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Generalized Jacobian elliptic functions *

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1 Introduction

This is a survey of the latest development in the generalized trigonometric functions and Jacobian elliptic functions.

First, we will review these functions shortly. In Section 2, we define new functions, which generalize Jacobian elliptic functions and coincide with the generalized trigonometric functions as their moduli vanish. Since a generalized Jacobian elliptic function satisfies a bistable ordinary differential equation with p-Laplacian, in Section 3 we will apply the new function to a bifurcation problem for the equation and show all the solutions by using its modulus as only one parameter. Moreover, it follows directly from representations of solutions by the new function that a kind of solutions of bistable problem of p-Laplacian is also an eigenfunction of $p/2$-Laplacian. To see this reduction in p in detail, in Section 4 we will try to extend generalized Jacobian elliptic functions more generally. Section 5 is devoted to give current topics in the generalized functions.

1.1 Jacobian Elliptic Functions

A set of basic elliptic functions was introduced by Carl Gustav Jacob Jacobi [16] in 1829. These functions are named the Jacobian elliptic functions after him.

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For any $k \in [0,1)$ we define $K(k)$ by the incomplete elliptic integral of the first kind.

$$K(k) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}.$$

Then, for any $k \in [0,1)$ and any $x \in [-K(k), K(k)]$ we define $\mathrm{sn}\,(x,k)$ by an inverse of the incomplete elliptic integral of the first kind.

$$x = \int_0^{\mathrm{sn}\,(x,k)} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}.$$

Clearly, $\mathrm{sn}\,(x,k)$ is an increasing odd function in x from $[-K(k), K(k)]$ to $[-1,1]$. We extend the domain of $\mathrm{sn}\,(x,k)$ to $\mathbb{R}$ by $\mathrm{sn}\,(x+2K(k),k) = -\mathrm{sn}\,(x,k)$, which implies that $\mathrm{sn}\,(x,k)$ has $4K(k)$-periodicity. We can see that $K(0)=\pi/2$, $\mathrm{sn}\,(x,0)=\sin x$, $K(k)\to\infty$, $\mathrm{sn}\,(x,k)\to\tanh x$ as $k\to 1$ and $\mathrm{sn}\,(\cdot\,,k)\in C^\infty(\mathbb{R})$.

Using $\mathrm{sn}\,(x,k)$, for $x \in [-K(k), K(k)]$ we also define

$$\mathrm{cn}\,(x,k) = \sqrt{1-\mathrm{sn}^2\,(x,k)},$$
$$\mathrm{dn}\,(x,k) = \sqrt{1-k^2\,\mathrm{sn}^2\,(x,k)}.$$

Clearly, $\mathrm{cn}\,(x,k)$ and $\mathrm{dn}\,(x,k)$ are even functions in x from $[-K(k), K(k)]$ to $[0,1]$. We extend the domains of $\mathrm{cn}\,(x,k)$ and $\mathrm{dn}\,(x,k)$ to the whole of $\mathbb{R}$ by $\mathrm{cn}\,(x+2K(k),k) = -\mathrm{cn}\,(x,k)$ and $\mathrm{dn}\,(x+2K(k),k)=\mathrm{dn}\,(x,k)$. This implies that $\mathrm{cn}\,(x,k)$ and $\mathrm{dn}\,(x,k)$ have $4K(k)$- and $2K(k)$-periodicity. It is shown that $\mathrm{cn}\,(x,0)=\cos x$, $\mathrm{dn}\,(x,0)=1$, $\mathrm{cn}\,(x,k)$, $\mathrm{dn}\,(x,k)\to\coth x$ as $k\to 1$ and $\mathrm{cn}\,(\cdot\,,k)$, $\mathrm{dn}\,(\cdot\,,k)\in C^\infty(\mathbb{R})$.

These functions satisfy

$$\mathrm{cn}^2\,(x,k)+\mathrm{sn}^2\,(x,k)=1,$$
$$\mathrm{dn}^2\,(x,k)+k^2\,\mathrm{sn}^2\,(x,k)=1,$$
$$(\mathrm{sn}\,(x,k))' = \mathrm{cn}\,(x,k)\,\mathrm{dn}\,(x,k),$$
$$(\mathrm{cn}\,(x,k))' = -\mathrm{sn}\,(x,k)\,\mathrm{dn}\,(x,k),$$
$$(\mathrm{dn}\,(x,k))' = -k^2\,\mathrm{sn}\,(x,k)\,\mathrm{cn}\,(x,k).$$

We can find many other properties of these functions in [28].

In particular, it is important that $y=\mathrm{sn}\,(x,k)$ satisfies

$$y''+y(1+k^2-2k^2y^2)=0,$$

which reminds that solutions of bistable problem

$$\begin{cases} u'' + \lambda u(1-u^2) = 0, & x \in (0, L), \\ u(0) = u(L) = 0 \end{cases}$$

can be represented explicitly by using $\operatorname{sn}(x, k)$. Indeed, for any $k \in (0, 1)$, the value of bifurcation parameter λ is given by

$$\lambda_n(k) = (1+k^2)\left(\frac{2nK(k)}{L}\right)^2 \tag{1.1}$$

for each $n \in \mathbb{N}$, with corresponding solutions $\pm u_{n,k}$, where

$$u_{n,k}(x) = \sqrt{\frac{2k^2}{1+k^2}}\operatorname{sn}\left(\frac{2nK(k)}{L}x, k\right). \tag{1.2}$$

Conversely, all nontrivial solutions are given by Eqs. (1.1) and (1.2). In particular, it follows from Eq. (1.2) that all solutions satisfy $|u| < 1$ in $[0, L]$.

1.2 Generalized Trigonometric Functions

Generalized Trigonometric functions were introduced in 1879 by E. Lundberg (see Lindqvist and Peetre [21, pp.113-141]). After that, these functions have been developed mainly by A. Elbert [13], P. Lindqvist [19], P. Drábek and R. Manásevich [11], J. Lang and D.E. Edmunds [18].

For any constants $p,\ q > 1$, we define π_{pq} by

$$\pi_{pq} = 2\int_0^1 \frac{dt}{\sqrt[p]{1-t^q}} = \frac{2}{q}B\left(1-\frac{1}{p}, \frac{1}{q}\right) = \frac{2\Gamma(1-1/p)\Gamma(1/q)}{q\Gamma(1-1/p+1/q)},$$

where B and Γ are the Beta- and the Gamma functions, respectively. Then, for any $x \in [-\pi_{pq}/2, \pi_{pq}/2]$ we define $\sin_{pq} x$ by

$$x = \int_0^{\sin_{pq} x} \frac{dt}{\sqrt[p]{1-|t|^q}}.$$

Clearly, $\sin_{pq} x$ is an increasing odd function in x from $[-\pi_{pq}/2, \pi_{pq}/2]$ to $[-1, 1]$. We extend the domain of $\sin_{pq} x$ to the whole of $\mathbb{R}$ by $\sin_{pq}(x + \pi_{pq}) = -\sin_{pq} x$, which implies that $\sin_{pq} x$ has $2\pi_{pq}$-periodicity. We can see that

$\pi_{22} = \pi$ and $\sin_{22} x = \sin x$. Moreover, $y = \sin_{pq} x$ satisfies that y, $|y'|^{p-2}y' \in C^1(\mathbb{R})$ and that $y \in C^2(\mathbb{R})$ if $1 < p \leq 2$.

We agree that π_p and $\sin_p x$ denote π_{pp} and $\sin_{pp} x$ when $p = q$, respectively. In that case, we can also refer to [7, 8, 9, 10].

Using $\sin_{pq} x$, for $x \in [-\pi_{pq}/2, \pi_{pq}/2]$ we also define

$$\cos_{pq} x = \sqrt[q]{1 - |\sin_{pq} x|^q}. \tag{1.3}$$

Clearly, $\cos_{pq} x$ is an even function in x from $[-\pi_{pq}/2, \pi_{pq}/2]$ to $[0, 1]$. We extend the domain of $\cos_{pq} x$ to the whole of $\mathbb{R}$ by $\cos_{pq}(x + \pi_{pq}) = -\cos_{pq} x$. These implies that $\cos_{pq} x$ has $2\pi_{pq}$-periodicity. We can see that $\cos_{22} x = \cos x$. An analogue of $\tan x$ is obtained by defining

$$\tan_{pq} x = \frac{\sin_{pq} x}{\cos_{pq} x}$$

for those values of x at which $\cos_{pq} x \neq 0$. This means that $\tan_{pq} x$ is defined for all $x \in \mathbb{R}$ except for the points $(k + 1/2)\pi_{pq}$ $(k \in \mathbb{Z})$. We denote by $\cos_p x$ and $\tan_p x$ as for the case $\sin_p x$. The functions $\sin_p x$ and $\cos_p x$ are useful for Prüfer transformation of half-linear differential equations (see [8, 9, 13, 22]).

These functions satisfy, for $x \in [0, \pi_{pq}/2)$

$$\begin{aligned} \cos_{pq}^q x + \sin_{pq}^q x &= 1, \qquad (1.4)\\ (\sin_{pq} x)' &= \cos_{pq}^{q/p} x,\\ (\cos_{pq} x)' &= -\sin_{pq}^{q-1} x \cos_{pq}^{q/p+1-q} x,\\ (\tan_{pq} x)' &= \cos_{pq}^{q/p-1} x (1 + \tan_{pq}^q x). \end{aligned}$$

Note that

$$\left(\frac{d}{dx}\sin_{pq} x\right)^p + \sin_{pq}^q x = 1. \tag{1.5}$$

We can find many other properties of these functions in [12, 18].

