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PRÜFER'S TRANSFORMATION

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ABSTRACT. A review is given of recent work on reducing periodic/antiperiodic Sturm-Liouville problems to analysis of the Prüfer angle. This provides an alternative to the more usual approaches via operator theory or the Hill discriminant in the definite case, and leads to new results in cases with semidefinite weight and more general coupling boundary conditions.

1. INTRODUCTION.

The Sturm-Liouville equation

$$-(p(x)y')' + q(x)y = \lambda r(x)y, \quad x \in [a, b] \tag{1.1}$$

has received widespread attention. From about 100 years before the time of Sturm and Liouville, until the present day, such equations have been studied (and applied) in a variety of contexts subject to separating (also called Sturmian) boundary conditions of the form

$$y(a)\cos\alpha = (py)'(a)\sin\alpha, \text{ and } y(b)\cos\beta = (py)'(b)\sin\beta. \tag{1.2}$$

For a long time, the basic eigenvalue existence and eigenfunction oscillation theory proceeded via analysis of the zeros of real solutions of (1.1) as functions of λ. Much of the relevant material up to the 1920s is covered in, e.g., [13, Ch. 10], first published in 1926. In the same year, Prüfer [17] (in what seems to be his only paper on differential equations) gave a new transformation of the problem leading to an analysis simpler in various ways than in previous works. Assuming that $p, r > 0$, which we shall call the "definite case", he rewrote (1.1) as an equivalent system

$$y' = z/p, z' = (q - \lambda r)y \tag{1.3}$$

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and then used polar coordinates in the corresponding phase plane to derive the basic eigenvalue existence and eigenfunction oscillation theory.

Since then, Prüfer's transformation has been the method of choice in most texts discussing the above basic theory. Many follow Prüfer's lead and consider continuous coefficients p, q and r (cf. [9, Ch. 8]), but we shall allow L_1 conditions, which are treated in [1] and [21], for example.

Following Hill's studies of planetary motion in the latter part of the 19th century, Sturm-Liouville equations with periodic (or antiperiodic) conditions

$$\mathbf{y}(a) = \pm\mathbf{y}(b), \tag{1.4}$$

(where $\mathbf{y} = [y\ y']^T$) became of interest, and we remark that such boundary conditions also appear in the study of wave motion, separation of variables in classical boundary value problems, etc. Two basic methods have been established for the existence of eigenvalues of (1.1) and (1.4). One proceeds via Floquet theory and Hill's discriminant $d(\lambda)$. It requires little background, but is somewhat involved, particularly in the analysis of $d''(\lambda)$. The other is variational in nature, and requires a significant amount of background in operator (or equivalent) theory. Here we shall outline another method, via Prüfer's transformation.

Indeed there seem to be various reasons to give a Prüfer treatment of (1.1) and (1.4). It depends on elementary analysis of initial value problems, builds on standard ideas from the separated case, and is less intricate (and shorter) than the Floquet/Hill theory (as in, say, [9, Ch. 8]). It requires much less background than the treatments of, say, Eastham [10] or Weidmann [21], who use significant amounts of operator theory. Also, it is versatile enough to allow a unified treatment not only of eigenvalue existence but also of further topics like oscillation and comparison principles, asymptotics and interlacing.

Our plan is as follows. In Section 2 we recall some basic properties of the Prüfer transformation, showing how both the (standard) case of (1.2), and also the case of (1.4), can be reduced in different ways to analysis of the Prüfer angle $\theta(\lambda, \alpha, x)$ at $x = b$. As a function of $\lambda \in \mathbb{R}$, this increases continuously from 0 to infinity. Thus, values of the form $n\pi+\beta$ for integers $n \geq 0$ are attained and generate eigenvalues for (1.1) and (1.2). It turns out that the boundary conditions (1.4) can be treated via two such functions m and M, thereby generating pairs (with the same oscillation count) of periodic and antiperiodic eigenvalues. These functions still depend on the Prüfer angle, but via more involved constructions, and the corresponding theory is developed in Section 3. For proofs, and further topics like comparison principles, interlacing of eigenvalues and eigenfunction zeros and the relationship with Floquet theory and Hill's discriminant, we refer to [6].

