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Existence and Nonexistence of solutions to the Emden-Fowler equation on a geodesic ball in $\mathbf{S}^N$

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1 Introduction and preceding studies

In this paper, we consider the following problem

$$\begin{cases} \Delta_{\mathbf{S}^N} u + u^p = 0 & \text{in } B_{\theta_0}, \\ u = 0 & \text{on } \partial B_{\theta_0}, \end{cases} \tag{1.1}$$

where $N \geq 3$, $\mathbf{S}^N = \{x \in \mathbf{R}^{N+1} \mid |x| = 1\}$, $\Delta_{\mathbf{S}^N}$ is the Laplace–Beltrami operator on $\mathbf{S}^N$. Here B_{θ_0} is a geodesic ball in $\mathbf{S}^N$ with its geodesic radius $\theta_0 \in (0, \pi)$, and its center is located at the North Pole $P_n = (x_1, x_2, ..., x_{N+1}) = (0, 0, ..., 1)$.

The above problem (1.1) is said to be the Emden equation. Usually, the above equation is considered on the Euclidean space $\mathbf{R}^N$. Namely similar problems to the following problem are studied by many mathematicians:

$$\begin{cases} \Delta u + u^p = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where Ω is a bounded domain in $\mathbf{R}^N$ with the smooth boundary. Under $p > 1$ (if $N = 2$) or $1 < p < p_* := (N+2)/(N-2)$ (if $N \geq 3$), it is not difficult to prove the existence of a solution to (1.2), e.g., by using the direct method for variational problems, we can find a solution $u \in H_0^1(\Omega)$ to (1.2). Moreover, by the elliptic regularity theorem and the Schauder regularity theorem (e.g. see [4]), we can prove that $u \in C^2(\Omega)$. On the other hand, we assume $p \geq p_*$ ($N \geq 3$). Under the assumption, if Ω is a star-shaped domain in $\mathbf{R}^N$ ($N \geq 3$), by the Pohozaev identity (e.g. see Chapter III-1. in [9]). Furthermore the structure of solutions to (1.2) is investigated in detail (e.g., the number of classical solutions or the existence of singular solutions) when $\Omega = B := \{x \in \mathbf{R}^N \mid |x| < 1\}$. Concerning this result, e.g., see [5].

On the other hand, for the Emden equation on the sphere $\mathbf{S}^N$, the existence of solutions is not generally known. In this paper, we focus our attention on the existence of positive solutions to the Emden equation on caps of $\mathbf{S}^N$.

First we explain preceding studies of our problem (1.1). Bandle, Brillard and Flucher [1] proved the existence of a positive and radial solution from the viewpoint of the Sobolev imbedding. Here a solution to (1.1) depending only on the geodesic distance from P_n is said to be a radial solution. They proved that if $N \geq 4$, then there exists a positive classical solution to (1.1) for any $\theta_0 \in (0, \pi)$. On the other hand, if $N = 3$, then there exists some constant $\theta_c \in (0, \pi)$ such that

(a) a positive classical solution to (1.1) exists if $\theta_0 \in (\theta_c, \pi)$;

(b) there exist no positive classical solution to (1.1) if $\theta_0 \in (0, \theta_c)$.

Remark 1.1 *By* [7] *and* [3], *any positive classical solution to* (1.1) *is radially symmetric.*

Next, to obtain θ_c exactly, Bandle and Peletier [2] investigated (1.1) with $N = 3$ more precisely. They proved that $\theta_c = \pi/2$, and moreover the case $\theta_0 = \pi/2$ is contained in (b).

We [8] also investigated the structure of solutions. For (1.1), we will treat not only a classical solution but also a solution having singularity at the North Pole P_n (the solution is of class C^2 on B_{θ_0} except for $\{P_n\}$), and the following result.