Remark 1.1. There are some different definitions of $\cos_{pq} x$ from Eq. (1.3). For example, Drábek and Manásevich [11] define $\cos_{pq} x$ by

$$\cos_{pq} x = \frac{d}{dx}\sin_{pq} x,$$

and so Eq. (1.5) gives

$$\cos_{pq}^p x + \sin_{pq}^q x = 1,$$

which is slightly different from Eq. (1.4). Independently of the definition of $\cos_{pq} x$, it is essential that $\sin_{pq} x$ satisfies Eq. (1.5).

In particular, it is important that $y = \sin_{pq} x$ satisfies

$$(|y'|^{p-2}y')' + \frac{(p-1)q}{p}|y|^{q-2}y = 0, \tag{1.6}$$

which reminds that solutions of eigenvalue problem

$$\begin{cases} (|u'|^{p-2}u')' + \lambda|u|^{q-2}u = 0, & x \in (0, L), \\ u(0) = u(L) = 0 \end{cases}$$

can be represented explicitly by using $\sin_{pq} x$. Indeed, for any $R \in (0, 1)$, the value of bifurcation parameter λ is given by

$$\lambda_n(k) = \frac{(p-1)q}{p}\left(\frac{n\pi_{pq}}{L}\right)^p R^{p-q} \tag{1.7}$$

for each $n \in \mathbb{N}$, with corresponding solutions $\pm u_{n,k}$, where

$$u_{n,k}(x) = R \sin_{pq}\left(\frac{n\pi_{pq}}{L}x\right). \tag{1.8}$$

Conversely, all nontrivial solutions are given by Eqs. (1.7) and (1.8).

2 Generalized Jacobian Elliptic Functions

The author [24] introduced a generalization of Jacobian elliptic functions, which includes both the Jacobian elliptic functions and the generalized trigonometric functions.

Let $p,\ q > 1$. For any $k \in [0, 1)$ we define $K_{pq}(k)$ by

$$K_{pq}(k) = \int_0^1 \frac{dt}{\sqrt[p]{(1-t^q)(1-k^q t^q)}}.$$

Then, for any $k \in [0, 1)$ and $x \in [-K_{pq}(k), K_{pq}(k)]$ we define $\mathrm{sn}_{pq}(x, k)$ by

$$x = \int_0^{\mathrm{sn}_{pq}(x,k)} \frac{dt}{\sqrt[p]{(1-|t|^q)(1-k^q|t|^q)}}.$$

Clearly, $\mathrm{sn}_{pq}(x, k)$ is an increasing odd function in x from $[-K_{pq}(k), K_{pq}(k)]$ to $[-1, 1]$. We extend the domain of $\mathrm{sn}_{pq}(x, k)$ to $\mathbb{R}$ by $\mathrm{sn}_{pq}(x + 2K_{pq}(k), k) =$

$-\operatorname{sn}_{pq}(x,k)$, which implies that $\operatorname{sn}_{pq}(x,k)$ has $4K_{pq}(k)$-periodicity. We can see that $K_{22}(k) = K(k)$, $\operatorname{sn}_{22}(x,k) = \operatorname{sn}(x,k)$, $K_{pq}(0) = \pi_{pq}/2$ and that $\operatorname{sn}_{pq}(x,0) = \sin_{pq} x$. Moreover, $y = \operatorname{sn}_{pq}(x,k)$ satisfies that y, $|y'|^{p-2}y' \in C^1(\mathbb{R})$ and that $y \in C^2(\mathbb{R})$ if $1 < p \leq 2$. The following properties are crucial for the new function.

$$K_{2p,q}(k) \to \frac{\pi_{pq}}{2}, \quad \operatorname{sn}_{2p,q}(x,k) \to \sin_{pq} x \quad \text{as } k \to 1. \tag{2.1}$$

The former follows from easy calculation as

$$K_{2p,q}(k) = \int_0^1 \frac{dt}{\sqrt[2p]{(1-t^q)(1-k^q t^q)}} \to \int_0^1 \frac{dt}{\sqrt[p]{1-t^q}} = \frac{\pi_{pq}}{2}$$

and the latter is proved similarly.

Using $\operatorname{sn}_{pq}(x,k)$, for $x \in [-K_{pq}(k), K_{pq}(k)]$ we also define

$$\begin{aligned}\operatorname{cn}_{pq}(x,k) &= \sqrt[q]{1 - |\operatorname{sn}_{pq}(x,k)|^q},\\ \operatorname{dn}_{pq}(x,k) &= \sqrt[q]{1 - k^q|\operatorname{sn}_{pq}(x,k)|^q}.\end{aligned}$$

Clearly, $\operatorname{cn}_{pq}(x,k)$ and $\operatorname{dn}_{pq}(x,k)$ are even functions in x from $[-K_{pq}(k), K_{pq}(k)]$ to $[0,1]$. We extend the domains of $\operatorname{cn}_{pq}(x,k)$ and $\operatorname{dn}_{pq}(x,k)$ to $\mathbb{R}$ by $\operatorname{cn}_{pq}(x + 2K_{pq}(k), k) = -\operatorname{cn}_{pq}(x,k)$ and $\operatorname{dn}_{pq}(x + 2K_{pq}(k), k) = \operatorname{dn}_{pq}(x,k)$, respectively. This implies that $\operatorname{cn}_{pq}(x,k)$ and $\operatorname{dn}_{pq}(x,k)$ have $4K_{pq}(k)$- and $2K_{pq}(k)$-periodicity. We can see that $\operatorname{cn}_{pq}(x,0) = \cos_{pq} x$, $\operatorname{dn}_{pq}(x,0) = 1$. Moreover, $\operatorname{cn}_{2p,q}(x,k)$, $\operatorname{dn}_{2p,q}(x,k) \to \cos_{pq} x$ as $k \to 1$.

These functions satisfy, for $x \in [0, K_{pq}(k))$

$$\begin{aligned}\operatorname{cn}_{pq}^q(x,k) + \operatorname{sn}_{pq}^q(x,k) &= 1,\\ \operatorname{dn}_{pq}^q(x,k) + k^q \operatorname{sn}_{pq}^q(x,k) &= 1,\\ (\operatorname{sn}_{pq}(x,k))' &= \operatorname{cn}_{pq}^{q/p}(x,k)\operatorname{dn}_{pq}^{q/p}(x,k),\\ (\operatorname{cn}_{pq}(x,k))' &= -\operatorname{sn}_{pq}^{q-1}(x,k)\operatorname{cn}_{pq}^{q/p+1-q}(x,k)\operatorname{dn}_{pq}^{q/p}(x,k),\\ (\operatorname{dn}_{pq}(x,k))' &= -k^q \operatorname{sn}_{pq}^{q-1}(x,k)\operatorname{cn}_{pq}^{q/p}(x,k)\operatorname{dn}_{pq}^{q/p+1-q}(x,k).\end{aligned}$$

In our study, it is important that $y = \operatorname{sn}_{pq}(x,k)$ satisfies

$$(|y'|^{p-2}y')' + \frac{(p-1)q}{p}|y|^{q-2}y(1 + k^q - 2k^q|y|^q) = 0. \tag{2.2}$$

Defining (p,q)-elliptic integrals, we can define the generalized Jacobian elliptic functions in a different way.

For simplicity, let $\varphi \in [0, \pi_{pq}/2]$. Using the generalized trigonometric function, we define the (p,q)-elliptic integral of the first kind:

$$F_{pq}(k,\varphi) = \int_0^{\varphi} \frac{d\theta}{\sqrt[p]{1-k^q \sin_{pq}^q \theta}} = \int_0^{\sin_{pq}\varphi} \frac{dt}{\sqrt[p]{(1-t^q)(1-k^q t^q)}},$$

and the complete (p,q)-elliptic integral of the first kind:

$$K_{pq}(k) = F_{pq}\left(k, \frac{\pi_{pq}}{2}\right) = \int_0^{\frac{\pi_{pq}}{2}} \frac{d\theta}{\sqrt[p]{1-k^q \sin_{pq}^q \theta}} = \int_0^1 \frac{dt}{\sqrt[p]{(1-t^q)(1-k^q t^q)}}.$$

We define the (p,q)-amplitude function $\mathrm{am}_{pq}(x,k)$ for $x \in [0, K_{pq}(k)]$ by

$$x = \int_0^{\mathrm{am}_{pq}(x,k)} \frac{d\theta}{\sqrt[p]{1-k^q \sin_{pq}^q \theta}}$$

and we define

$$\begin{aligned}
\mathrm{sn}_{pq}(x,k) &= \sin_{pq}(\mathrm{am}_{pq}(x,k)),\\
\mathrm{cn}_{pq}(x,k) &= \cos_{pq}(\mathrm{am}_{pq}(x,k)),\\
\mathrm{dn}_{pq}(x,k) &= \sqrt[q]{1-k^q \sin_{pq}^q (\mathrm{am}_{pq}(x,k))}.
\end{aligned}$$