In Section 4 we extend (1.4) to boundary conditions of the form

$$\mathbf{y}(b) = K\mathbf{y}(a), \tag{1.5}$$

where $K = (k_{ij})$ is a real 2×2 matrix with determinant 1. These are the most general (real) self-adjoint boundary conditions for (1.1). The best known examples of coupling boundary conditions are the periodic and antiperiodic cases, where $K = I$ and $K = -I$, respectively. For the definite case, see, e.g., [9, 10, 21] for various conditions on the coefficients, via a combination of Hill discriminant and operator theoretic methods. Results include existence of eigenvalues and their interlacing with those for corresponding Dirichlet and Neumann problems. More general cases of K have been treated for over a century, again for the definite case, in, e.g., [8] (which was later corrected in some respects). Most of the work in this area (which is reviewed in [5]) involves generalised versions of the Hill discriminant and the Dirichlet and Neumann problems noted above. Proofs for the Prüfer method used here can be found in [7].

In Section 5 we relax definiteness to allow $r \geq 0$ a.e. This topic, often studied under the title of "semidefinite weight", has again been investigated for over a century – see, e.g., [12]. Later results on more general cases, e.g. [4, 11], examine a "loss of eigenvalues" if r vanishes on a "large enough" set. This case is also referred to as "right semidefinite", but recently "singular indefinite" has been used by some authors, e.g. in [15]. Here we shall extend the Prüfer transformation analysis of the definite case from the previous sections to the right semidefinite problem. Superficially, little changes except for the "loss" of a certain number (which we make precise, at least in the periodic/antiperiodic cases) of eigenvalues. On the other hand, several aspects of the analysis become more involved, and for details, and various related results, we refer to [7].

The methods here can in fact be extended to a more general situation in which (1.3) is replaced by

$$y' = sz, z' = (q - \lambda r)y. \tag{1.6}$$

This problem, under additional conditions, was considered by Atkinson [1, Chapter 8]. For the definite case, and more generally when $p > 0$, this simply involves a change of notation in (1.3) to $s = 1/p$ with $z = py'$. (Despite some advantages, this has not been widely used).

Atkinson, however, also took the further step of relaxing definiteness to the "semi-definiteness" conditions

$$r \geq 0 \text{ and } s \geq 0 \tag{1.7}$$

a.e. on $[a, b]$. The second inequality in (1.7) extends the meaning of (1.3) to (1.6), and is a proper extension when s vanishes nontrivially. Indeed, Atkinson used this idea to incorporate certain difference equations into a generalised Sturm-Liouville framework. For further work on such problems we cite [4] for separating boundary conditions, and special cases of Atkinson's problem (equivalent to equations with piecewise constant coefficients) with finitely many eigenvalues have been studied by various authors, cf. [14] and its references. The setting has been shown in [19] to be powerful enough to include work of Krein on strings, Feller on certain diffusion processes, and certain equations involving measures. It is shown in [7] that the Prüfer method used here can be extended to Atkinson's problem, although certain concepts like oscillation count have to be redefined since eigenfunctions can have intervals of zeros.

Moreover, related methods have been used to derive existence of periodic eigenvalues for various (definite) equations generalising (1.1), for example for Fučík spectra, the half-eigenvalue problem and the p-Laplacian. The results are then not as sharp, however, and arbitrarily many eigenvalues can have the same oscillation count; see, e.g., [2].

2. The basic Prüfer transformation.

In this section we assume that $p, r > 0$ and $\frac{1}{p}, q, r \in L_1(a, b)$. We recall that Prüfer's transformation for a nonzero solution y of (1.1) takes the form

$$y = \rho \sin\theta, \text{ and } p\,y' = \rho\cos\theta. \tag{2.1}$$

Given the initial conditions

$$y(a) = \sin\alpha, \quad (py')(a) = \cos\alpha, \tag{2.2}$$

for (1.1), standard manipulations (e.g., [9, Ch. 8]) give

$$\theta' = \frac{\cos^2\theta}{p} + (\lambda r - q)\sin^2\theta \text{ and } \rho' = \rho\left(\frac{1}{p} + q - \lambda r\right)\sin\theta\cos\theta \tag{2.3}$$

with initial conditions

$$\theta(\lambda, \alpha, 0) = \alpha \in \mathbb{R} \text{ and } \rho(\lambda, \alpha, 0) = 1. \tag{2.4}$$

Conversely, (2.3) and (2.4) imply (1.1) and (2.2).

