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ON GENERALIZED POWERS-STØRMER'S INEQUALITY

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ABSTRACT. A generalization of Powers-Størmer's inequality for operator monotone functions on $[0,+\infty)$ and for positive linear functional on general C^*-algebras will be proved. It also will be shown that the generalized Powers-Størmer inequality characterizes the tracial functionals on C^*-algebras.

1. INTRODUCTION

Powers-Størmer's inequality (see, for example, [16, Lemma 2.4], [4, Theorem 11.19]) asserts that for $s\in[0,1]$ the following inequality

$$2\,\mathrm{Tr}(A^sB^{1-s})\geq \mathrm{Tr}(A+B-|A-B|) \tag{1}$$

holds for any pair of positive matrices A, B. This is a key inequality to prove the upper bound of Chernoff bound, in quantum hypothesis testing theory [1]. This inequality was first proven in [1], using an integral representation of the function t^s. After that, N. Ozawa gave a much simpler proof for the same inequality, using fact that for $s\in[0,1]$ function $f(t)=t^s$ $(t\in[0,+\infty))$ is an operator monotone ([11, Proposition 1.1]). Recently, Y. Ogata in [13] extended this inequality to standard von Neumann algebras. The motivation of this paper is that if the function $f(t)=t^s$ is replaced by another operator monotone function (this class is intensively studied, see [8][14]), then $\mathrm{Tr}(A+B-|A-B|)$ may get smaller upper bound than what is used in quantum hypothesis testing. Based on N. Ozawa's proof we formulate Powers-Størmer's inequality for an arbitrary operator monotone function on $[0,+\infty)$ in the context of general C^*-algebras.

2. DOUBLE PILING STRUCTURES FOR MATRIX FUNCTIONS

Throughout this note, M_n stands for the algebra of all $n\times n$ matrices, M_n^+ denote the set of positive semi-definite matrices. We call a function

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f *matrix convex of order n* or *n-convex* in short (resp. *matrix concave of order n* or *n-concave*) whenever the inequality

$$f(\lambda A + (1-\lambda)B) \leq \lambda f(A) + (1-\lambda)f(B), \ \lambda \in [0,1]$$

(resp. $f(\lambda A + (1-\lambda)B) \geq \lambda f(A) + (1-\lambda)f(B), \ \lambda \in [0,1]$) holds for every pair of selfadjoint matrices $A, B \in M_n$ such that all eigenvalues of A and B are contained in I. *Matrix monotone functions* on I are similarly defined as the inequality

$$A \leq B \Longrightarrow f(A) \leq f(B)$$

for any pair of selfadjoint matrices $A, B \in M_n$ such that $A \leq B$ and all eigenvalues of A and B are contained in I. We call a function f *operator convex* (resp. *operator concave*) if for each $k \in \mathbb{N}$, f is k-convex (resp. k-concave) and *operator monotone* if for each $k \in \mathbb{N}$ f is k-monotone.

In [15] Tomiyama and the author discussed about the following 3 assertions at each level n among them in order to see clear insight of the double piling structure of matrix monotone functions and of matrix convex functions:

Theorem 2.1. *Let $n \in \mathbb{N}$ and $f : [0,\alpha) \to \mathbb{R}$ and consider the following assertions.*

(i) $f(0) \leq 0$ *and* f *is n-convex in* $[0,\alpha)$,
(ii) *For each matrix a with its spectrum in* $[0,\alpha)$ *and a contraction c in the matrix algebra* M_n,

$$f(c^*ac) \leq c^*f(a)c,$$

(iii) *The function* $\frac{f(t)}{t}$ $(= g(t))$ *is n-monotone in* $(0,\alpha)$.

Then we have

$$(i)_{n+1} \prec (ii)_n \sim (iii)_n \prec (i)_{[\frac{n}{2}]},$$

where the denotation $(A)_m \prec (B)_n$ *means that "if* (A) *holds for the matrix algebra* M_m*, then* (B) *holds for the matrix algebra* M_n*".*

The following result is proved in [5].