We also define the (p,q)-elliptic integrals of the second- and the third kinds by

$$\begin{aligned}
E_{pq}(k,\varphi) &= \int_0^{\varphi} \sqrt[p]{1-k^q \sin_{pq}^q \theta} = \int_0^{\sin_{pq}\varphi} \sqrt[p]{\frac{1-k^q t^q}{1-t^q}}\,dt,\\
\Pi_{pq}(k,n,\varphi) &= \int_0^{\varphi} \frac{d\theta}{(1+n\sin_{pq}^q \theta)\sqrt[p]{1-k^q \sin_{pq}^q \theta}}\\
&= \int_0^{\sin_{pq}\varphi} \frac{dt}{(1+nt^q)\sqrt[p]{(1-t^q)(1-k^q t^q)}},
\end{aligned}$$

and the corresponding complete (p,q)-elliptic integrals by

$$E_{pq}(k) = E_{pq}\left(k, \frac{\pi_{pq}}{2}\right) = \int_0^{\frac{\pi_{pq}}{2}} \sqrt[p]{1 - k^q \sin_{pq}^q \theta} = \int_0^1 \sqrt[p]{\frac{1-k^qt^q}{1-t^q}}\, dt,$$

$$\begin{aligned}\Pi_{pq}(k,n) &= \Pi_{pq}\left(k, n, \frac{\pi_{pq}}{2}\right) \\ &= \int_0^{\frac{\pi_{pq}}{2}} \frac{d\theta}{(1+n\sin_{pq}^q\theta)\sqrt[p]{1-k^q\sin_{pq}^q\theta}} \\ &= \int_0^1 \frac{dt}{(1+nt^q)\sqrt[p]{(1-t^q)(1-k^qt^q)}}.\end{aligned}$$

Finally, we show the relation between the generalized functions in Tables 1 and 2, where $\phi_s(u) = |u|^{s-2}u$ and $\Delta_s u = (\phi_s(u'))'$ and $C = (p-1)q/p$.

	$k \to 0$	$0 < k < 1$
$p,\ q > 1$	$\sin_{pq} x$ $\Delta_p u + C\phi_q(u) = 0$	$\mathrm{sn}_{pq}(x,k)$ $\Delta_p u + C\phi_q(u)(1+k^q-2k^q\lvert u\rvert^q) = 0$
$p = q = 2$	$\sin x$ $u'' + u = 0$	$\mathrm{sn}\,(x,k)$ $u'' + u(1+k^2-2k^2u^2) = 0$

Table 1: $k \to 0$

	$0 < k < 1$	$k \to 1$
$p,\ q > 1$	$\mathrm{sn}_{pq}(x,k)$ $\Delta_p u + C\phi_q(u)(1+k^q-2k^q\lvert u\rvert^q) = 0$	$\sin_{p/2,q} x$ $\Delta_p u + 2C\phi_q(u)(1-\lvert u\rvert^q) = 0$
$p = q = 2$	$\mathrm{sn}\,(x,k)$ $u'' + u(1+k^2-2k^2u^2) = 0$	$\tanh x$ $u'' + 2u(1-u^2) = 0$

Table 2: $k \to 1$

3 Application to Bistable Problem

The results of this section have been obtained in [24].

Let L, $\lambda > 0$ and p, $q > 1$. We consider the following (p, q)-bistable problem with p-Laplacian.

$$\begin{cases} (\phi_p(u'))' + \lambda\phi_q(u)(1 - |u|^q) = 0, & x \in (0, L), \\ u(0) = u(L) = 0, \end{cases} \tag{B_{pq}}$$

where $\phi_p(s) = |s|^{p-2}s$.

Problem (B_{pq}) has been studied by Berger and Fraenkel [1] and Chafee and Infante [5] ($p = q = 2$), Wang and Kazarinoff [27] and Korman, Li and Ouyang [17] ($p = 2 < q$), Guedda and Véron [14] ($p = q > 1$), and Takeuchi and Yamada [25] ($p > 2$, $q \geq 2$). However, there is no study providing explicit forms of the values of bifurcation parameter and the corresponding solutions for (B_{pq}).

3.1 Solutions of (B_{pq})

We follow closely the ideas of Drábek and Manásevich [11]. It will be convenient to find first the solution to the initial value problem

$$\begin{cases} (\phi_p(u'))' + \lambda\phi_q(u)(1 - |u|^q) = 0, \\ u(0) = 0, \ u'(0) = \alpha, \end{cases} \tag{3.1}$$

where without loss of generality we may assume $\alpha > 0$. Eq. (2.2) reminds that the solutions of Problem (3.1) can be represented by using $\mathrm{sn}_{pq}(x, k)$.

Let u be a solution to Eq. (3.1) and let $X(\alpha)$ be the first zero point of $u'(x)$. On interval $(0, X(\alpha))$, u satisfies $u(x) > 0$ and $u'(x) > 0$, and thus

$$\frac{u'(x)^p}{p'} + \lambda\frac{F(u)}{q} = \lambda\frac{F(R)}{q} = \frac{\alpha^p}{p'},$$

where $p' = \frac{p}{p-1}$, $F(s) = s^q - \frac{1}{2}s^{2q}$ and $R = u(X(\alpha))$. Since we are interested in functions satisfying the boundary condition of (B_{pq}), it suffices to assume $0 < R \leq 1$, which means $|u| \leq 1$. Moreover, we restrict to $0 < R < 1$ and concentrate solutions satisfying $|u| < 1$ for a while.

Solving for u' and integrating, we find

$$\left(\frac{q}{\lambda p'}\right)^{\frac{1}{p}} \int_0^x \frac{u'(s)}{\sqrt[p]{F(R) - F(u(s))}}\, ds = x,$$

which after a change of variable can be written as

$$x = \left(\frac{q}{\lambda p'}\right)^{\frac{1}{p}} \int_0^{\frac{u(x)}{R}} \frac{R}{\sqrt[p]{F(R) - F(Rs)}}\, ds. \tag{3.2}$$

It is easy to verify that

$$F(R) - F(Rs) = F(R)(1 - s^q)\left(1 - \frac{R^q}{2 - R^q} s^q\right),$$

and hence

$$\begin{aligned} x &= \left(\frac{q}{\lambda p'}\right)^{\frac{1}{p}} \frac{R}{F(R)^{\frac{1}{p}}} \int_0^{\frac{u(x)}{R}} \frac{ds}{\sqrt[p]{(1 - s^q)(1 - k^q s^q)}} \qquad \left(k^q := \frac{R^q}{2 - R^q}\right) \\ &= \left(\frac{q}{\lambda p'}\right)^{\frac{1}{p}} \frac{R}{F(R)^{\frac{1}{p}}} \operatorname{sn}_{pq}^{-1}\left(\frac{u(x)}{R}, k\right). \end{aligned}$$

Then we obtain that the solution to Eq. (3.1) can be written as

$$u(x) = R \operatorname{sn}_{pq}\left(\left(\frac{\lambda p'}{q}\right)^{\frac{1}{p}} \frac{F(R)^{\frac{1}{p}}}{R} x, k\right), \tag{3.3}$$

where

$$k = \left(\frac{R^q}{2 - R^q}\right)^{\frac{1}{q}}. \tag{3.4}$$

We first observe the structure of the set of all nontrivial solutions of (B_{pq}) satisfying $|u| < 1$.

Theorem 3.1 ($|u| < 1$). *All nontrivial solutions of* (B_{pq}) *for* $p \in (1, 2]$ *and all nontrivial solutions of* (B_{pq}) *with* $|u| < 1$ *for* $p > 2$ *are given as follows. For any given* $k \in (0, 1)$*, the value of bifurcation parameter* λ *of* (B_{pq}) *is given by*

$$\lambda_n(k) = \frac{q}{p'}(1 + k^q)\left(\frac{2k^q}{1 + k^q}\right)^{\frac{p}{q} - 1}\left(\frac{2nK_{pq}(k)}{L}\right)^p \tag{3.5}$$

for each $n \in \mathbb{N}$*, with corresponding solutions* $\pm u_{n,k}$*, where*

$$u_{n,k}(x) = \left(\frac{2k^q}{1 + k^q}\right)^{\frac{1}{q}} \operatorname{sn}_{pq}\left(\frac{2nK_{pq}(k)}{L} x, k\right). \tag{3.6}$$

Proof. For $k \in (0,1)$ given, we impose that Function (3.3) with $R \in (0,1)$, where R is uniquely decided from Eq. (3.4), satisfies the boundary conditions in (B_{pq}). Then, we obtain

$$\left(\frac{\lambda p'}{q}\right)^{\frac{1}{p}} \frac{F(R)^{\frac{1}{p}}}{R} L = 2nK_{pq}(k), \quad n \in \mathbb{N},$$

where from Eq. (3.4)

$$\frac{F(R)^{\frac{1}{p}}}{R} = \left(\frac{2k^q}{1+k^q}\right)^{\frac{1}{p}-\frac{1}{q}} (1+k^q)^{-\frac{1}{p}}.$$

Thus λ is given by Eq. (3.5). Expression (3.6) for the solutions follows then directly from Eq. (3.3).

