. Thus part (ii) follows immediately by properties (1)–(7) and (3.5).

(iii) For $0 < L < L^*$, there exist positive numbers $\hat{\lambda} < \check{\lambda} < \lambda^*$ such that $g_p(\hat{\lambda}) = g_p(\check{\lambda}) = L$ and $h_p(\lambda^*) = L$. Thus part (iii) follows immediately by properties (1)–(7) and (3.5).

The proof of Theorem 2.1 is now complete. ■

Proof of Theorem 2.2. Let $p \in (0,1)$ be fixed. By Lemmas 3.1–3.4 and 3.8, we have the following properties:

(1) $\lim_{r\to 0^+} T_\lambda(r) = 0$ for all $\lambda > 0$.

(2) $\lim_{r\to 0^+} T'_\lambda(r) = \infty$ for all $\lambda > 0$.

(3) $T'_\lambda(F^{-1}(1/\lambda)) < 0$ for $\lambda > 1 - p$.

(4) $\lim_{r\to 1^-} T'_\lambda(r) = -\infty$ for $0 < \lambda \le 1 - p$.

(5) $T_\lambda(r)$ has exactly one critical point in $(0, F^{-1}(\frac{1}{\lambda}))$ for $\lambda > 1 - p$.

(6) $T_\lambda(r)$ has exactly one critical point in $(0,1)$ for $0 < \lambda \le 1 - p$.

(7) For fixed $r \in (0,1)$, $T_\lambda(r)$ is a continuous, strictly decreasing function of $\lambda > 0$, and $\lim_{\lambda\to 0^+} T_\lambda(r) = \infty$.

(8) $h_p(\lambda)$ is a continuous, strictly decreasing function of λ, $\lim_{\lambda\to 0^+} h_p(\lambda) = \infty$ and $\lim_{\lambda\to\infty} h_p(\lambda) = 0$.

(9) $g_p(\lambda)$ has exactly two critical points, one local minimum at $\underline{\lambda} = 1 - p$ and one local maximum at $\bar{\lambda} \in (\underline{\lambda}, 1)$, on $(0,\infty)$, $\lim_{\lambda\to 0^+} g_p(\lambda) = \infty$ and $\lim_{\lambda\to\infty} g_p(\lambda) = 0$.

See, e.g., Fig. 4 for numerical computations of $T_\lambda(r)$ with $p = 1/2$ with varying $\lambda > 0$.

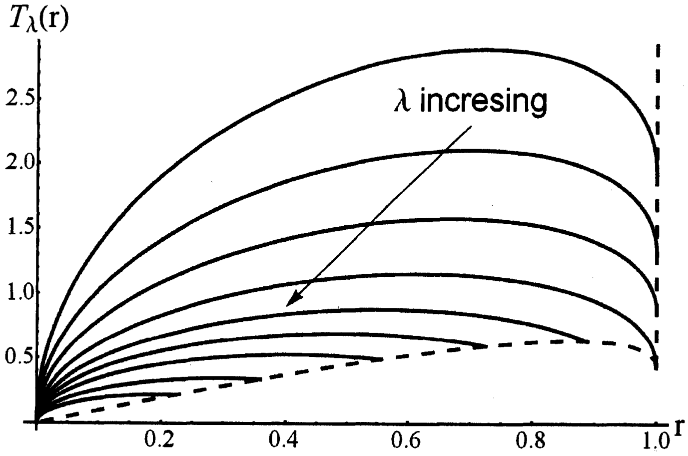

Fig. 4. Numerical computations of $T_\lambda(r)$ with $p = 1/2$.
$\lambda = 0.1, 0.18, 0.3, 0.5, 0.75, 1.05, 1.5, 2.5, 4.2$.

Let $L^* = T_{\bar{\lambda}}(F^{-1}(1/\bar{\lambda}))$ and $L_* = \lim_{r\to 1^-} T_{\underline{\lambda}}(r) = T_{1-p}(1)$. We obtain that:

(i) For $L > L^*$, there exist positive numbers $\lambda_* < \lambda^*$ such that $g_p(\lambda_*) = L$ and $h_p(\lambda^*) = L$. Thus part (i) follows immediately by properties (1)–(9) and (3.5).

(ii) For $L = L^*$, there exist positive numbers $\lambda_* < \bar{\lambda} < \lambda^*$ such that $g_p(\lambda_*) = g_p(\bar{\lambda}) = L^*$ and $h_p(\lambda^*) = L^*$. Thus part (ii) follows immediately by properties (1)–(9) and (3.5).

(iii) For $L_* < L < L^*$, there exist positive numbers $\lambda_* < \hat{\lambda} < \check{\lambda} < \lambda^*$ such that $g_p(\lambda_*) = g_p(\hat{\lambda}) = g_p(\check{\lambda}) = L$ and $h_p(\lambda^*) = L$. Thus part (iii) follows immediately by properties (1)–(9) and (3.5).

(iv) For $L = L_*$, there exist $0 < \underline{\lambda} = 1 - p < \check{\lambda} < \lambda^*$ such that $g_p(\underline{\lambda}) = g_p(\check{\lambda}) = L_*$ and $h_p(\lambda^*) = L_*$. For $0 < L < L_*$, there exist $0 < \check{\lambda} < \lambda^*$ such that $g_p(\check{\lambda}) = L$ and $h_p(\lambda^*) = L$. Thus part (iv) follows immediately by properties (1)–(9) and (3.5).

The proof of Theorem 2.2 is now complete. ■

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