We use (2.3), (2.4) to define θ and ρ as functions of (λ, α, x). We remark that the above differential equations are all to be understood in the sense of Carathéodory and are Lipschitz in the dependent variables. Basic theory for such equations can be found in various books, e.g., [9, 18, 20].

For separating conditions (1.2), α is given, and the eigenvalue condition can be written in the form

$$\theta(\lambda, \alpha, b) = \beta + k\pi, \text{ for } \alpha, \beta \in [0, \pi), \tag{2.5}$$

where k is the "oscillation count" of λ. We define this as the number of zeros in $(a, b]$ of any eigenfunction corresponding to λ. It suffices, then, to study θ alone and we now list some of the standard properties that we need (see, e.g., [9, 21]. First,

$$\theta(\lambda, \alpha, x) \quad \text{increases with } x \text{ through multiples of } \pi. \tag{2.6}$$

Next,

$$\theta(\lambda, \alpha, b) \quad \text{increases strictly and continuously with } \lambda, \tag{2.7}$$

and finally

$$\theta(\lambda, \alpha, b) \to 0 \text{ (resp. } +\infty) \text{ as } \lambda \to -\infty \text{ (resp. } +\infty). \tag{2.8}$$

Via well known arguments, e.g., in [9, 21] these suffice to give existence of a unique $\lambda = \lambda_k(\alpha, \beta)$ with oscillation count k for each $k \geq 0$ except, because of (2.5), that $k \geq 1$ when $\beta = 0$, e.g., for Dirichlet conditions $y(0) = y(\pi) = 0$). In this case the above oscillation result differs from the usual one, where $\beta \in (0, \pi]$ and eigenfunction zeros are counted in (a, b), but our convention will allow a more unified treatment of the results below.

We now lay the groundwork for the periodic/antiperiodic conditions where $K = \pm I$ in (1.5). First we note that $\theta(\lambda, \alpha, x)$ is C^1 in α, and indeed θ_α (the subscript denoting partial differentiation by α) satisfies the variational initial value problem

$$\theta_\alpha' = -2\theta_\alpha(\frac{1}{p} + q - \lambda r) \sin\theta \cos\theta, \text{ where } \theta_\alpha(\lambda, \alpha, 0) = 1,$$

obtained from the θ equations in (2.3) and (2.4) (see, e.g., [9], [18]). Using the ρ equation in (2.3) we see that $(\rho^2\theta_\alpha)' = 0$, and from (2.4) we derive

$$\rho^2\theta_\alpha = 1. \tag{2.9}$$

We now use (2.4) (where α is as yet undetermined) to rewrite (1.4) in the form

$$\theta(\lambda, \alpha, \pi) = \alpha + k\pi,$$

where k is even (resp. odd) for a periodic (resp. antiperiodic) condition, together with $\rho(\lambda, \alpha, \pi) = 1$. This last equation and (2.9) yield

$$\theta_\alpha(\lambda, \alpha, \pi) = 1$$

so again it is enough to study θ without ρ. Actually, it will be more convenient to work with the difference

$$\delta(\lambda, \alpha) = \theta(\lambda, \alpha, \pi) - \alpha \tag{2.10}$$

between final and initial values, and then the eigenvalue conditions take the form

$$\delta(\lambda, \alpha) = k\pi, \text{ and } \delta_\alpha(\lambda, \alpha) = 0. \tag{2.11}$$

3. Periodic and antiperiodic eigenvalues.

In this section we assume that $p, r > 0$ and $\frac{1}{p}, q, r \in L_1(a, b)$. From (2.11), $k\pi$ is a critical value (with respect to α) of $\delta(\lambda, \alpha)$, and one key for us will be to replace "critical" by "extreme". We start by extending the definition of $\delta(\lambda, \alpha)$ from $\alpha \in [0, \pi)$ to $\alpha \in \mathbb{R}$, noting that δ is then π-periodic in α, by virtue of

$$\theta(\lambda, \alpha + \pi, x) = \theta(\lambda, \alpha, x) + \pi. \tag{3.1}$$

In particular, the minimum $m(\lambda)$ and maximum $M(\lambda)$ of $\delta(\lambda, \alpha)$ over $\alpha \in \mathbb{R}$ equal those over $\alpha \in [0, \pi]$ and are attained. The following result describes some basic properties of m and M.