It remains to show that no other nontrivial solution of (B_{pq}) is obtained when $1 < p \leq 2$. Assume the contrary. Then there exist $x_* > 0$ and a nontrivial solution u of (B_{pq}) with $R = u(x_*) = 1$. However, the right-hand side of Eq. (3.2) with $x = x_*$ diverges because $\sqrt[p]{F(1) - F(s)} = O((1-s^q)^{\frac{2}{p}})$ as $s \to 1-0$. Thus, $x_* = \infty$, which is a contradiction. $\square$

Next we find solutions of (B_{pq}) with $|u| \leq 1$, except the solutions given by Theorem 3.1. Now we assume $p > 2$. From Property (2.1), one of solutions of Eq. (3.1) can be obtained by $k \to 1-0$ in Eq. (3.3) with Eq. (3.4), namely

$$u(t) = \sin_{\frac{p}{2},q}\left(\left(\frac{\lambda p'}{2q}\right)^{\frac{1}{p}} x\right).$$

We take a number x_* as $(\frac{\lambda p'}{2q})^{\frac{1}{p}} x_* = \pi_{\frac{p}{2},q}/2$, then u attains 1 at $x = x_*$ (note that x_* is well-defined if and only if $p > 2$). Using this u, we can make the other solutions of Eq. (3.1) as follows. In the phase-plane, the orbit $(u(x), u'(x))$ arrives at the equilibrium point $(1,0)$ at $x = x_*$ and can stay there for any finite time ℓ before it begins to leave there. Then, the interval $[x_*, x_* + \ell]$ is a flat core of the solution. Similarly, there is the other equilibrium point $(-1,0)$, where the orbit can stay, and the solution has another flat core of any finite length. Thus we have solutions of Eq. (3.1) attaining ± 1 with any number of flat cores.

Theorem 3.2 ($|u| \le 1$). *Let $p > 2$, then all nontrivial solutions of* (B_{pq}) *without $|u| < 1$ (that is, $|u|$ attains 1) are given as follows. For any given $\ell \in [0, L)$, the value of bifurcation parameter λ of* (B_{pq}) *is given by*

$$\Lambda_n(\ell) = \frac{2q}{p'}\left(\frac{n\pi_{\frac{p}{2},q}}{L-\ell}\right)^p$$

for each $n \in \mathbb{N}$, with corresponding sets $\pm U_{n,\ell}$ of solutions, where $U_{n,\ell}$ consists of all functions given as follows: for any $\{\ell_i\}_{i=1}^n$ with $\ell_i \ge 0$ and $\sum_{i=1}^n \ell_i = \ell$

$$u(x) = \begin{cases} (-1)^{j-1}\sin_{\frac{p}{2},q}\left(\frac{n\pi_{\frac{p}{2},q}}{L-\ell}(x - L_{j-1})\right) & \text{if } L_{j-1} \le x \le L_{j-1} + \frac{L-\ell}{2n}, \\ (-1)^{j-1} & \text{if } L_{j-1} + \frac{L-\ell}{2n} \le x \le L_j - \frac{L-\ell}{2n}, \\ (-1)^{j-1}\sin_{\frac{p}{2},q}\left(\frac{n\pi_{\frac{p}{2},q}}{L-\ell}(L_j - x)\right) & \text{if } L_j - \frac{L-\ell}{2n} \le x \le L_j, \\ j = 1, 2, \dots, n, \end{cases} \tag{3.7}$$

where $L_0 = 0$ and $L_j = \frac{(L-\ell)j}{n} + \sum_{i=1}^j \ell_i$ for $j = 1, 2, \dots, n$. (Figure 1)

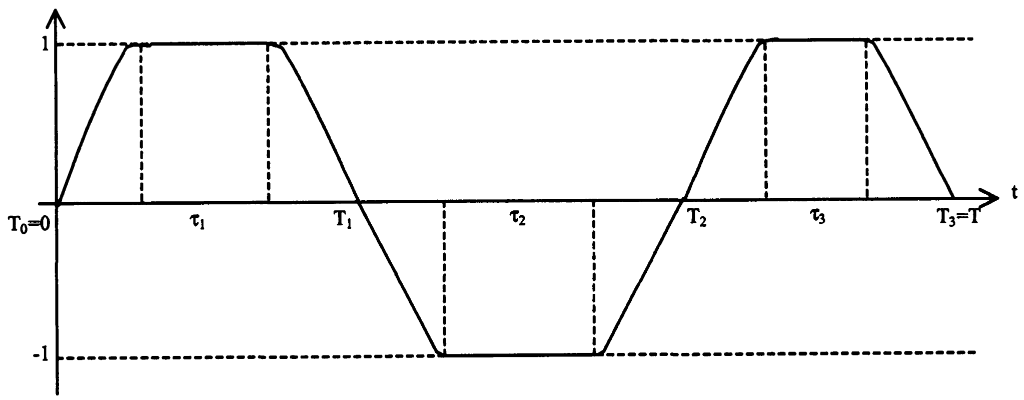

Figure 1: A solution in $U_{3,\ell}$. It has 3-flat cores with total length of ℓ.

Proof. For each $n \in \mathbb{N}$, it suffices to construct solutions with $(n-1)$-zeros. Let $\ell \in [0, L)$. They are all generated by the value of bifurcation parameter and the corresponding solution of (B_{pq}) with L replaced by $L-\ell$, namely,

$$\Lambda_n(\ell) = \frac{2q}{p'}\left(\frac{n\pi_{\frac{p}{2},q}}{L-\ell}\right)^p,$$
$$u_{n,\ell}(x) = \sin_{\frac{p}{2},q}\left(\frac{n\pi_{\frac{p}{2},q}}{L-\ell}x\right),$$

which are obtained from Eqs. (3.5) and (3.6) with $k \to 1-0$, respectively. In the phase-plane, the orbit $(u_{n,\ell}(x), u'_{n,\ell}(x))$ goes through the equilibrium points $(\pm 1, 0)$ in n-times without staying there as x increases from 0 to $L-\ell$. Therefore, if the orbit stays the i-th equilibrium point for time ℓ_i, where $\ell_1 + \ell_2 + \cdots + \ell_n = \ell$, then we can obtain Solution (3.7) with n-flat cores in $[0, L]$. □

We notice that for any $\ell \in [0, L)$, the sets $\pm U_{n,\ell}$ consist of solutions of (B_{pq}) having n-flat cores with total length of ℓ. Clearly, $\pm U_{n,\ell}$ are continua for $\ell > 0$ and singletons for $\ell = 0$. We call the (unique) elements of $\pm U_{n,0}$ special solutions $\pm U_n(x)$, respectively.

In Theorems 3.1 and 3.2, we gave parameters k and ℓ to obtain the value of bifurcation parameter and the corresponding solution of (B_{pq}). Conversely, giving any $\lambda > 0$, we can observe the set S_λ of all solutions of (B_{pq}) by considering the inverses of λ_n and Λ_n.

Theorem 3.3. *Let $p > 1$ and $q > 1$.*

Case $p > q$ (Figure 2). For any $\lambda > 0$ there exists a strictly decreasing positive sequence $\{k_j\}_{j=1}^{\infty}$ such that $k_j \to 0$ as $j \to \infty$ and

$$S_\lambda = \{0\} \cup \bigcup_{j=1}^{\infty} \{\pm u_{j,k_j}\}.$$

Case $p = q$ (Figure 3). Let $\lambda_n = (p-1)(n\pi_p/L)^p$ with $\pi_p = 2\pi/(p\sin(\pi/p))$. If $0 < \lambda \le \lambda_1$, then $S_\lambda = \{0\}$. If $\lambda_n < \lambda \le \lambda_{n+1}$, $n \in \mathbb{N}$, then there exists a strictly decreasing positive sequence $\{k_j\}_{j=1}^{n}$ such that

$$S_\lambda = \{0\} \cup \bigcup_{j=1}^{n} \{\pm u_{j,k_j}\}.$$

Case $p < q$ (Figure 4). There exists $\lambda_* > 0$ such that if $0 < \lambda < \lambda_*$, then $S_\lambda = \{0\}$. If $n^p\lambda_* \le \lambda < (n+1)^p\lambda_*$, $n \in \mathbb{N}$, then there exist a *strictly decreasing positive sequence* $\{k_j\}_{j=1}^n$ *and a strictly increasing positive sequence* $\{l_j\}_{j=1}^n$ *such that* $k_j > l_j$, $j = 1, 2, \dots, n-1$ *and*

$$S_\lambda = \{0\} \cup \bigcup_{j=1}^{n} \{\pm u_{j,k_j}\} \cup \bigcup_{j=1}^{n} \{\pm u_{j,l_j}\},$$

where $u_{n,k_n} = u_{n,l_n}$ *with* $k_n = l_n$ *for* $\lambda = n^p\lambda_1$ *and* $|u_{n,k_n}| > |u_{n,l_n}|$ $(x \neq jL/n,\ j = 1, 2, \dots, n-1)$ *with* $k_n > l_n$ *otherwise.*

In any case, each k_j, l_j *is calculated by Eq.* (3.5) *for* λ *and* j, *and the corresponding solution is given in Form* (3.6).