Theorem A *Assume $p = p_*$. If $N \geq 4$, then, for any $\theta_0 \in (0, \pi)$, there exists a unique classical solution and a continuum of singular solutions to* (1.1). *On the other hand, under $N = 3$,*

(a) *if $\theta_0 \in (0, \pi/2]$, then there does not exist either a classical solution or a singular solution to* (1.1);

(b) *if $\theta_0 \in (\pi/2, \pi)$, then there exist a unique classical solution and a continuum of singular solutions to* (1.1).

Remark 1.2 *Although we only proved Theorem A under $N = 3$ in* [8], *we can prove Theorem A under $N \geq 4$ by using the same idea with some modifications. Furthermore, in* [8], (1.1) *was treated under more general boundary conditions, that is, we assumed, instead of the Dirichlet boundary condition, the Robin boundary condition*

$$u + \kappa \frac{\partial u}{\partial \nu} = 0 \qquad \partial B_{\theta_0},$$

where ν is the outer unit normal vector to on ∂B_{θ_0} and $\kappa \geq 0$.

The above results are proved in the case of $p = p_*$. On the other hand, our aim in this paper is to explain results on (1.1) with $p > p_*$.

2 Main results

In this paper, we investigate (1.1) with a supercritical case $p > p_*$, and we only treat positive radial solutions here. For this purpose, we introduce the polar coordinates. Namely, let

$$\begin{cases} x_1 = \sin\theta \sin\varphi_1 \sin\varphi_2 \ldots \sin\varphi_{N-1} \\ x_2 = \sin\theta \sin\varphi_1 \sin\varphi_2 \ldots \cos\varphi_{N-1} \\ x_3 = \sin\theta \sin\varphi_1 \sin\varphi_2 \ldots \cos\varphi_{N-2} \\ \vdots \\ x_N = \sin\theta \cos\varphi_1 \\ x_{N+1} = \cos\theta \end{cases} \tag{2.1}$$

with, $\theta, \varphi_i \in [0,\pi]$ $(i = 1, 2, \ldots, N-2)$ and $\varphi \in [0, 2\pi]$. By (2.1), the Laplace–Beltrami operator $\Delta_{\mathbf{S}^N}$ is expressed by

$$\Delta_{\mathbf{S}^N} u = \frac{1}{\sin^{N-1}} \frac{\partial}{\partial\theta}\left(\sin^{N-1}\theta \frac{\partial u}{\partial\theta}\right) + \sum_{i=1}^{N-1} \frac{1}{\gamma_i} \frac{\partial}{\partial\varphi_i}\left(\sin^{N-i-1}\varphi_i \frac{\partial u}{\partial\varphi_i}\right)$$

with

$$\gamma_i = \sin^2\theta \sin^{N-i-1}\varphi_i \prod_{j=1}^{i-1} \sin^2\varphi_j.$$

Hence a positive and radial solution u to (1.1) satisfies

$$\begin{cases} \dfrac{1}{\sin^{N-1}\theta}\left(u_\theta \sin^{N-1}\theta\right)_\theta + u^p = 0 & \text{in } (0,\theta_0), \\ u(\theta) > 0 & \text{in } (0,\theta_0), \\ u(\theta_0) = 0. \end{cases} \tag{2.2}$$

Here we define two kinds of solutions to (2.2); a solution u to (1.1) is said to be a *regular* solution to (2.2) if $C^2((0,\theta_0))$ and $u(\theta)$ converging to some positive constant as $\theta \to 0$; a solution u to (1.1) is said to be a *singular* solution to (2.2) if $C^2((0,\theta_0))$ and $u(\theta)$ tends to $+\infty$ as $\theta \to 0$. Our main theorems are as follows.