Lemma 2.1. *Let f be a strictly positive, continuous function on $[0,\infty)$. If the function f is $2n$-monotone, then for any positive semi-definite A and a contraction C in M_n we have*

$$C^*f(A)C \leq f(C^*AC).$$

The following result is essentially proved in [7, Theorem 2.4], but for the reader's convenience we will include a proof.

Proposition 2.1. *Let f be a strictly positive, continuous function on $[0,\infty)$. If f is $2n$-monotone, the function $g(t) = \frac{t}{f(t)}$ is n-monotone on $[0,\infty)$.*

Proof. Let A, B be positive matrices in M_n such that $0 < A \leq B$.

Let $C = B^{-\frac{1}{2}}A^{\frac{1}{2}}$. Then $\|C\| \leq 1$. Since f is $2n$-monotone, $-f$ satisfies the Jensen type inequality from Lemma 2.1, that is,

$$-f(A) = -f(C^*BC) \leq -C^*f(B)C$$
$$-f(A) \leq -A^{\frac{1}{2}}B^{-\frac{1}{2}}f(B)B^{-\frac{1}{2}}A^{\frac{1}{2}}$$
$$-A^{-\frac{1}{2}}f(A)A^{-\frac{1}{2}} \leq -B^{-\frac{1}{2}}f(B)B^{-\frac{1}{2}}$$
$$-A^{-1}f(A) \leq -B^{-1}f(B)$$

Therefore, since $-1/t$ is operator monotone, $-1/(-f(t)/t) = t/f(t)$ is n-monotone. □

Remark 1. The condition of $2n$-monotonicity of f is needed to guarantee the n-monotonicity of g. Indeed, it is well-known that t^3 is monotone, but not 2-monotone. In this case the function $g(t) = \frac{t}{t^3} = \frac{1}{t^2}$ is obviously not 1-monotone.

Corollary 2.1. *Let f be a $2n$-monotone, continuous function on $[0,\infty)$ such that $f((0,\infty)) \subset (0,\infty)$, and let g be a Borel function on $[0,\infty)$ defined by $g(t) = \begin{cases} \frac{t}{f(t)} & (t \in (0,\infty)) \\ 0 & (t = 0) \end{cases}$. Then for any pair of positive matrices $A, B \in M_n$ with $A \leq B$, $g(A) \leq g(B)$.*

Similarly, we can get the concave version of the above observation [10].

Theorem 2.2. *For $n \in \mathbb{N}$ and $f : [0,\alpha) \to \mathbb{R}$ we consider the following assertions:*

(iv) $f(0) \geq 0$ *and f is n-concave in $[0,\alpha)$,*

(v) *For each matrix a with spectrum in $[0,\alpha)$ and a contraction c in the matrix algebra M_n,*

$$f(c^\star ac) \geq c^\star f(a)c,$$

(vi) *The function $\frac{t}{f(t)}$ is n-monotone in $(0,\alpha)$.*

We can show that

$$(iv)_{n+1} \prec (v)_n \sim (vi)_n \prec (iv)_{[\frac{n}{2}]}.$$

3. Generalized Powers-Størmer's Inequality

In this section we investigate the generalized Powers-Størmer inequality from the point of matrix functions. Note that the $2n$-monotonicity of a function f on $[0,\infty)$ implies the n-concavity of f by [2, Theorem V.2.5].

Theorem 3.1. *Let* Tr *be the canonical trace on* M_n *and* f *be a* $(n+1)$*-concave (or* $2n$*-monotone) function on* $[0,\infty)$ *such that* $f((0,\infty)) \subset (0,\infty)$*. Then for any pair of positive matrices* $A, B \in M_n$

$$\mathrm{Tr}(A) + \mathrm{Tr}(B) - \mathrm{Tr}(|A-B|) \leq 2\,\mathrm{Tr}(f(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}}), \tag{2}$$

where $g(t) = \begin{cases} \frac{t}{f(t)} & (t \in (0,\infty)) \\ 0 & (t = 0) \end{cases}$.

Proof. Note that we know that a function g is n-monotone from Corollary 2.1 and Theorem 2.2.

Let A, B be any positive matrices in M_n.