When $1 < p \leq 2$, *we have* $k_j < 1$ *(i.e.,* $\pm u_{j,k_j}$ *have no flat core).*

When $p > 2$, *in addition, if*

$$\lambda \geq \frac{2q}{p'} \left(\frac{m\pi_{\frac{p}{2},q}}{L} \right)^p, \quad m \in \mathbb{N}, \tag{3.8}$$

then for each $j = 1, 2, \dots, m$, *the set* $\{\pm u_{j,k_j}\}$ *in* S_λ *above is replaced by* $\pm U_{j,\ell}$, *where*

$$\ell = L - j\pi_{\frac{p}{2},q} \left(\frac{2q}{\lambda p'} \right)^{\frac{1}{p}}. \tag{3.9}$$

Proof. First we assume $1 < p \leq 2$. In this case, we have already known that all nontrivial solutions of (B_{pq}) are obtained by Theorem 3.1.

Now we fix $\lambda > 0$. We obtain that λ is the j-th smallest value for which (B_{pq}) has a solution if and only if from Eq. (3.5) there exists $k \in (0, 1)$ such that $\lambda = \lambda_j(k)$, that is,

$$\Phi(k) = c(\lambda), \tag{3.10}$$

where

$$\Phi(k) = (1 + k^q)^{\frac{1}{p}} \left(\frac{2k^q}{1 + k^q} \right)^{\frac{1}{q} - \frac{1}{p}} K_{pq}(k),$$
$$c(\lambda) = \frac{L}{2j} \left(\frac{\lambda p'}{q} \right)^{\frac{1}{p}}.$$

Case $p > q$. $\Phi(k)$ is strictly increasing in $(0, 1)$ and $\Phi(0) = 0$ and $\lim_{k \to 1-0} \Phi(k) = \infty$. Thus, there exists a unique $k = k_j(\lambda)$ satisfying

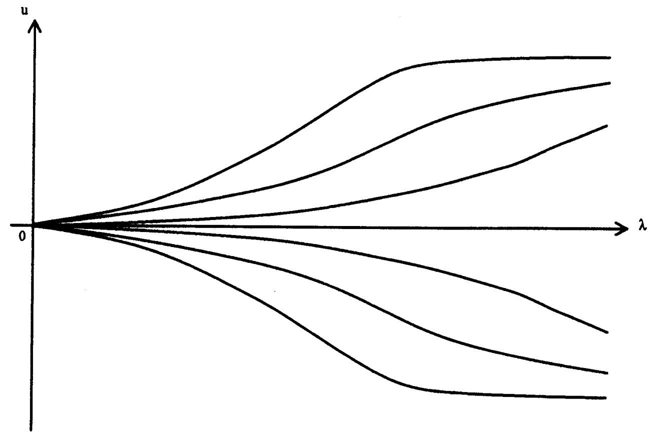

Figure 2: Bifurcation diagram for $p > q$.

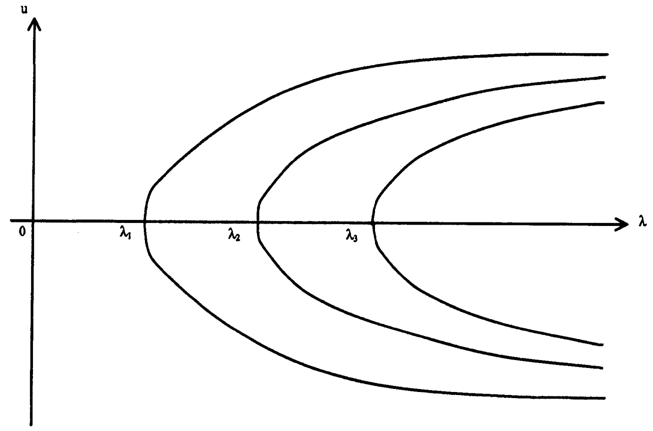

Figure 3: Bifurcation diagram for $p = q$, where $\lambda_n = (p-1)\left(\frac{n\pi_p}{T}\right)^p$.

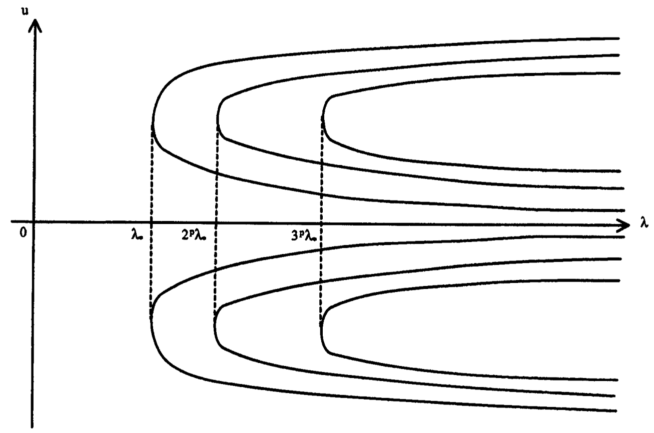

Figure 4: Bifurcation diagram for $p < q$.

Eq. (3.10). For j and k_j, a unique solution u_{j,k_j} of (B_{pq}) is obtained by Eq. (3.6).

Case $p = q$. $\Phi(k)$ is strictly increasing in $(0,1)$ and $\Phi(0) = \pi_p/2$ and $\lim_{k\to 1-0}\Phi(k) = \infty$. Thus, if $c(\lambda) > \pi_p/2$, namely, $\lambda > \lambda_j$, then there exists a unique $k = k_j(\lambda)$ satisfying Eq. (3.10). For j and k_j, a unique solution u_{j,k_j} of (B_{pq}) is obtained by Eq. (3.6).

Case $p < q$. It is clear that $\lim_{k\to +0}\Phi(k) = \lim_{k\to 1-0}\Phi(k) = \infty$. Changing variable $r = \frac{k^q}{1+k^q}$, we can write Φ as

$$\Psi(r) = \int_0^1 \frac{(1+s^q)^{\frac{1}{p}-\frac{1}{q}}}{(1-s^q)^{\frac{1}{p}}}\psi((1+s^q)r)\,ds, \quad r \in (0,1/2),$$

where $\psi(t) = (2t)^{\frac{1}{q}-\frac{1}{p}}(1-t)^{-\frac{1}{p}}$. It is easy to see that ψ is convex in $(0,1)$ because $\psi(t) > 0$ and

$$(\log\psi(t))'' = \left(\frac{1}{p}-\frac{1}{q}\right)\frac{1}{t^2} + \frac{1}{p}\frac{1}{(1-t)^2} > 0.$$

Then, Ψ is twice-differentiable in $(0,1/2)$ and

$$\Psi''(r) = \int_0^1 \frac{(1+s^q)^{\frac{1}{p}-\frac{1}{q}+2}}{(1-s^q)^{\frac{1}{p}}}\psi''((1+s^q)r)\,ds > 0.$$

Thus, Ψ is convex and there exists $k_* \in (0,1)$ such that $\Phi(k_*)$ is the only one critical value, and hence the minimum of Φ in $(0,1)$.

If $c(\lambda) = \Phi(k_*)$, namely, $\lambda = j^p\lambda_*$, where $\lambda_* = (2\Phi(k_*)/L)^p q/p'$, then k_* satisfies Eq. (3.10). For j and k_*, a unique solution u_{j,k_*} of (B_{pq}) is obtained by Eq. (3.6). If $c(\lambda) > \Phi(k_*)$, namely, $\lambda > j^p\lambda_*$, then there exist $k = k_j(\lambda)$ and $l_j(\lambda)$ such that $k_j(\lambda) = \Phi^{-1}(c(\lambda)) \in (k_*,1)$, $l_j(\lambda) = \Phi^{-1}(c(\lambda)) \in (0,k_*)$. For j, k_j and l_j, solutions u_{j,k_j} and u_{j,l_j} of (B_{pq}) are obtained by Eq. (3.6) (Figure 5).

Next, we assume $p > 2$. In any case, a similar proof as above with $\lim_{k\to 1-0}\Phi(k) = 2^{\frac{1}{p}-1}\pi_{\frac{p}{2},q}$ instead of $\lim_{k\to 1-0}\Phi(k) = \infty$ implies that it is impossible to find $k_m \in (0,1)$ above satisfying Eq. (3.10), provided λ satisfies Inequality (3.8). Then, however, for each $j = 1,2,\ldots,m$, we can take $\ell \in [0,L)$ as Eq. (3.9) so that $\lambda = \Lambda_j(\ell)$. Therefore, Theorem 3.2 yields the set $U_{n,\ell}$ of solutions (Figure 6). $\square$

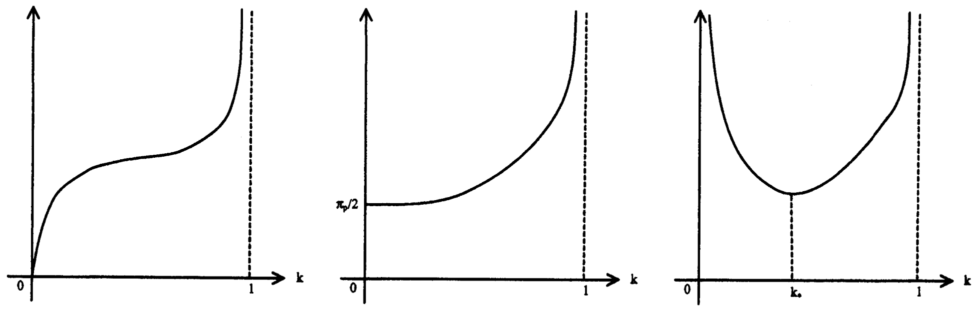

Figure 5: Graphs of $\Phi(k)$ for $p > q$, $p = q$ and $p < q$ in case $1 < p \le 2$.