Lemma 3.1.
(a) *The functions m and M are continuous and strictly increasing.*
(b) *For each λ, we have the inequalities $m(\lambda) \le M(\lambda) < m(\lambda) + \pi$.*
(c) *If $\lambda \to -\infty$, then $m(\lambda) \to -\pi$ and $M(\lambda) \to 0$.*
(d) *If $\lambda \to +\infty$, then $m(\lambda) \to +\infty$.* The proof can be found in [6].

Now we are ready for our central construction. From Lemma 3.1, m (resp. M) attains each value $k\pi$ for $k \ge 0$ (resp. $k \ge 1$) so we can define intervals

$$I_k = \{\lambda : m(\lambda) \le k\pi \le M(\lambda)\} = \{\lambda : \delta(\lambda, \alpha) = k\pi \text{ for some } \alpha\} \tag{3.2}$$

with end points $\lambda_k^- \le \lambda_k^+$, for each $k \ge 0$. Apart from $\lambda_0^- = -\infty$, each $\lambda_k^\pm$ is finite, and

$$m(\lambda_0^+) = 0, \text{ and } m(\lambda_k^+) = k\pi = M(\lambda_k^-) \quad \text{for all} \quad k \ge 1. \tag{3.3}$$

We are now ready for the main result of this section.

Theorem 3.2. (a) *Except for λ_0^-, each $\lambda_k^\pm$ is an eigenvalue of the periodic eigenvalue problem when k is even and of the antiperiodic eigenvalue problem when k is odd.*

(b) *There are no other eigenvalues.*
(c) *The I_k are disjoint, so $\lambda_k^+ < \lambda_{k+1}^-$ for each $k \geq 0$.*
(d) *An eigenfunction y belonging to the eigenvalue $\lambda_k^\pm$ has k zeros in $(a, b]$.*

The proof can be found in [6], along with various additional results concerning comparison principles, interlacing of eigenvalues and eigenfunction zeros and the relationship with Floquet theory and Hill's discriminant.

4. Coupling boundary conditions

In this section we assume that $p, r > 0$ and $\frac{1}{p}, q, r \in L_1(a, b)$. We consider the system (1.6) subject to the boundary conditions (1.5) where $K = (k_{ij})$ is a real 2×2 matrix with determinant 1.

It will be convenient to treat the boundary conditions

$$\begin{pmatrix} y(b) \\ (py')(b) \end{pmatrix} = -K \begin{pmatrix} y(a) \\ (py')(a) \end{pmatrix} \tag{4.1}$$

as well as (1.5). Since we may replace K by $-K$ we will assume, without loss of generality, that

$$k_{12} < 0 \quad \text{or} \quad k_{12} = 0 < k_{11}. \tag{4.2}$$

For $\alpha \in \mathbb{R}$, $i = 1, 2$, we set

$$k_i(\alpha) := k_{i1} \sin \alpha + k_{i2} \cos \alpha.$$

The curve $(k_2(\alpha), k_1(\alpha))$ does not pass through $(0, 0)$, and we denote its continuously chosen polar angle by $\beta(\alpha)$. By virtue of (4.2), we may determine the function β uniquely by the condition

$$\beta(0) \in (-\pi, 0]. \tag{4.3}$$

If $k_1(\alpha) = 0$ then $\beta(\alpha)$ is an integer multiple of π (for example, $k_{12} = 0$ is equivalent to $\beta(0) = 0$); otherwise we have

$$\cot \beta(\alpha) = \frac{k_2(\alpha)}{k_1(\alpha)}. \tag{4.4}$$

In the special case of periodic/antiperiodic boundary conditions, K is the identity matrix and $\beta(\alpha) = \alpha$.

The function $\beta : \mathbb{R} \to \mathbb{R}$ is continuously differentiable with derivative

$$\beta'(\alpha) = [k_1(\alpha)^2 + k_2(\alpha)^2]^{-1} > 0. \tag{4.5}$$

Therefore, β is an increasing function and

$$\beta(\alpha + \pi) = \beta(\alpha) + \pi. \tag{4.6}$$

We can now extend the definition of δ from Section 2 to

$$\delta(\alpha, \lambda) := \theta(b, \alpha, \lambda) - \beta(\alpha). \tag{4.7}$$

As a function of α, $\delta(\alpha, \lambda)$ is continuously differentiable and has period π. The proof of the following lemma can be found in [7].