To investigate the structure of solutions to (2.2), we apply results on [6], [10] and [11]. First we transform (2.2) into the Emden-Fowler equation on $\mathbf{R}^N$. Namely we define

$$\tau = g(\theta) := \int_\theta^{\theta_0} \frac{d\psi}{\sin^{N-1}\psi}. \tag{2.3}$$

By the new variable τ and a solution u to (2.2), we define

$$w := \frac{u(\theta)}{\tau}.$$

Then the new function w satisfies the following problem

$$\begin{cases} \dfrac{1}{\tau^2}\left(\tau^2 w_\tau\right)_\tau + K(\tau) w_+^p(\tau) = 0 & \text{for } \tau \in (0,+\infty), \\ w(0) = \alpha, \\ w_\tau(0) = 0, \end{cases} \tag{2.4}$$

where $w_+ = \max\{w, 0\}$ and

$$\begin{aligned} K(\tau) &:= \tau^{p-1} \sin^{2N-2}\theta, \\ \alpha &:= -u_\theta(\theta_0)\sin^2\theta_0. \end{aligned} \tag{2.5}$$

Thus it suffices to investigate the behavior of solutions to (2.4) instead of (2.2).

Next we explain methods to investigate (2.4). First let

$$\begin{cases} \dfrac{1}{\tau^2}\left(\tau^2 w_\tau\right)_\tau + L(\tau) w_+^p(\tau) = 0 & \text{for } \tau \in (0,+\infty), \\ w(0) = \beta, \end{cases} \tag{2.6}$$

where $w_+ := \max\{w, 0\}$ and β is a positive constant, and a solution to (2.6) is sometimes denoted by $w(\tau;\beta)$. Here the function L satisfies

$$\begin{cases} L(\tau) \in C^1((0,+\infty)), \\ L(\tau) \geq 0 \ \text{ and } \ L(\tau) \not\equiv 0 & \text{on } (0,+\infty), \\ \tau L(\tau) \in L^1(0,1), \\ \tau^{1-p} L(\tau) \in L^1(1,+\infty). \end{cases} \tag{L}$$

Hereafter the solution to (2.6) with an initial data β is denoted by $w(\tau;\beta)$. From (2.3) and (2.5), we can easily confirm that $K(\tau)$ satisfies (L). Therefore any result on (2.6) is valid for (2.4).

Second we define three types of solutions to (2.6) as follows:

Definition 2.1 (i) *A solution w to* (2.4) *is said to be a rapidly decaying solution if $w > 0$ on $[0,+\infty)$ and $\tau w(\tau)$ converges to some positive constant as $\tau \to +\infty$.*

(ii) *A solution w to* (2.4) *is said to be a slowly decaying solution if $w > 0$ on $[0,+\infty)$ and $\tau w(\tau) \to +\infty$ as $\tau \to +\infty$.*

(iii) *A solution w to* (2.4) *is said to be a crossing solution if w has a zero in $(0,+\infty)$.*

Here we remark that a rapidly decaying solution and a slowly decaying solution to (2.6) are corresponding to a regular solution and a singular solution to (2.2).

Next we refer to some lemmas concerning the structure of solutions to (2.6). We introduce the following identity

$$P(\tau; w) := \frac{1}{2}\tau^2 w_\tau\{\tau w_\tau + w\} + \frac{\tau^3}{p+1} L(\tau) w_+^{p+1}. \tag{2.7}$$

The behavior of (2.7) for sufficiently large τ depends on a kind of w defined above. Namely the next lemma is known.

Lemma 2.1 (Lemma 2.6 in [11]) (a) *If $w(\tau;\beta)$ is a crossing solution to* (2.6), *then $P(\tau;w) > 0$ for $\tau \in [z(\beta), +\infty)$, where $z(\beta)$ is a zero of $w(\tau;\beta)$.*

(b) *If $w(\tau;\beta)$ is a slowly decaying solution to* (2.6), *then there exists a sequence $\{\hat{\tau}_j\}$ such that $\hat{\tau}_j \to +\infty$ as $j \to +\infty$ and $P(\hat{\tau}_j;w) < 0$ for any j.*