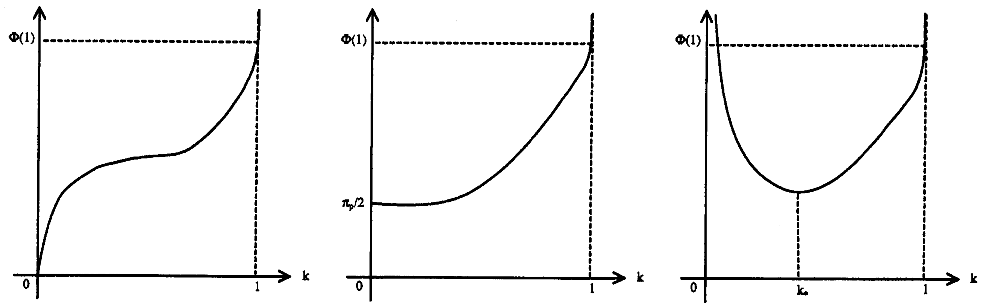

Figure 6: Graphs of $\Phi(k)$ for $p > q$, $p = q$ and $p < q$ in case $p > 2$, where $\Phi(1) = 2^{\frac{1}{p}-1}\pi_{\frac{p}{2},q}$.

3.2 Property of Special Solutions of (B_{pq})

It follows directly from Representation (3.7) of Theorem 3.2 that the special solutions $\pm U_n$ of (B_{pq}) for p-Laplacian are also solutions of $(\mathrm{E}_{\frac{p}{2},q})$, eigenfunctions of $p/2$-Laplacian. That is,

Corollary 3.1. *Let $p > 2$. For each $n \in \mathbb{N}$ and $\ell \in [0, L)$, any solution in $\pm U_{n,\ell}$ of (B_{pq}) also satisfies*

$$(\phi_{\frac{p}{2}}(u'))' + \frac{(p-2)q}{p}\left(\frac{n\pi_{\frac{p}{2},q}}{L-\ell}\right)^{\frac{p}{2}}\phi_q(u) = 0$$

in the intervals where $|u| < 1$. In particular, for each $n \in \mathbb{N}$, the special *solutions* $\pm U_n$ *of* (B_{pq}) *are also solutions of* $(\mathrm{E}_{\frac{p}{2},q})$, *that is,*

$$\begin{cases} (\phi_{\frac{p}{2}}(u'))' + \frac{(p-2)q}{p}\left(\frac{n\pi_{\frac{p}{2},q}}{L}\right)^{\frac{p}{2}}\phi_q(u) = 0, & x \in (0, L), \\ u(0) = u(L) = 0, \end{cases}$$

and hence, the solution is characterized by $\pm u_{n,R}$ *with* $R = 1$ *in Solution* (1.8) *with* p *replaced by* $p/2$.

We will give an example of Corollary. It follows from Theorem 3.2 that the $(4, 2)$-bistable problem

$$\begin{cases} (|u'|^2u')' + 3\pi^4 u(1 - |u|^2) = 0, \ 0 < x < 1, \\ u(0) = u(1) = 0, \ u \not\equiv 0 \end{cases} \tag{B$_{42}$}$$

has a special solution $U_1(x) = \sin_{\frac{4}{2},2}\pi x = \sin \pi x$. Clearly, U_1 also satisfies the $(2, 2)$-eigenvalue problem

$$\begin{cases} u'' + \pi^2 u = 0, \ 0 < x < 1, \\ u(0) = u(1) = 0, \ u \not\equiv 0. \end{cases} \tag{E$_{22}$}$$

In the above, we have seen that the special solutions $\pm U_n$ of (p, q)-bistable problem are also the $(p/2, q)$-eigenfunctions if and only if $p > 2$. We can also another formal explanation why the reduction arises.

For simplicity, we replace p by $2p$ and show the reduction from $2p$ to p for $p > 1$. The special solution of $(2p, q)$-bistable problem satisfies

$$(\phi_{2p}(y'))' + \frac{(2p-1)q}{p}\phi_q(y)(1 - |y|^q) = 0 \tag{3.11}$$

and attains the maximum 1 for some $x > 0$. Therefore, we have

$$\frac{2p-1}{2p}|y'|^{2p} + \frac{2p-1}{p}(|y|^q - \frac{1}{2}|y|^{2q}) = \frac{2p-1}{2p},$$

which implies

$$1 - |y|^q = |y'|^p.$$

Moreover, formal calculation gives that $(\phi_{2p}(y'))' = (2p-1)|y'|^{2p-2}y''$. Thus, canceling $|y'|^p$ in Eq. (3.11), we have

$$(2p-1)|y'|^{p-2}y'' + \frac{(2p-1)q}{p}\phi_q(y) = 0,$$

which implies the (p,q)-eigenvalue problem:

$$(\phi_p(y'))' + \frac{(p-1)q}{p}\phi_q(y) = 0.$$

4 Further Generalization

We will try to extend Jacobian elliptic functions more generally.

Let $p,\ q,\ r > 1$. For any $k \in [0,1)$ we define $K_{pqr}(k)$ as

$$K_{pqr}(k) = \int_0^1 \frac{dt}{\sqrt[p]{1-t^q}\sqrt[r]{1-k^q t^q}}.$$

Then, for any $k \in [0,1)$ and any $x \in [-K_{pqr}(k), K_{pqr}(k)]$ we define $\mathrm{sn}_{pqr}(x,k)$ as an inverse of the incomplete elliptic integral of the first kind.

$$x = \int_0^{\mathrm{sn}_{pqr}(x,k)} \frac{dt}{\sqrt[p]{1-|t|^q}\sqrt[r]{1-k^q|t|^q}}.$$

We can also give another definition. For simplicity, let $\varphi \in [0, \pi_{pq}/2]$. Using the generalized trigonometric function, we define the (p,q,r)-elliptic integral of the first kind:

$$F_{pqr}(k,\varphi) = \int_0^{\varphi} \frac{d\theta}{\sqrt[r]{1-k^q \sin_{pq}^q \theta}} = \int_0^{\sin_{pq}\varphi} \frac{dt}{\sqrt[p]{1-t^q}\sqrt[r]{1-k^q t^q}},$$

and the complete (p,q,r)-elliptic integral of the first kind:

$$K_{pqr}(k) = F_{pqr}\left(k, \frac{\pi_{pq}}{2}\right) = \int_0^{\frac{\pi_{pq}}{2}} \frac{d\theta}{\sqrt[r]{1-k^q \sin_{pq}^q \theta}} = \int_0^1 \frac{dt}{\sqrt[p]{1-t^q}\sqrt[r]{1-k^q t^q}}.$$

We define the (p,q,r)-amplitude function $\mathrm{am}_{pqr}(x,k)$ for $x \in [0, K_{pqr}(k)]$ as

$$x = \int_0^{\mathrm{am}_{pqr}(x,k)} \frac{d\theta}{\sqrt[r]{1-k^q \sin_{pq}^q \theta}}$$

and

$$\mathrm{sn}_{pqr}(x,k) = \sin_{pq}(\mathrm{am}_{pqr}(x,k)).$$

We can see that $\mathrm{sn}_{pqr}(x,k) = \mathrm{sn}_{pq}(x,k)$ if $p = r$ and that $K_{pqr}(0) = \pi_{pq}/2$, $\mathrm{sn}_{pqr}(x,0) = \sin_{pq} x$. We have

$$K_{2p,q,2r}(k) \to \frac{\pi_{sq}}{2}, \quad \mathrm{sn}_{2p,q,2r}(x,k) \to \sin_{sq} x \quad \text{as } k \to 1,$$

where s is the harmonic mean of p and r, i.e., $2/s = 1/p + 1/r$. The former follows from easy calculation as

$$K_{2p,q,2r}(k) = \int_0^1 \frac{dt}{\sqrt[2p]{1-t^q}\,\sqrt[2r]{1-k^q t^q}} \to \int_0^1 \frac{dt}{\sqrt[s]{1-t^q}} = \frac{\pi_{sq}}{2}.$$

Moreover, the function $y = \mathrm{sn}_{pqr}(x,k)$ satisfies

$$(|y'|^{p-2}u')' + \frac{(p-1)q}{p}|y|^{q-2}y(1-k^q|y|^q)^{p/r-1}(1+\frac{p}{r}k^q - (1+\frac{p}{r})k^q|y|^q) = 0.$$

Thus, $y = \mathrm{sn}_{2p,q,2r}(x,k)$ satisfies

$$(|y'|^{2p-2}u')' + \frac{(2p-1)q}{2p}|y|^{q-2}y(1-k^q|y|^q)^{p/r-1}(1+\frac{p}{r}k^q - (1+\frac{p}{r})k^q|y|^q) = 0.$$

By letting $k \to 1$ in above and Eq. (1.6), we see that the function $y = \sin_{sq} x$ $(2/s = 1/p + 1/r)$ satisfies the following two equations simultaneously.

$$(|y'|^{2p-2}y')' + \frac{(2p-1)q}{s}|y|^{q-2}y(1-|y|^q)^{p/r} = 0,$$
$$(|y'|^{s-2}y')' + \frac{(s-1)q}{s}|y|^{q-2}y = 0.$$

We have seen the case $p = r = s$ in the previous section.