Lemma 4.1. *A real number λ is an eigenvalue of* (1.6) *subject to boundary conditions* (1.5) *(resp.* (4.1)*) if and only if there is an even (resp. odd) integer k such that $k\pi$ is a critical value of the function $\delta(\ ,\lambda) : \alpha \mapsto \delta(\alpha, \lambda)$.*

Following the procedure from Section 3 we define

$$m(\lambda) := \min_{\alpha \in \mathbb{R}} \delta(\alpha, \lambda),$$
$$M(\lambda) := \max_{\alpha \in \mathbb{R}} \delta(\alpha, \lambda)$$

and we extend the reasoning of Section 3 to give the following result.

Lemma 4.2.

(a) *The functions $m, M : \mathbb{R} \to \mathbb{R}$ are continuous and strictly increasing.*

(b) *For every $\lambda \in \mathbb{R}$,*

$$-\pi < m(\lambda) \leq M(\lambda) < m(\lambda) + \pi. \tag{4.8}$$

(c) *For every $\lambda \in \mathbb{R}$, $m(\lambda)$ and $M(\lambda)$ are the only critical values of the function $\delta(\ ,\lambda)$.*

The proof can be found in [7]. The following theorem now follows from Lemmas 4.1 and 4.2(c).

Theorem 4.3. *A real number λ is an eigenvalue of* (1.6) *subject to boundary conditions* (1.5) *(resp.* (4.1)*) if and only if there is an even (resp. odd) integer k such that either $m(\lambda) = k\pi$ or $M(\lambda) = k\pi$.*

In order to obtain analogues of Lemma 3.1(c) and (d), we note that $\beta(\alpha)$ attains its minimum and maximum for $0 \leq \alpha \leq \pi$ at $\beta(0)$ and $\beta(\pi) = \beta(0)+\pi$ respectively. Thus if $\lambda \to -\infty$, then $m(\lambda) \to -\beta(0)-\pi$ and $M(\lambda) \to -\beta(0)$, while if $\lambda \to +\infty$, then $m(\lambda) \to +\infty$. Since $-\beta(0) \in [0, \pi)$ by (4.3), we may argue as in [7] to obtain

Theorem 4.4. (a) *Except for λ_0^-, each $\lambda_k^\pm$ is an eigenvalue corresponding to* (1.5) *when k is even and to* (4.1) *when k is odd.*

(b) *There are no other eigenvalues.*

(c) *The I_k are disjoint, so $\lambda_k^+ < \lambda_{k+1}^-$ for each $k \geq 0$.*

This corresonds to Theorem 3.2(a)–(c), but statement (d) of Theorem 3.2 regarding oscillation counts now becomes more complicated. Eigenfunctions y corresponding to the eigenvalue $\lambda = \lambda_k^{\pm}$ may be taken in the form (2.1), (2.4) with $\alpha \in [0, \pi)$. Then

$$\theta(b, \alpha, \lambda) - \beta(\alpha) = k\pi.$$

Note that $\beta(\alpha) \in (-\pi, \pi)$. If $\beta(\alpha) < 0$ then y has $k - 1$ zeros in $(a, b]$, and if $\beta(\alpha) \geq 0$ then there are k zeros. For further analysis of this topic in the definite case, via different methods, we refer to [5] and its references.

5. Semidefinite weight function

In this section we assume the previous integrability conditions on $1/p, q$ and r with $p > 0$ but additionally we allow $r \geq 0$ a.e. In this case (2.7) (and (2.8), as we shall see) may fail. For example, if $r = 0$ a.e. then the eigenvalue problem is independent of λ and in what follows we shall assume the nondegeneracy condition

$$\int_a^b r > 0. \tag{5.1}$$

Initially we shall assume periodic/antiperiodic boundary conditions, i.e., $K = \pm I$ in (1.5), and we extend the coefficients p, q, r periodically over $\mathbb{R}$. Assumption (5.1) allows us to translate the independent variable if necessary so that

$$\int_a^c r > 0, \quad \int_c^b r > 0 \quad \text{for every } c \in (a, b). \tag{5.2}$$

Accordingly we shall assume that θ, δ, m, M are defined using such an a.