For operator $(A-B)$ let us denote by $P = (A-B)^+$ and $Q = (A-B)^-$ its positive and negative part, respectively. Then we have

$$A - B = P - Q \quad \text{and} \quad |A-B| = P + Q, \tag{3}$$

from that it follows that

$$A + Q = B + P. \tag{4}$$

On account of (4) the inequality (2) is equivalent to the following

$$\mathrm{Tr}(A) - \mathrm{Tr}(f(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}}) \leq \mathrm{Tr}(P).$$

Since $B + P \geq B \geq 0$ and $B + P = A + Q \geq A \geq 0$ and g is n-monotone, we have $g(A) \leq g(B+P)$ and

$$\begin{aligned}
&\mathrm{Tr}(A) - \mathrm{Tr}(f(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}}) \\
&= \mathrm{Tr}(f(A)^{\frac{1}{2}} g(A) f(A)^{\frac{1}{2}}) - \mathrm{Tr}(f(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}}) \\
&\leq \mathrm{Tr}(f(A)^{\frac{1}{2}} g(B+P) f(A)^{\frac{1}{2}}) - \mathrm{Tr}(f(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}}) \\
&= \mathrm{Tr}(f(A)^{\frac{1}{2}} (g(B+P) - g(B)) f(A)^{\frac{1}{2}}) \\
&\leq \mathrm{Tr}(f(B+P)^{\frac{1}{2}} (g(B+P) - g(B)) f(B+P)^{\frac{1}{2}}) \\
&= \mathrm{Tr}(f(B+P)^{\frac{1}{2}} g(B+P) f(B+P)^{\frac{1}{2}}) \\
&- \mathrm{Tr}(f(B+P)^{\frac{1}{2}} g(B) f(B+P)^{\frac{1}{2}}) \\
&\leq \mathrm{Tr}(B+P) - \mathrm{Tr}(f(B)^{\frac{1}{2}} g(B) f(B)^{\frac{1}{2}}) \\
&= \mathrm{Tr}(B+P) - \mathrm{Tr}(B) \\
&= \mathrm{Tr}(P).
\end{aligned}$$

Hence, we have the conclusion.

□

Corollary 3.1. *Let f be an operator monotone function on $[0,\infty)$ such that $f((0,\infty)) \subset (0,\infty)$. Then for any pair of positive matrices $A, B \in M_n$*

$$\text{(5)} \qquad \mathrm{Tr}(A) + \mathrm{Tr}(B) - \mathrm{Tr}(|A-B|) \le 2\,\mathrm{Tr}(f(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}}),$$

where $g(t) = \begin{cases} \frac{t}{f(t)} & (t \in (0,\infty)) \\ 0 & (t = 0) \end{cases}$.

Note that the operator monotonicity is equivalent to the operator concavity in the case that $f([0,\infty)) \subset [0,\infty)$ [2, Theorem V. 2.5].

Corollary 3.2. [1, Theorem 1] *Let A and B be positive matrices, then for all $s \in [0,1]$*

$$\mathrm{Tr}(A + B - |A-B|) \le \mathrm{Tr}(A^s B^{1-s}).$$

Remark 2. As pointed in Proposition 2.1, 2-monotonicity of f is needed to guarantee the inequality (2). Indeed, let $f(t) = t^3$ and $n = 1$. Then, for any $a, b \in (0,\infty)$, the inequality (2) would imply

$$a \le f(a)^{\frac{1}{2}} g(b) f(a)^{\frac{1}{2}},$$

that is,

$$\frac{a}{f(a)} \le \frac{b}{f(b)}.$$

Since $\frac{t}{f(t)}$ is, however, not 1-monotone, the latter inequality is impossible.

Remark 3. For matrices $A, B \in M_n^+$ let us denote

$$\text{(6)} \qquad Q(A,B) = \min_{s\in[0,1]} \mathrm{Tr}(A^{(1-s)/2} B^s A^{(1-s)/2})$$

and

$$\text{(7)} \qquad Q_{\mathcal{F}_{2n}}(A,B) = \inf_{f\in\mathcal{F}_{2n}} \mathrm{Tr}(f(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}}),$$

where $\mathcal{F}_{2n}$ is the set of all $2n$-monotone functions on $[0,+\infty)$ satisfy condition of the Theorem 3.1 and $g(t) = tf(t)^{-1}$ $(t \in [0,+\infty))$.

