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Blow-up and pointwise comparison principles for the generalized Riccati differential equation and application to oscillation of some nonlinear ode's

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Abstract We give a short presentation on the blow-up and pointwise comparison principles for the generalized Riccati differential equation recently applied to a study of oscillation of a class of general second-order differential equations with damping term. A part of this review article was presented on the RIMS workshop: Global qualitative theory of ordinary differential equations and its applications, Kyoto 2012. All details have been published in author's paper [4].

1 Introduction

We consider the following class of the second-order differential equation with damping:

$$\big(r(t)\Phi(x,x')\big)' + p(t)\Psi(x,x') + q(t)f(x) = 0, \quad t \geq t_0, \tag{1.1}$$

where we adopt the following:

- solutions: $x = x(t)$, $x \in C([t_0,\infty),\mathbb{R}) \cap C^2((t_0,\infty),\mathbb{R})$;
- nonlinear or linear functions:
$\Phi = \Phi(u,v)$, $\Psi = \Psi(u,v)$, $f = f(u)$, $\Phi,\Psi : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$, are smooth enough;
- coefficients: $r \in C^1([t_0,\infty),(0,\infty))$, $p,q \in C([t_0,\infty),\mathbb{R})$.

Definition 1.1 *A function $x(t)$ is oscillatory if there is a sequence $t_n \geq t_0$ such that $x(t_n) = 0$ and $t_n \to \infty$ as $n \to \infty$. A differential equation is oscillatory if all its solutions are oscillatory.*

The main assumptions on the differential operators $\big(r(t)\Phi(x,x')\big)'$ and the damped term $p(t)\Psi(x,x')$ are:

$$|u|^{\gamma-2}v\Phi(u,v) \geq g\big(|\Phi(u,v)|\big) \quad \text{and} \quad \Psi(u,v)u \geq 0, \tag{1.2}$$

or

$$\Phi(u,v)v \geq 0 \quad \text{and} \quad u|u|^{\gamma-2}\Psi(u,v) \geq g\big(|\Phi(u,v)|\big) \tag{1.3}$$

where $\gamma \geq 2$ and $g : \mathbb{R}_+ \to \mathbb{R}_+$:

$$\begin{cases} g(cs) \geq c^\gamma g_0(s) \quad \text{for all } c > 0 \text{ and } s > 0, \\ g_0 : \mathbb{R}_+ \to \mathbb{R}_+ \text{ is a locally Lipschitz function on } \mathbb{R}_+, \\ \exists M_0 > 0 \ \text{ such that } g_0(s) + M_0 \geq s^2, \ \forall s \in \mathbb{R}_+. \end{cases} \tag{1.4}$$

For instance, if $g(s) = g_0(s) = s^\gamma$, $\gamma \geq 2$, then (1.2) is:

$$|u|^{\gamma-2}v\Phi(u,v) \geq |\Phi(u,v)|^\gamma \quad \text{and} \quad \Psi(u,v)u \geq 0,$$

Next we give the main examples for the second-order differential operators $\big(r(t)\Phi(x,x')\big)'$ that satisfy required assumption (1.2) or (1.3).

Example 1.1 (linear second-order differential operators in x') We consider the equation:

$$\big(r(t)A(x)x'\big)' + p(t)B(x)\psi(x') + q(t)f(x) = 0, \quad t \geq t_0.$$

Condition (1.2) for $\gamma = 2$ is fulfilled provided functions $A(u)$, $B(u)$ and $\psi(v)$ satisfy:

$$0 \leq A(u) \leq 1, \;\; A(u) \neq 0, \;\; uB(u) \geq 0 \quad \text{and } \psi(v) \geq 0.$$