(c) *If $w(\tau;\beta)$ is a rapidly decaying solution to* (2.6), *then there exists a sequence $\{\tilde{\tau}_j\}$ such that $\tilde{\tau}_j \to +\infty$ and $P(\tilde{\tau}_j;w) \to 0$ as $j \to +\infty$.*

By using properties of $P(\tau;w)$, Yanagida and Yotsutani [10], [11] proved a structure theorem for (2.6). To explain the theorem, we require some preliminaries. First we introduce the function

$$G(\tau) := \frac{1}{p+1}\tau^3 L(\tau) - \frac{1}{2}\int_0^\tau s^2 L(s)ds.$$

The function $G(\tau)$ is related to $P(\tau;w)$ by the following lemma.

Lemma 2.2 (Lemma 3.2 in [5]) *Any solution w to* (2.6) *satisfies the identity*

$$\frac{d}{d\tau}P(\tau;w) = G_\tau(\tau)w_+^{p+1}(\tau)$$

and its integral form

$$P(\tau;w) = G(\tau)w_+^{p+1}(\tau) - (p+1)\int_0^\tau G(s)w_+^p w_s(s)ds.$$

Next we define the following function

$$H(\tau) := \frac{1}{p+1}\tau^{2-p}L(\tau) - \frac{1}{2}\int_\tau^{+\infty} s^{1-p}L(s)ds.$$

The function $H(\tau)$ is corresponding to $G(\tau)$ by

$$G_\tau(\tau) = \frac{\tau^{(p+1)/2}}{p+1}(\tau^{-\xi}L)_\tau = \tau^{p+1}H_\tau(\tau)$$

with

$$\xi = \frac{p-5}{2}.$$

By using $G(\tau)$ and $H(\tau)$, we define

$$\tau_G := \inf\{\tau \in [0,+\infty) \mid G(\tau) < 0\},$$
$$\tau_H := \sup\{\tau \in [0,+\infty) \mid H(\tau) < 0\}.$$

Here $\tau_G = +\infty$ if $G(\tau) \geq 0$ on $(0,+\infty)$ and $\tau_H = 0$ if $H(\tau) \geq 0$ on $(0,+\infty)$. From the above preliminaries, following proposition holds.

Lemma 2.3 (Theorem 1 in [10]) *Assume* (L) *and* $G \not\equiv 0$ *on* $(0, +\infty)$. *Then the following three statements hold.*

(i) *If* $\tau_G = +\infty$, *then the structure of solutions to* (2.6) *is of type C:* $w(\tau; \beta)$ *is a crossing solution for any* $\beta > 0$.

(ii) *If* $\tau_H = 0$, *then the structure of solutions to* (2.6) *is of type S:* $w(\tau; \beta)$ *is a slowly decaying solution for any* $\beta > 0$.

(iii) *If* $0 < \tau_H \leq \tau_G < +\infty$, *then the structure of solutions to* (2.6) *is of type M: there exists a constant* $\beta_* > 0$ *such that* $w(\tau; \beta)$ *is a slowly decaying solution for* $\beta \in (0, \beta_*)$, $w(\tau; \beta_*)$ *is a rapidly decaying solution, and* $w(\tau; \beta)$ *is a crossing solution for* $\beta \in (\beta_*, +\infty)$.

By Lemma 2.3, we obtain the next proposition.

Proposition 2.1 *Under* $p = p_*$, *let* $w(\tau; \alpha)$ *be a solution to* (2.4). *Assume* $\theta_0 \in (\pi/2, \pi)$ $(N = 3)$ *or* $\theta_0 \in (0, \pi)$ $(N \geq 4)$. *Then there exists some* $\alpha_* > 0$ *such that*

(i) *if* $\alpha < \alpha_*$, *then* w *is a slowly decaying solution;*

(ii) *if* $\alpha = \alpha_*$, *then* w *is a rapidly decaying solution;*

(iii) *if* $\alpha > \alpha_*$, *then* w *is a crossing solution.*

On the other hand, assume $N = 3$ *and* $\theta_0 \in (0, \pi]$. *Then, for any* $\alpha > 0$, w *is a crossing solution.*

Proposition 2.1 is equivalent to Theorem A.