5 Topics in the Generalized Functions

5.1 Isoperimetric Problem

The isoperimetric property of the circle is well-known among scientists. Of all curves with the same length, the circle has the largest area.

We replace the Euclidean metric with the p-metric according to which the distance between two points is measured as

$$\sqrt[p]{|x_2 - x_1|^p + |y_2 - y_1|^p}.$$

Then, the q-circle with radius R and centre (x_0, y_0) has the equation

$$|x - x_0|^q + |y - y_0|^q = R^q.$$

The problem is to find a curve which has the largest area, of all curves with the same p-length, that is,

$$A = \frac{1}{2}\oint (x\,dy - y\,dx) = \text{Maximum};$$

$$L_p = \int (|x'(t)|^p + |y'(t)|^p)^{1/p}\,dt = \text{Constant}.$$

Theorem 5.1 ([20]). *Among all closed curves of the same p-length, the q-circle encloses the largest area where $q = p' = \dfrac{p}{p-1}$.*

Several of the proofs in the literature of the classical isoperimetric inequality can be adapted to the present more general situation. The isoperimetric inequality reads

$$L_p^2 \geq 4\omega_q A, \quad \omega_q = \pi_{pq} = \frac{2\Gamma(1/q)^2}{q\Gamma(2/q)}. \tag{5.1}$$

The case $p = q = 2$ reduces to the familiar

$$L^2 \geq 4\pi A.$$

Lindqvist and Peetre [20] (see also [18]) calculate the arc-length and area of q-circle, using generalized trigonometric functions. Let us turn our attention to the unit q-circle, where q is conjugate to p. The first quarter of it can be parametrized by

$$x = \cos_{pq} t, \quad y = \sin_{pq} t,$$

where $0 \leq t \leq \pi_{pq}/2$. Then, noting that

$$\begin{aligned} x'(t) &= -\sin_{pq}^{q-1} t \cos_{pq}^{q/p+1-q} t = -\sin_{pq}^{q-1} t, \\ y'(t) &= \cos_{pq}^{q/p} t = \cos_{pq}^{q-1} t, \end{aligned}$$

we have

$$\begin{aligned} \int_0^t (|x'(t)|^p + |y'(t)|^p)^{1/p}\,dt &= \int_0^t (|-\sin_{pq}^{q-1} t|^p + |\cos_{pq}^{q-1} t|^p)^{1/p}\,dt \\ &= \int_0^t (\sin_{pq}^q t + \cos_{pq}^q t)^{1/p}\,dt \\ &= \int_0^t dt = t. \end{aligned}$$

Moreover,

$$\begin{aligned}\frac{1}{2}\oint(x\,dy-y\,dx)&=\frac{1}{2}\int_0^t(\cos_{pq}t\cos_{pq}^{q-1}t-\sin_{pq}t\cdot(-\sin_{pq}^{q-1}t))\,dt\\&=\frac{1}{2}\int_0^t(\cos_{pq}^q t+\sin_{pq}^q t)\,dt\\&=\frac{1}{2}\int_0^t dt=\frac{1}{2}t.\end{aligned}$$

Thus, q-circle attains the equality of Isoperimetric Inequality (5.1), indeed

$$\begin{aligned}L_p^2&=\left(4\cdot\frac{\pi_{pq}}{2}\right)^2=4\pi_{pq}^2,\\4\pi_{pq}A&=4\pi_{pq}\cdot4\cdot\frac{\pi_{pq}}{4}=4\pi_{pq}^2.\end{aligned}$$

5.2 Superellipse

In a similar way to the above, we can obtain the p-length L_p and area A of q-superellipse or q-Lamé curve

$$\left|\frac{x}{a}\right|^q+\left|\frac{y}{b}\right|^q=1,$$

where $a\geq b>0$ and $q=p'=\frac{p}{p-1}$ for $p>1$.

The first quarter of q-superellipse can be parametrized by

$$x=a\cos_{pq}t,\quad y=b\sin_{pq}t,$$

where $0\leq t\leq\pi_{pq}/2$. Then, as above, we have

$$\begin{aligned}\int_0^t(|x'(t)|^p+|y'(t)|^p)^{1/p}\,dt&=\int_0^t(|-a\sin_{pq}^{q-1}t|^p+|b\cos_{pq}^{q-1}t|^p)^{1/p}\,dt\\&=\int_0^t(a^p\sin_{pq}^q t+b^p\cos_{pq}^q t)^{1/p}\,dt\\&=a\int_0^t\left(1-k^q\sin_{pq}^q t\right)^{1/p}\,dt\\&=aE_{pq}(k,t),\end{aligned}$$

where $k^q = (a^p - b^p)/a^p$ and $E_{pq}(k,t)$ is the (p,q)-elliptic integral of the second kind. Moreover,

$$\begin{aligned}\frac{1}{2}\oint(x\,dy - y\,dx) &= \frac{1}{2}\int_0^t (a\cos_{pq} t\cdot b\cos_{pq}^{q-1} t - b\sin_{pq} t\cdot(-a\sin_{pq}^{q-1} t))\,dt\\ &= \frac{ab}{2}\int_0^t (\cos_{pq}^q t + \sin_{pq}^q t)\,dt\\ &= \frac{ab}{2}\int_0^t dt = \frac{ab}{2}t.\end{aligned}$$

Thus,

$$\begin{aligned}L_p &= 4aE_{pq}\left(k, \frac{\pi_{pq}}{2}\right) = 4aE_{pq}(k),\\ A &= 4\cdot\frac{ab}{2}\cdot\frac{\pi_{pq}}{2} = \pi_{pq}ab.\end{aligned}$$

Remark 5.1. We do not use $q = p'$ to obtain A.

5.3 Catalan's Constant

Bushell and Edmunds [4] (see also [18]) have obtained a new representation of Catalan's constant G, which is defined as

$$G = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2} = 0.915965594\cdots.$$

It is well known that

$$\int_0^{\pi/2}\frac{x}{\sin x}\,dx = 2G. \tag{5.2}$$

There are connections between the generalized trigonometric functions and those of classical analysis. Since the integral formula of Euler type (see [28, Section 14.6]) yields

$$\sin_p^{-1} x = \frac{x}{p}\int_0^1 t^{-1/p'}(1 - x^p t)^{-1/p}\,dt = \frac{x}{p}F\left(\frac{1}{p}, \frac{1}{p}; 1 + \frac{1}{p}; x^p\right),$$

where F is the hypergeometric function, we have

$$\sin_p^{-1} x = x\sum_{n=0}^{\infty}\frac{\Gamma(n+1/p)}{\Gamma(1/p)(np+1)}\frac{x^{pn}}{n!}.$$

This leads to: for $x \in (0, \pi_p/2)$,

$$x = \sin_p x \sum_{n=0}^{\infty} \frac{\Gamma(n+1/p)}{\Gamma(1/p)(np+1)} \frac{\sin_p^{pn} x}{n!}.$$

Thus,

$$\int_0^{\pi_p/2} \frac{x}{\sin_p x}\,dx = \sum_{n=0}^{\infty} \frac{\Gamma(n+1/p)}{\Gamma(1/p)(pn+1)n!} \int_0^{\pi_p/2} \sin_p^{pn} x\,dx.$$

Using

$$\int_0^{\pi_p/2} \sin_p^{pn} x\,dx = \frac{1}{p} B\left(n+\frac{1}{p}, \frac{1}{p'}\right) = \frac{\Gamma(n+1/p)\Gamma(1/p')}{p\Gamma(n+1)} = \frac{\pi_p \Gamma(n+1/p)}{2\Gamma(1/p)n!},$$

we have

$$\int_0^{\pi_p/2} \frac{x}{\sin_p x}\,dx = \frac{\pi_p}{2} \sum_{n=0}^{\infty} \left(\frac{\Gamma(n+1/p)}{\Gamma(1/p)n!}\right)^2 \frac{1}{pn+1}.$$

In particular, the case $p = 2$ with Eq. (5.2) reduces to

$$2G = \frac{\pi}{2} \sum_{n=0}^{\infty} \left(\frac{\Gamma(n+1/2)}{\sqrt{\pi}n!}\right)^2 \frac{1}{2n+1}.$$

Since $\Gamma(n+1/2) = (2n)!\sqrt{\pi}/(2^{2n}n!)$ from the duplication formula of Γ in [28, Section 12.15], we obtain

$$G = \frac{\pi}{4} \sum_{n=0}^{\infty} \left(\frac{(2n)!}{2^{2n}(n!)^2}\right)^2 \frac{1}{2n+1}.$$

5.4 Addition Theorem

Edmunds, Gurka and Lang [12] have obtained an addition formula for $\sin_{pp'} x$ in a very special case.