For instance: $A(u) = |\sin(u)|$ and $A(u) = |u|/(1+|u|)$. Indeed, for $\Phi(u,v) = A(u)v$, $\Psi(u,v) = B(u)\psi(v)$, $\gamma = 2$ we have:

$$|u|^{\gamma-2}v\Phi(u,v) = A(u)v^2 \geq A^2(u)v^2 = [A(u)v]^2 = |\Phi(u,v)|^{\gamma},$$

$$\Psi(u,v)u = uB(u)\psi(v) \geq 0. \;\square$$

Example 1.2 (quasilinear prescribed mean-curvature differential operators in x') We consider the equation:

$$\Big(r(t)A(x)\frac{x'}{\sqrt{1+x'^2}}\Big)' + p(t)B(x)\psi(x') + q(t)f(x) = 0, \quad t \geq t_0.$$

Condition (1.2) for any $\gamma \geq 2$ is fulfilled provided $A(u)$, $B(u)$ and $\psi(v)$ satisfy:

$$0 \leq A^{\gamma-1}(u) \leq |u|^{\gamma-2}, \;\; A(u) \neq 0, \;\; uB(u) \geq 0 \quad \text{and } \psi(v) \geq 0.$$

For instance: $A(u) = |\sin(u)|$ and $A(u) = |u|/(1+|u|)$.

Indeed, for

$$\Phi(u,v) = A(u)\frac{v}{(1+v^2)^{1/2}} \quad \text{and} \quad \Psi(u,v) = B(u)\psi(v),$$

we have:

$$|u|^{\gamma-2}v\Phi(u,v) = |u|^{\gamma-2}A(u)\frac{v^2}{(1+v^2)^{1/2}} \geq A^{\gamma}(u)\frac{v^2}{(1+v^2)^{1/2}}$$
$$\geq A^{\gamma}(u)\frac{|v|^{\gamma}}{(1+v^2)^{\gamma/2}} = |\Phi(u,v)|^{\gamma}. \;\square$$

Example 1.3 (half-linear second-order differential operators in x') We consider the equation:

$$\big(r(t)A(x)|x'|^{\beta-1}x'\big)' + p(t)B(x)|x'|^{\beta\gamma} + q(t)f(x) = 0, \quad t \geq t_0,$$

where $\beta \geq 1$. Condition (3) for any $\gamma \geq 2$ is fulfilled provided $A(u)$ and $B(u)$ satisfy:

$$0 \leq A(u), \;\; A(u) \neq 0 \quad \text{and} \quad u|u|^{\gamma-2}B(u) \geq A^{\gamma}(u).$$

Indeed, for $\Phi(u,v) = A(u)|v|^{\beta-1}v$ and $\Psi(u,v) = B(u)|v|^{\beta\gamma}$, we have:

$$\Phi(u,v)v = A(u)|v|^{\beta-1}vv = A(u)|v|^{\beta+1} \geq 0$$

and

$$|u|^{\gamma-1}\Psi(u,v) = |u|^{\gamma-1}B(u)|v|^{\beta\gamma} \geq A^{\gamma}(u)|v|^{\beta\gamma}$$
$$= \big|A(u)|v|^{\beta-1}v\big|^{\gamma} = |\Phi(u,v)|^{\gamma}. \;\square$$

2 Riccati transformation under assumption (1.2)

In this section we recall the well known Riccati transformation of the equation (1.1) by supposing the main assumption (1.2).

Lemma 2.1 *Let $p(t) \geq 0$ and $q(t) > 0$. Let $\Phi(u,v)$ and $\Psi(u,v)$ satisfy* (1.2), *and $f(u)$ satisfy: $f(u)/u \geq K > 0$ for all $u \neq 0$. Let $x(t)$ be a nonoscillatory solution of equation* (1.1). *Then $\overline{w}(t)$ defined by:*

$$\overline{w}(t) = -\frac{r(t)\Phi(x(t), x'(t))}{x(t)}, \ t \geq T, \tag{2.1}$$

satisfies the inequality:

$$\overline{w}' \geq (r(t))^{1-\gamma} g_0(|\overline{w}|) + Kq(t) \qquad t > T,$$