Arguing as in [7], we see that enough properties of m and M extend to the case of semidefinite weight to allow us to deduce Theorem 4.3 verbatim. Since m and M are easily seen to be nondecreasing, it remains to consider their limits at $\pm\infty$ and whether they increase strictly.

We start by noting that, for each α, $\theta(b, \alpha, \lambda) \to +\infty$ as $\lambda \to +\infty$, see, e.g., [3]. Also $\theta(b, \alpha, \lambda)$ is nondecreasing and analytic in λ for each α. Arguing as in [7], one can use the analogue of Lemma 3.1(b) to deduce

Lemma 5.1. (a) *The functions $m, M : \mathbb{R} \to \mathbb{R}$ are continuous and strictly increasing.*

(b) *$m(\lambda)$ and $M(\lambda)$ tend to $+\infty$ as $\lambda \to +\infty$.*

It remains to consider the limits of $m(\lambda)$ and $M(\lambda)$ as $\lambda \to -\infty$. To this end we let $\mathcal{J}$ be the collection of maximal closed intervals $[c,d] \subset [a,b]$ with $c < d$ for which $\int_c^d r = 0$.

For $J = [c,d] \in \mathcal{J}$ let $\theta_J(x)$, $x \in J$, be the solution of the first equation in (2.3), that is,

$$\theta_J' = \frac{1}{p}\cos^2\theta_J - q\sin^2\theta_J \tag{5.3}$$

with initial value

$$\theta_J(c) = 0.$$

We define

$$\ell_J^- := \max\{k\pi : k \in \mathbb{Z}, k\pi \leq \theta_J(d)\}$$

Then as in [4, Theorem 3.2] we see that if $\alpha \in [0,\pi)$ then

$$\theta(b,\alpha,\lambda) \to \ell_- := \sum_{J\in\mathcal{J}} \ell_J^- \text{ as } \lambda \to -\infty, \tag{5.4}$$

where the empty sum is understood as 0. Note that the above sum is always finite since $\frac{1}{p}, q$ and r are integrable.

Reasoning as in [7], we then reach

Lemma 5.2. *$m(\lambda) \to \ell_- - \pi$ and $M(\lambda) \to \ell_-$ as $\lambda \to -\infty$.*

Combining Lemmas 5.1 and 5.2 with the reasoning used in [7], we obtain the following result, where $\kappa := \ell_-/\pi$.

Theorem 5.3. (a) *Except for λ_κ^-, each $\lambda_k^\pm$ with $k \geq \kappa$ is an eigenvalue of the periodic eigenvalue problem when k is even and of the antiperiodic eigenvalue problem when k is odd.*

(b) *There are no other eigenvalues.*

(c) *The I_k are disjoint, so $\lambda_k^+ < \lambda_{k+1}^-$ for each $k \geq 0$.*

(d) *An eigenfunction y belonging to the eigenvalue $\lambda_k^\pm$ has k zeros in $(a,b]$.*

Finally, we discuss the changes needed when the coupling boundary conditions involve general K in (1.5). Then (5.1) no longer allows us to assume (5.2), but it is still possible to adopt the same general strategy. This involves the analysis of several cases, which we omit, but we shall indicate an analogue of Theorem 5.3.

This requires modifications to the definition of κ via ℓ_- to allow for intervals $[c,d] \in \mathcal{J}$ with either $a = c$ or $b = d$ (but not both, because of (5.1)). If there is an interval $T = [c,b] \in \mathcal{J}$, then we write $\ell_T^- := \theta_T(b)$, where θ_T obeys the differential equation (5.3) for θ_J. If there is $I = [a,d] \in \mathcal{J}$ then we let θ_R obey the same differential equation (5.3) as θ_J, but with terminal condition $\theta_R(d) = 0$.

Then ℓ_- in Lemma 5.2 must be modified to account for θ_T, θ_R and the function β of Section 4. See [7] for details of this and the final result, which we can express loosely as follows.

For general K and with κ as the greatest integer not exceeding ℓ_-/π, conclusions (a)–(c) of Theorem 5.3 hold, and conclusion (d) is replaced by the paragraph following Theorem 4.4. For related results on comparison, interlacing etc., we refer to [7].

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