We can apply Lemma 2.3 (i) for (2.4) with $p > p_*$ when p is sufficiently large. In fact, by direct calculation, we obtain

$$G_\tau(\tau) = \frac{1}{p+1} r(\theta, p) \tau^{p+1} \sin^{2N-2} \theta \tag{2.8}$$

with

$$r(\theta, p) = \frac{p+3}{2} - (2N-2)\tau \sin^{N-2} \theta \cos \theta. \tag{2.9}$$

From (2.3), we see that $\tau \sin^{N-2} \theta$ is bounded for $\tau \in (0, +\infty)$. Hence if p is sufficiently large, then $r(\theta, p) > 0$. Therefore, by Lemmas 2.2 and 2.3 (i), the following theorem holds.

Theorem 2.1 *Assume* $\theta_0 \in (\pi/2, \pi)$ $(N = 3)$ *or* $\theta_0 \in (0, \pi)$ $(N \geq 4)$. *Then there exists some* $p_c(\theta_0) > p_*$ *such that, for any* $p > p_c$, *there does not exist either a regular or a singular solution to* (2.2).

Especially, in the case of $N = 3$, we can obtain

$$p_c(\theta_0) = 4\left(1 + \frac{1}{\sin\theta_0}\right) - 3.$$

On the other hand, it is not easy to investigate the existence of solutions to (2.2) if $p > p_*$ is sufficiently near p_*. To explain the reason, we introduce

$$\begin{aligned}
\mathcal{C}(p) &:= \{\beta > 0 \mid w(\tau;\beta) \text{ is a crossing solution to (2.6)}\},\\
\mathcal{R}(p) &:= \{\beta > 0 \mid w(\tau;\beta) \text{ is a rapidly decaying solution to (2.6)}\},\\
\mathcal{S}(p) &:= \{\beta > 0 \mid w(\tau;\beta) \text{ is a slowly decaying solution to (2.6)}\}.
\end{aligned}$$

For the above sets, the next lemma holds.

Lemma 2.4 (Lemma 2.7 in [6]) *The set $\mathcal{C}(p)$ is open.*

Moreover we introduce the following condition

$$\begin{cases} \text{there exists } \eta_1 \in [0, +\infty) \text{ such that} \\ G(\tau) \geq 0 \text{ for } (0, \eta_1) \text{ and } G_\tau(\tau) \leq 0 \text{ for } (\eta_1, +\infty). \end{cases} \tag{G}$$

Under (G), the following lemma holds.

Lemma 2.5 (Lemma 2.6 in [6]) *Under* (G), *$\mathcal{S}(p)$ is an open set.*

Remark 2.1 *Assume $\mathcal{C}(p)$ and $\mathcal{S}(p)$ are open. Then there exists a rapidly decaying solution between $\mathcal{C}(p)$ and $\mathcal{S}(p)$.*

If $p = p_*$, then G defined by (2.4) satisfies (G), and Lemma 2.5 holds. On the other hand, if $p > p_*$, then (G) is not satisfied. Hence it seems difficult to investigate the openness of $\mathcal{S}(p)$ for $p > p_*$.

Moreover the next lemma implies that the structure of solutions to (2.4) with $p > p_*$ is qualitatively different from the structure of solutions to (2.4) with $p = p_*$.