Proof. Taking the first derivative in (2.1), we obtain:

$$\overline{w}'(t) = \frac{r(t)\Phi(x(t), x'(t))}{x^2(t)} x'(t) - \frac{\left(r(t)\Phi(x(t), x'(t))\right)'}{x(t)}$$

and using equation (1.1) we get:

$$\overline{w}'(t) = \frac{r(t)\Phi(x(t), x'(t))}{x^2(t)} x'(t) + \frac{p(t)\Psi(x, x')}{x(t)} + q(t)\frac{f(x(t))}{x(t)}$$

that is:

$$\overline{w}'(t) = \frac{r(t)\Phi(x(t), x'(t))|x(t)|^{\gamma-2}x'(t)}{|x(t)|^{\gamma}} + \frac{p(t)\Psi(x, x')x(t)}{x^2(t)} + q(t)\frac{f(x(t))}{x(t)}.$$

Using assumptions (1.2) and $f(u)/u \geq K > 0$ for all $u \neq 0$, where $\gamma \geq 2$ and $g(s)$ satisfies (1.4), we get:

$$\overline{w}'(t) \geq \frac{r(t)g(|\Phi(x(t), x'(t))|)}{|x(t)|^{\gamma}} + Kq(t) \quad \text{for } t > T.$$

From (2.1) we get:

$$|\Phi(x(t), x'(t))| = \frac{|\overline{w}(t)||x(t)|}{r(t)}, \ t \geq T,$$

and putting it into previous inequality we obtain:

$$\overline{w}'(t) \geq \frac{r(t)}{|x(t)|^{\gamma}} g\Big(\frac{|\overline{w}(t)||x(t)|}{r(t)}\Big) + Kq(t) \quad \text{for } t > T.$$

Now we can use assumption (1.4) in previous inequality, and hence, we obtain:

$$\begin{aligned}\overline{w}'(t) &\geq \frac{r(t)}{|x(t)|^{\gamma}} \frac{|x(t)|^{\gamma}}{r(t)^{\gamma}} g_0(|\overline{w}(t)|) + Kq(t) \\ &= (r(t))^{1-\gamma} g_0(|\overline{w}(t)|) + Kq(t) \quad \text{for } t > T,\end{aligned}$$

that proves this lemma. □

3 Riccati transformation under assumption (1.3)

In this section, we repeat the consideration from previous section but supposing the main assumption (1.3) instead of (1.2).

Lemma 3.1 *Let $p(t) \geq 0$ and $q(t) > 0$. Let $\Phi(u,v)$ and $\Psi(u,v)$ satisfy* (1.3), *and $f(u)$ satisfy: $f(u)/u \geq K > 0$ for all $u \neq 0$. Let $x(t)$ be a nonoscillatory solution of equation* (1.1). *Then $\overline{w}(t)$ defined by:*

$$\overline{w}(t) = -\frac{r(t)\Phi(x(t),x'(t))}{x(t)}, \; t \geq T,$$

satisfies the inequality:

$$\overline{w}' \geq \frac{p(t)}{r(t)^\gamma} g_0(|\overline{w}|) + Kq(t), \quad t > T.$$

Proof. We start as before. Taking the first derivative in (2.1), we obtain:

$$\overline{w}'(t) = \frac{r(t)\Phi(x(t),x'(t))}{x^2(t)} x'(t) - \frac{\big(r(t)\Phi(x(t),x'(t))\big)'}{x(t)},$$

that is,

$$\overline{w}'(t) = \frac{r(t)\Phi(x(t),x'(t))x'(t)}{|x(t)|^2} + \frac{p(t)\Psi(x,x')x(t)|x(t)|^{\gamma-2}}{|x(t)|^\gamma} + q(t)\frac{f(x(t))}{x(t)}.$$

