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Semi-operator monotonicity for operator monotone functions

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We review results on operator monotone functions. For details, we refer [11].

Loewner and Kwong matrices

Let $f(t)$ be a continuously differentiable function from the interval $(0,\infty)$ into itself. For distinct $t_1,\ldots,t_n$ in $(0,\infty)$, we define the $n\times n$ matrix $L_{f(t)}(t_1,\ldots,t_n)$ as

$$L_{f(t)}(t_1,\ldots,t_n):=\left[\frac{f(t_i)-f(t_j)}{t_i-t_j}\right],$$

where the diagonal entries are understood as the first derivatives $f'(t_i)$. This matrix is called a *Loewner matrix*. Similarly we define the $n\times n$ matrix $K_{f(t)}(t_1,\ldots,t_n)$ as

$$K_{f(t)}(t_1,\ldots,t_n):=\left[\frac{f(t_i)+f(t_j)}{t_i+t_j}\right],$$

which we call an *Kwong matrix*. (In [2, 8] it is called an anti-Loewner matrix.) See [3, 4, 5] on Loewner and Kwong matrices.

We also define the $n\times n$ matrix $L_{f(t)}^{(m)}(t_1,\ldots,t_n)$ and $K_{f(t)}^{(m)}(t_1,\ldots,t_n)$ as

$$\begin{aligned} L_{f(t)}^{(m)}(t_1,\ldots,t_n) &:= \left[\frac{\{f(t_i)\}^m-\{f(t_j)\}^m}{t_i^m-t_j^m}\right], \\ K_{f(t)}^{(m)}(t_1,\ldots,t_n) &:= \left[\frac{\{f(t_i)\}^m+\{f(t_j)\}^m}{t_i^m+t_j^m}\right] \end{aligned}$$

for a positive integer m.

It is well-known that $f(t)$ is operator monotone if and only if for all n and $t_1,\ldots,t_n$, the Loewner matrices $L_{f(t)}(t_1,\ldots,t_n)$ are positive semidefinite; see [10]. If $f(t)$ is operator monotone, the Kwong matrices $K_{f(t)}(t_1,\ldots,t_n)$ are positive semidefinite; see [9]. The latter is recently characterized by

Audenaert [2]. On the other hand, it is known that if $f(t)$ is operator monotone, so is the function $t \mapsto \{f(t^{1/m})\}^m$ for any positive integer m. See [1, 7]. Hence, combining them, we see that if f is operator monotone, then the Loewner matrices $L_{\{f(t^{1/m})\}^m}(t_1, \ldots, t_n)$ and the Kwong matrices $K_{\{f(t^{1/m})\}^m}(t_1, \ldots, t_n)$ are positive semidefinite; therefore, so are $L^{(m)}_{f(t)}(t_1, \ldots, t_n)$ and $K^{(m)}_{f(t)}(t_1, \ldots, t_n)$.

We have an alternative proof for operator monotonicity of the function $t \mapsto \{f(t^{1/m})\}^m$ by Theorem 1.6 that if f is operator monotone, then $L^{(m)}_{f(t)}(t_1, \ldots, t_n)$ are positive semidefinite for all n and $t_1, \ldots, t_n$ in $(0, \infty)$. We also have in Theorem 1.5 that if f is operator monotone, then $K^{(m)}_{f(t)}(t_1, \ldots, t_n)$ are positive semidefinite for all n and $t_1, \ldots, t_n$ in $(0, \infty)$.

We recall several facts:

Theorem 1.1 (Löwner [10]) Let f be a C^1 function on $(0, \infty)$. Then f is operator monotone if and only if $L_{f(t)}(t_1, \ldots, t_n)$ are positive semidefinite for all positive integers n and $t_1, \ldots, t_n$ in $(0, \infty)$.

Theorem 1.2 (Kwong [9]) Let f be a positive C^1 function on $(0, \infty)$. If f is operator monotone, then $K_{f(t)}(t_1, \ldots, t_n)$ are positive semidefinite for all positive integers n and $t_1, \ldots, t_n$ in $(0, \infty)$.

Theorem 1.3 (Audenaert [2]) Let f be a positive C^1 function on $(0, \infty)$. For all positive integers n and $t_1, \ldots, t_n$ in $(0, \infty)$ $K_{f(t)}(t_1, \ldots, t_n)$ are positive semidefinite if and only if $f(\sqrt{t})\sqrt{t}$ is operator monotone.

Theorem 1.4 (Ando [1], Fujii-Fujii [7]) Let f be an operator monotone function from $(0, \infty)$ into itself. Then so is the function $t \mapsto \{f(t^{1/m})\}^m$ for any positive integer m.

We have the following theorems in [11]:

Theorem 1.5 Let f be an operator monotone function from $(0, \infty)$ into itself. Then for any positive integer m, $K^{(m)}_{f(t)}(t_1, \ldots, t_n)$ are positive semidefinite for all positive integers n and $t_1, \ldots, t_n$ in $(0, \infty)$: or $K_{\{f(t^{1/m})\}^m}(t_1, \ldots, t_n)$ are positive semidefinite for all positive integers n and $t_1, \ldots, t_n$ in $(0, \infty)$.

Theorem 1.6 Let f be an operator monotone function from $(0, \infty)$ into itself. Then for any positive integer m, $L^{(m)}_{f(t)}(t_1, \ldots, t_n)$ are positive semidefinite for all positive integers n and $t_1, \ldots, t_n$ in $(0, \infty)$: or $L_{\{f(t^{1/m})\}^m}(t_1, \ldots, t_n)$ are positive semidefinite for all positive integers n and $t_1, \ldots, t_n$ in $(0, \infty)$.

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