We consider the case $p = \frac{4}{3}$ and $q = p' = 4$. Then, we have

$$\sin_{\frac{4}{3},4}^{-1} u = \int_0^u \frac{dt}{(1-t^4)^{\frac{3}{4}}}.$$

The change of variable $w = (\frac{1-t^2}{1+t^2})^{\frac{1}{2}}$ leads to

$$\sin^{-1}_{\frac{4}{3},4} u = \frac{1}{\sqrt{2}} \int_{(\frac{1-u^2}{1+u^2})^{\frac{1}{2}}}^{1} \frac{dw}{\sqrt{w(1-w^2)}}.$$

The further change of variable $1 - w = z^2$ leads to

$$\sin^{-1}_{\frac{4}{3},4} u = \int_0^{\phi(u)} \frac{dz}{\sqrt{(1-z^2)(1-\frac{z^2}{2})}},$$

where $\phi(u) = (1 - (\frac{1-u^2}{1+u^2})^{\frac{1}{2}})^{\frac{1}{2}}$. Therefore,

$$\phi(u) = \mathrm{sn}\left(\sin^{-1}_{\frac{4}{3},4} u, \frac{1}{\sqrt{2}}\right).$$

Setting $y = \sin^{-1}_{\frac{4}{3},4} u$, we have

$$\sin_{\frac{4}{3},4} y = \phi^{-1}\left(\mathrm{sn}\left(y, \frac{1}{\sqrt{2}}\right)\right)$$

Thus, using the addition theorem for $\mathrm{sn}(x,k)$ (see [28, Section 22.2]) , we obtain

$$\begin{aligned}&\sin_{\frac{4}{3},4}(u+v)\\ &= \phi^{-1}\left(\frac{\phi(U)\sqrt{1-\phi(V)^2}\sqrt{1-\frac{1}{2}\phi(V)^2} + \phi(V)\sqrt{1-\phi(U)^2}\sqrt{1-\frac{1}{2}\phi(U)^2}}{1-\frac{1}{2}\phi(U)^2\phi(V)^2}\right),\end{aligned}$$

where $U = \sin_{4/3,4} u$, $V = \sin_{4/3,4} v$. For simplicity, putting $u = v$, we have the following duplication formula.

Proposition 5.1 ([12]).

$$\sin_{\frac{4}{3},4} 2x = \frac{2\sin_{\frac{4}{3},4} x \cos_{\frac{4}{3},4} x}{(1 + 4\sin^4_{\frac{4}{3},4} x \cos^4_{\frac{4}{3},4} x)^{\frac{1}{2}}} \quad \left(0 \leqq x < \frac{\pi_{\frac{4}{3},4}}{4}\right)$$

Finally it should be noted that we still have no addition formula for $\sin_{pq} x$ in the other cases even if $q = p'$.

Remark 5.2. In [12, Proposition 3.4], this formula is written in a slightly different form as

$$\sin_{\frac{4}{3},4} 2x = \frac{2\sin_{\frac{4}{3},4} x(\cos_{\frac{4}{3},4} x)^{\frac{1}{3}}}{(1+4\sin^4_{\frac{4}{3},4} x \cos^{\frac{4}{3}}_{\frac{4}{3},4} x)^{\frac{1}{2}}} \qquad \left(0 \leqq x < \frac{\pi_{\frac{4}{3},4}}{4}\right).$$

The difference is caused by that of the definitions of $\cos_{pq} x$. They define $\cos_{pq} x$ in [12] as

$$\cos_{pq} x = \frac{d}{dx}\sin_{pq} x,$$

while we have defined in Eq. (1.3)

$$\cos_{pq} x = \sqrt[q]{1-|\sin_{pq} x|^q} = \left(\frac{d}{dx}\sin_{pq} x\right)^{\frac{p}{q}}.$$

5.5 Basis Property

Given any element in $L^q(0,1)$ for every $q \in (1,\infty)$, its odd extension to $L^q(-1,1)$ has a unique representation in terms of the functions $\sin(n\pi x)$, which means that $\sin(n\pi x)$ is a basis of $L^q(0,1)$. Binding, Boulton, Čepička, Drábek and Girg [2] showed that the functions $\sin_p(n\pi_p x)$ have a similar property, provided that p is not too close 1.

Stating their result, we recall the definitions of Schauder and Riesz bases. For details, see Singer [23] and Higgins [15].

Definition 5.1. $\{g_n\}$ is a *Schauder basis of a Banach space* X, if for any $g \in X$, there exist unique coefficients c_n, depending continuously on g, so that $g = \sum_{n=1}^{\infty} c_n g_n$, i.e., $\|\sum_{n=1}^{N} c_n g_n - g\|_X \to 0$ as $N \to \infty$.

Definition 5.2. $\{g_n\}$ is a *Riesz basis of a Hilbert space* H, if it is a Schauder basis of H and there exist $0 < C_1 \leqq C_2$ such that $C_1(\sum_{n=1}^{N}|c_n|^2)^{1/2} \leqq \|\sum_{n=1}^{N} c_n g_n\|_H \leqq C_2(\sum_{n=1}^{N}|c_n|^2)^{1/2}$ for all finite sequences $\{c_n\}$ of scalars.

Remark 5.3. Riesz basis is also defined as a basis being simultaneously Besselian and Hilbertian. See [23, Corollary 11.2].

Then, we can obtain (omitting the proof)

Theorem 5.2 ([2]). *Let $f_n(t) := \sin_p(n\pi_p t)$. If $\frac{12}{11} \leqq p < \infty$, then the family $\{f_n\}_{n=1}^{\infty}$ forms a Riesz basis of $L^2(0,1)$ and a Schauder basis of $L^q(0,1)$ whenever $1 < q < \infty$.*

The condition $\frac{12}{11} \leqq p < \infty$ can be relaxed as

$$\pi_p = \frac{2\pi}{p \sin \frac{\pi}{p}} < \frac{4\sqrt{2}\pi^2}{\pi^2 - 8} = 29.86242088 \cdots .$$

We can see that the number $\frac{12}{11} = 1.\dot{0}\dot{9}$ satisfies this condition, indeed

$$\pi_{\frac{12}{11}} = \frac{2\pi}{\frac{12}{11} \sin \frac{11\pi}{12}} = \frac{11\pi\sqrt{2}}{3(\sqrt{3}-1)} = 22.25333351 \cdots .$$

The optimal value for the condition is $p = 1.067415277 \cdots$, i.e., $\pi_{1.067415277\cdots} = \frac{4\sqrt{2}\pi^2}{\pi^2-8}$, but this is not so for the conclusion of theorem.

Recently, Edmunds, Gurka and Lang [12] proved that $\sin_{pq}(n\pi_{pq}x)$ also has the basis property. This does not include Theorem 5.2 completely, but holds for cases $q \neq p$. In particular, it assures that $\sin_{pp'}(n\pi_{pp'}x)$ is a basis for all $1 < p < \infty$.

Theorem 5.3 ([12, 18]). *Let $f_n(t) := \sin_{pq}(n\pi_{pq}t)$. If $\pi_{pq} < \frac{16}{\pi^2-8} = 8.557960158\cdots$, then the family $\{f_n\}_{n=1}^{\infty}$ is a basis in $L^r(0,1)$ for any $r \in (1,\infty)$.*

For Jacobian elliptic functions, we have known a result of Craven [6, Theorem 2]. In that paper, Stone's theorem is used to prove a more general completeness theorem, which can also be applied to certain expansions in Jacobian elliptic functions, analogous to trigonometric Fourier series.

Theorem 5.4 ([6]). *The sequence of Jacobian elliptic functions*

$$1,\ \mathrm{cn}\,(x,k),\ \mathrm{cn}\,(2x,k),\ \cdots,\ \mathrm{cn}\,(nx,k),\ \cdots$$

is complete in $L^2(0,2K(k))$, provided that $0 < k < k_c$, where $k_c = 0.99$ approximately.

A similar proof, and conclusion, applies to the sequence

$$1,\ \mathrm{sn}\,(x,k),\ \mathrm{sn}\,(2x,k),\ \cdots,\ \mathrm{sn}\,(nx,k),\ \cdots$$

in $L^2(0,2K(k))$, with a similar restriction on k.

In the forthcoming paper [26], the author will give the basis property for generalized Jacobian elliptic functions, which includes the results of [2] and [12], partially of [6].

5.6 Arithmetic-Geometric Mean

Let $a_0 = a > 0,\ b_0 = b > 0$. We consider the sequences

$$a_{n+1} = \frac{a_n + b_n}{2}, \quad b_{n+1} = \sqrt{a_n b_n}.$$

Then, it was shown by Gauss in 1799 that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \frac{\pi}{\displaystyle\int_{-\infty}^{\infty} \frac{dx}{\sqrt{(x^2+a^2)(x^2+b^2)}}},$$

where the limit is called the arithmetic-geometric mean.

Concerning this, D. Borwein and P. B. Borwein [3] gave the problem:

> The following conjecture appears in Les Annales des Sciences Mathématiques du Québec, 6 (1982), p. 79: Let $\alpha + \beta = 1,\ \alpha > 0,\ \beta > 0$. If $a_0 = a > 0,\ b_0 = b > 0$ and
>
> $$a_{n+1} = \alpha a_n + \beta b_n, \quad b_{n+1} = a_n^{\alpha} b_n^{\beta},$$
>
> then
>
> $$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \frac{\pi}{\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^{\alpha}(x^2+b^2)^{\beta}}}.$$
>
> Show that this conjecture is false unless $\alpha = \beta = 1/2$.

In [3] they give a proof by themselves, but they mention nothing about the correct limit of these sequences. It seems to be still open and interesting to study the relation between π_{pq}, $K_{pq}(k)$ and this generalized arithmetic-geometric mean.

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