Using assumptions $f(u)/u \geq K > 0$ for all $u \neq 0$, and (1.3), where $\gamma \geq 2$ and $g(s)$ satisfy (1.4), we get:

$$\overline{w}'(t) \geq \frac{p(t)g(|\Phi(x(t),x'(t))|)}{|x(t)|^\gamma} + Kq(t) \quad \text{for } t > T.$$

From (2.1) we have in particular that:

$$|\Phi(x(t),x'(t))| = \frac{|\overline{w}(t)||x(t)|}{r(t)}, \quad t \geq T,$$

and puting it into

$$\overline{w}'(t) \geq \frac{p(t)g(|\Phi(x(t),x'(t))|)}{|x(t)|^\gamma} + Kq(t) \quad \text{for } t > T,$$

we obtain:

$$\overline{w}'(t) \geq \frac{p(t)}{|x(t)|^\gamma} g\Big(\frac{|\overline{w}(t)||x(t)|}{r(t)}\Big) + Kq(t) \quad \text{for } t > T.$$

Now we use assumption (1.4) in previous inequality and so, we conclude that:

$$\begin{aligned} \overline{w}'(t) &\geq \frac{p(t)}{|x(t)|^\gamma} \frac{|x(t)|^\gamma}{r(t)^\gamma} g_0(|\overline{w}(t)|) + Kq(t) \\ &= \frac{p(t)}{r(t)^\gamma} g_0(|\overline{w}(t)|) + Kq(t) \quad \text{for } t > T, \end{aligned}$$

that shows this lemma. □

4 A pointwise comparison principle and a blow-up argument

We consider the generalized Riccati differential equation:

$$w'(t) = a(t)g_0(|w(t)|) + Kq(t), \quad t > T, \tag{4.1}$$

where

$$a(t) = \begin{cases} (r(t))^{1-\gamma} & \text{under condition (1.2),} \\ p(t)/r(t)^{\gamma} & \text{under condition (1.3).} \end{cases}$$

For T_0 and T^*, $T \leq T_0 < T^*$, we associate to equation (4.1) the corresponding sub- and supersolutions: $\underline{w}, \overline{w} \in C^1([T_0, T^*), \mathbb{R})$ defined respectively by:

$$\underline{w}'(t) \leq a(t)g_0(|\underline{w}(t)|) + Kq(t) \quad \text{and} \quad \overline{w}'(t) \geq a(t)g_0(|\overline{w}(t)|) + Kq(t) \quad \text{in } [T_0, T^*).$$

Definition 4.1 We says that the comparison principle holds for equation (4.1) with arbitrary T_0 and T^*, $T \leq T_0 < T^*$, if the following statement holds for all sub- and supersolutions $\underline{w}, \overline{w}$ of equation (4.1):

$$\underline{w}(T_0) \leq \overline{w}(T_0) \quad \text{implies} \quad \underline{w}(t) \leq \overline{w}(t) \text{ for all } t \in [T_0, T^*).$$

For a supersolution $\overline{w} \in C^1([T_0, \infty), \mathbb{R})$ of the Riccati differential equation (4.1), let find:
- two real numbers T_0 and T^*, $T \leq T_0 < T^*$,
- a subsolution $\underline{w} \in C^1([T_0, T^*), \mathbb{R})$ of equation (4.1),

such that the following initial and blow-up arguments are satisfied at the sam time:

$$\underline{w}(T_0) \leq \overline{w}(T_0) \quad \text{and} \quad \lim_{t \to T^*} \underline{w}(t) = \infty.$$

By a combination of the preceding comparison principle and the initial and blow-up arguments we can conclude:

$$\lim_{t \to T^*} \underline{w}(t) \leq \lim_{t \to T^*} \overline{w}(t),$$

and hence, every supersoluton $\overline{w}(t)$ satisfies:

$$\lim_{t \to T^*} \overline{w}(t) = \infty.$$

It shows the nonexistence of a global supersolution of the Riccati differential equation (4.1).