Lemma 2.6 (Theorem 3 in [11]) *If $\liminf_{\tau\to+\infty} G(\tau) > 0$, then there exists $\beta_c > 0$ such that $w(\tau;\beta)$ is a crossing solution to* (2.6) *for any $\beta \in (0, \beta_c)$.*

In fact if $p > p_*$, then $\lim_{\tau\to+\infty} G(\tau) > 0$. Hence, for a sufficiently small $\beta > 0$, $w(\tau;\beta)$ is a crossing solution, that is, the structure of solutions to (2.4) with $p > p_*$ is different from the structure of solutions to (2.4) with $p = p_*$ (see Proposition 2.1). Furthermore, from Lemma 2.6, we expect that there exist at least two rapidly decaying solution to (2.4) if $p > p_*$ is sufficiently near p_*. Hereafter we state this result.

To investigate the structure of solutions to (2.4) with $p = p_* + \epsilon$ (ϵ is sufficiently small), we considered a transformed problem from (2.4). Namely let

$$t := \frac{1}{\tau}, \quad v := \frac{w}{t}. \tag{2.10}$$

The transformation (2.10) is said to be the Kelvin transformation. By (2.10), we obtain the next lemma.

Lemma 2.7 *Assume w is a rapidly decaying solution to* (2.4). *Then v defined in* (2.10) *satisfies*

$$\begin{cases} \frac{1}{t^2}(t^2 v_t)_t + \tilde{K}(t)v_+^p = 0 & \text{for } t \in (0,+\infty), \\ v(0) = \eta > 0, \\ v_t(0) = 0, \end{cases} \tag{2.11}$$

where $v_+ = \max\{v, 0\}$ and

$$\tilde{K}(t) := t^{-4}\sin^{2N-2}\theta. \tag{2.12}$$

Similarly, for a rapidly decaying solution v to (2.11), *$w(\tau) = v(t)/\tau$ is a rapidly decaying solution to* (2.4).

Lemma 2.7 implies that the number of rapidly decaying solutions to (2.4) is the same as the number of rapidly decaying solutions to (2.11).

Next, for (2.11), we define

$$\begin{aligned} \tilde{P}(t;v) &:= \frac{1}{2}t^2 v_t\{tv_t + v\} + \frac{t^3}{p+1}\tilde{K}(t)v_+^{p+1}, \\ \tilde{G}(t) &:= \frac{1}{p+1}t^3\tilde{K}(t) - \frac{1}{2}\int_0^t s^2\tilde{K}(s)ds \\ \tilde{H}(t) &:= \frac{1}{p+1}t^{2-p}\tilde{K}(t) - \frac{1}{2}\int_t^{+\infty} s^{1-p}\tilde{K}(s)ds, \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} \tilde{\mathcal{C}}(p) &:= \{\beta > 0 \mid w(\tau;\beta) \text{ is a crossing solution to (2.11)}\}, \\ \tilde{\mathcal{R}}(p) &:= \{\beta > 0 \mid w(\tau;\beta) \text{ is a rapidly decaying solution to (2.11)}\}, \\ \tilde{\mathcal{S}}(p) &:= \{\beta > 0 \mid w(\tau;\beta) \text{ is a slowly decaying solution to (2.11)}\}. \end{aligned}$$

Then, from (2.5), (2.10) and (2.12), it follows that

$$\begin{aligned} P(\tau;w) &= \tilde{P}(t;v), \\ G(\tau) &= \tilde{H}(t), \\ H(\tau) &= \tilde{G}(t). \end{aligned} \tag{2.14}$$

From the above properties, we can obtain the following corollary.

Corollary 2.1 *Under $p = p_*$, the structure of solutions to* (2.11) *is of type M:, that is, the structure of solutions to* (2.11) *is the same as the structure of solutions to* (2.4).

Therefore, in the case of $p = p_*$, (2.11) has a unique rapidly decaying solution. Moreover, for (2.11), it is not so difficult to prove the openness of $\tilde{\mathcal{S}}(p)$ under $p > p_*$. Namely the following two lemmas hold.