Since the class of $2n$-monotone functions is large enough [14], we know that $Q_{\mathcal{F}_{2n}}(A,B) \le Q(A,B)$. Hence, we hope on finding another $2n$-monotone function f on $[0,+\infty)$ such that

$$\text{(8)} \qquad \mathrm{Tr}(f(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}}) < Q(A,B).$$

If we can find such a function, then we may get smaller upper bound than what is used in quantum hypothesis testing [1]. For example, considering the trace distance $T(A,B) = \dfrac{\mathrm{Tr}(|A-B|)}{2}$, we might have the following better estimate

$$\frac{1}{2}\mathrm{Tr}(A+B) - Q_{\mathcal{F}_{2n}}(A,B) \leq T(A,B) \leq \sqrt{\{\frac{1}{2}\mathrm{Tr}(A+B)\}^2 - Q_{\mathcal{F}_{2n}}(A,B)^2}.$$

(See the estimate (6) in [1].)

4. Characterizations of the trace property

In this section the generalized Powers-Størmer inequality in the previous section implies the trace property for a positive linear functional on operator algebras.

Lemma 4.1. *Let φ be a positive linear functional on M_n and f be a continuous function on $[0,\infty)$ such that $f(0) = 0$ and $f((0,\infty)) \subset (0,\infty)$. If the following inequality*

$$\varphi(A+B) - \varphi(|A-B|) \leq 2\varphi(f(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}}) \tag{9}$$

holds true for all $A, B \in M_n^+$, then φ should be a positive scalar multiple of the canonical trace Tr *on M_n, where $g(t) = \begin{cases} \frac{t}{f(t)} & (t \in (0,\infty)) \\ 0 & (t = 0) \end{cases}$.*

Theorem 4.1. *Let φ be a positive normal linear functional on a von Neumann algebra $\mathcal{M}$ and f be a continuous function on $[0,\infty)$ such that $f(0) = 0$ and $f((0,\infty)) \subset (0,\infty)$. If the following inequality*

$$\varphi(A) + \varphi(B) - \varphi(|A-B|) \leq 2\varphi(f(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}}) \tag{10}$$

holds true for any pair $A, B \in \mathcal{M}^+$, then φ is a trace, where $g(t) = \begin{cases} \frac{t}{f(t)} & (t \in (0,\infty)) \\ 0 & (t = 0) \end{cases}$.

Corollary 4.1. *Let φ be a positive linear functional on a C^*-algebra $\mathcal{A}$ and f be a continuous function on $[0,\infty)$ such that $f(0) = 0$ and $f((0,\infty)) \subset (0,\infty)$. If the following inequality*

$$\varphi(A) + \varphi(B) - \varphi(|A-B|) \leq 2\varphi(f(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}}) \tag{11}$$

holds true for any pair $A, B \in \mathcal{A}^+$, then φ is a tracial functional, where $g(t) = \begin{cases} \frac{t}{f(t)} & (t \in (0,\infty)) \\ 0 & (t = 0) \end{cases}$.

Remark 4. Let $\mathcal{A}$ be a von Neumann algebra and φ be a positive normal linear functional on $\mathcal{A}$. The set $P(\mathcal{A})$ of all orthogonal projections from $\mathcal{A}$ is enough as a testing space for some inequality to characterize the trace property of φ (see [3]). But, in the case of the inequality (10) the set $P(\mathcal{A})$ is not enough as a testing set.

Indeed, let P, Q be arbitrary orthogonal projections from a von Neumann algebra $\mathcal{M}$. Since $Q \geq P \wedge Q$ it follows that $PQP \geq P(P \wedge Q)P = P \wedge Q$. So $PQP \geq P \wedge Q$ holds for any pair of projections. From that it follows

$$\varphi(P+Q-|P-Q|) = 2\varphi(P \wedge Q) \leq 2\varphi(PQP) = 2\varphi(f(P)^{\frac{1}{2}} g(Q) f(P)^{\frac{1}{2}})$$

whenever φ is an arbitrary positive linear functional on $\mathcal{M}$.

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