In conclusion:

1°th step: if there is a nonoscillatory solution $x(t)$ of the main equation (1.1):

$$\left(r(t)\Phi(x, x')\right)' + p(t)\Psi(x, x') + q(t)f(x) = 0, \quad t \geq t_0,$$

then the function $\overline{w}(t)$ defined by:

$$\overline{w}(t) = -\frac{r(t)\Phi(x(t), x'(t))}{x(t)}, \quad t \geq T,$$

is a GLOBAL supersolution of the Riccati differential equation (4.1);

2°th and 3°th steps: for some sufficient "oscillation" conditions, the comparison and blow-up principles for equation (4.1) hod and it implies: $\lim_{t \to T^*} \overline{w}(t) = \infty$, that is, $\overline{w}(t)$ is a LOCAL supersolution of (4.1), which implies that there is NO any nonoscillatory solution of the main equation (1.1). By this contradiction, we conclude that equation (1.1) is oscillatory.

Now we present the main results of this section.

Lemma 4.1 (a pointwise comparison principle) *Let* $a(t) = (r(t))^{1-\gamma}$ *in the case of condition* (1.2) *or* $a(t) = p(t)/r(t)^\gamma$ *in the case of condition* (1.3) *and let* $a(t)$ *be a locally integrable function on* $\mathbb{R}_+$. *Then for every two points* T_0 *and* T^*, $T \le T_0 < T^*$, *and for every sub- and supersolution* $\underline{w}(t), \overline{w}(t) \in C^1([T_0, T^*), \mathbb{R})$ *of the generalized Riccati differential equation* (4.1) *we have:* $\underline{w}(T_0) \le \overline{w}(T_0)$ *implies* $\underline{w}(t) \le \overline{w}(t)$ *for all* $t \in [T_0, T^*)$. *That is:*

- $\underline{w}(T_0) \le \overline{w}(T_0)$,
- $\underline{w}'(t) \le a(t)g_0(|\underline{w}(t)|) + Kq(t),\ t > T$,
- $\overline{w}'(t) \ge a(t)g_0(|\overline{w}(t)|) + Kq(t),\ t > T$,

gives: $\underline{w}(t) \le \overline{w}(t)$ *for all* $t \in [T_0, T^*)$.

Lemma 4.2 (a blow-up principle) *Let the coefficients* $a(t)$ *and* $Kq(t)$ *of the generalized Riccati differential equation* (4.1) *satisfy the following "oscillation" condition: there is a continuous function* $C(t)$ *and a point* $T_1 \ge t_0$ *such that:*

$$C(t) \le \min\{a(t), Kq(t)\},\ t \ge T_1 \quad \text{and} \quad \limsup_{t\to\infty} \int_T^t C(\tau)d\tau = \infty. \tag{4.2}$$

Then there are two points T_0 *and* T^*, $T_0 < T^*$, *and a subsolution* $\underline{w}(t)$ *of equation* (4.1) *such that:*

$$\underline{w}(T_0) \le \overline{w}(T_0) \quad \text{and} \quad \lim_{t\to T^*} \underline{w}(t) = \infty.$$

5 Main results and examples

In this section we present the main results and their consequences. Also, a few examples are given to illustrate the importance of our main results.

Theorem 5.1 *Let* $\Phi(u,v)$ *and* $\Psi(u,v)$ *satisfy condition* (1.2) *or* (1.3). *Let* $f(u)/u \ge K > 0$ *for* $u \ne 0$, *and let coefficients* $r(t)$ *and* $q(t)$ *satisfy "oscillatory condition"* (4.2). *Then equation* (1.1) *is oscillatory.*

The main consequence is the following.

Corollary 5.1 *Let* $\Phi(u,v)$ *and* $\Psi(u,v)$ *satisfy condition* (1.2) *or* (1.3), *and let* $f(u)/u \ge K > 0$ *for* $u \ne 0$. *Let* $\mu \le 1/(\gamma-1)$ *or* $\nu \ge \gamma\mu - 1$ *and* $\sigma \le 1$, *where* $\gamma \ge 2$ *Then equation:*

$$(t^\mu \Phi(x,x'))' + t^\nu \Psi(x,x') + t^{-\sigma} f(x) = 0, \quad t \ge t_0, \tag{5.1}$$

is oscillatory.