Lemma 2.8 *For $p > p_*$, $\tilde{S}(p)$ is an open set.*

In fact, from (2.14), we see that $\tilde{G}_t(t) < 0$ for sufficiently large t. By this properties and Lemma 2.2, we can prove that $\lim_{t\to+\infty} \tilde{P}(t; v) < 0$ for $p = p_* + \epsilon$ (ϵ is sufficiently small) if $\lim_{t\to+\infty} \tilde{P}(t; v) < 0$ for $p = p_*$.

Now we investigate structures of $\tilde{C}(p)$. Let $v(t;\eta)$ be a solution to (2.11), and we define

$$p_\dagger(\eta) := \sup\{p' > p_* \mid v(t;\eta) \text{ is a crossing solution for any } p \in (p_*, p')\}.$$

Here if a solution $v(t;\eta)$ is not a crossing solution for any $p > p*$, then $p_\dagger(\eta) = p_*$.The function $p_\dagger(\eta)$ is bounded. In fact by the same argument as in Theorem 2.1, the following proposition is proved.

Proposition 2.2 *Assume $\theta_0 \in (\pi/2, \pi)$ $(N = 3)$ or $\theta_0 \in (0, \pi)$ $(N \geq 4)$. Then there exists some $p_c'(\theta_0) > p_*$ such that, for any $p > p_c'$ and $\eta > 0$, a solution $v(t;\eta)$ to (2.11) is a slowly decaying solution.*

Proposition 2.2 implies that $p_\dagger(\eta)$ is bounded.

For $p_\dagger(\eta)$, we can prove some properties. First, we refer to the following lemma.

Lemma 2.9 (Theorem 2 in [11]) *If $\limsup_{\tau\to+\infty} G(\tau) < 0$, then there exists $\beta_s > 0$ such that $w(\tau;\beta)$ is a slowly decaying solution to (2.6) for any $\beta \in (0, \beta_s)$.*

If $p > p_*$, then we can apply Lemma 2.9 for (2.11). Thus, for sufficiently small $\eta > 0$, it holds that $p_\dagger(\eta) = p_*$. On the other hand, $p_\dagger(\eta) \not\equiv p_*$ for $\eta > 0$. In fact let $\eta_0 \in \tilde{C}(p_*)$. Then, by the continuity of solutions to (2.11) concerning parameters, it holds that $\eta_0 \in \tilde{C}(p_* + \epsilon)$ (ϵ is sufficiently small), that is, $p_\dagger(\eta_0) > p_*$. Therefore, from Remark 2.1, we see that there exists at least one rapidly decaying solution to (2.11).

Next we state our main result, that is, there exists at least two rapidly decaying solutions to (2.11) when $p > p_*$ is sufficiently near p_*. The following lemma is essential in the proof of our main result.

Lemma 2.10 *As $\eta \to +\infty$, $p_\dagger(\eta) \to p_*$.*

By Lemma 2.10, sufficiently large $\eta > 0$ is not contained in $\tilde{C}(p_* + \epsilon)$ (ϵ is sufficiently small). Hence, for sufficiently large $\eta > 0$, $v(t;\eta)$ is a rapidly decaying solution or a slowly decaying solution. In addition if $v(t;\eta_0)$ is a slowly decaying solution for some $\eta_0 > \eta_M := \max\{\eta > 0 \mid \eta \in \tilde{C}(p)\}$, then, from Lemma 2.8, there exists $\eta_1 \in (\eta_M, \eta_0)$ such that $v(t;\eta_1)$ is a rapidly decaying solution. Thus we see that there exists at least two rapidly decaying solutions to (2.11). Therefore, by Lemma 2.7, the following theorem is obtained.

Theorem 2.2 *Assume $\theta_0 \in (\pi/2, \pi)$ $(N = 3)$ or $\theta_0 \in (0, \pi)$ $(N \geq 4)$. Then there exists some $\epsilon_0(\theta_0) > 0$ such that, for any $\epsilon \in (0, \epsilon_0)$, there exists at least two regular solutions to (2.2) with $p = p_* + \epsilon$.*

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