Proof. The hypotheses on $\Phi(u,v)$, $\Psi(u,v)$, and $f(u)$ are the same as in Theorem 5.1. Hence, we need only to show that the coefficients:

$$r(t) = t^\mu, \quad p(t) = t^\nu \quad \text{and} \quad q(t) = t^{-\sigma},\ t \ge t_0,$$

where $\mu \le 1/(\gamma-1)$ or $\nu \ge \gamma\mu - 1$ and $\sigma \le 1$, satisfy the required oscillatory conditon (4.2). Indeed, in both cases (1.2) and (1.3), if $C(t) = c/t$ for some $c > 0$ and all $t \ge t_0 > 0$, then:

$$\frac{c}{t} \le \left(\frac{1}{t}\right)^{\mu(\gamma-1)}, \quad \frac{c}{t} \le \left(\frac{1}{t}\right)^{\mu\gamma-\nu} \quad \text{and} \quad \frac{c}{t} \le \left(\frac{1}{t}\right)^{\sigma},$$

and

$$\limsup_{t\to\infty} \int_T^t C(\tau)d\tau = \lim_{t\to\infty} \int_T^t \frac{c}{\tau}d\tau = \infty,$$

which proves this corollary. □

Finally, we present some concrete examples which can be shown by previous corollary.

Example 5.1 Let $K > 0$, $\mu \le 1$ or $\nu \ge 2\mu - 1$ and $\sigma \le 1$. Then the equation:

$$\left(t^{\mu}\frac{x^2}{1+x^2}x'\right)' + t^{\nu}x^3x'^2 + Kt^{-\sigma}x = 0,\ t \ge t_0 > 0,$$

is oscillatory.

Example 5.2 Let $K > 0$, $\mu \le 1$ or $\nu \ge 2\mu - 1$ and $\sigma \le 1$. Then the equation:

$$(t^{\mu}(\sin x)^2 x')' + t^{\nu}x^3x'^2 + Kt^{-\sigma}x = 0,\ t \ge t_0 > 0,$$

is oscillatory.

Example 5.3 Let $\alpha \ge 1$, $n \in \mathbb{N}$, $K > 0$, $\mu \le 1$ or $\nu \ge 2\mu - 1$ and $\sigma \le 1$. Then the equation:

$$\left(t^{\mu}\frac{x^2}{1+x^2}\frac{x'}{(1+x'^2)^{\frac{\alpha}{2}}}\right)' + t^{\nu}x\left(\frac{xx'}{(1+x^2)(1+x'^2)^{\frac{\alpha}{2}}}\right)^{2n} + Kt^{-\sigma}x = 0,$$

is oscillatory.

Example 5.4 Let $\beta \ge 1$, $K > 0$, $\nu \ge 2\mu - 1$ and $\sigma \le 1$. Then the equation:

$$(t^{\mu}(\sin x)^2 x'^{\beta})' + t^{\nu}x^3x'^{2\beta} + Kt^{-\sigma}x = 0,\ t \ge t_0 > 0,$$

is oscillatory.

Example 5.5 Let $K > 0$, $\mu \le 1$, $\nu \ge 0$, $\lambda \ge 0$, and $\sigma \le 1$. Then the equation:

$$\left(t^{\mu}\frac{x^2}{1+x^2}x'\right)' + t^{\nu}|x|^{\lambda}x\,\mathrm{sh}(x')x' + Kt^{-\sigma}x = 0,\ t \ge t_0 > 0,$$

is oscillatory.

The oscillation criterion presented in Theorem 5.1 can be called the Fite-Wintner-Leighton type criterion. The reason for that can be found in papers by Fite [1], Wintner [2], Leighton [3], and Pasic [4].

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