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GEOMETRIC MEAN MAJORIZATIONS

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ABSTRACT. We extend the notion of the classical majorization of real numbers to positive definite matrices by taking a multivariable weighted geometric mean of positive definite matrices. Some connections between our geometric mean majorizations and classical results of the standard majorization of real numbers are presented.

1. INTRODUCTION

The Hardy-Littlewood-Pólya-Rado majorization theorem (cf. [2]) says in particular that for an $n \times n$ doubly stochastic matrix $W = (w_{ij})$ and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $y = Wx$ is a convex combination of the $n!$ vectors $x_\sigma = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ where σ varies over the permutation group of n-letters (Rado's theorem) and equivalently for every continuous convex function f defined on an interval I containing x_i and $y_i, i = 1, \ldots, n$, $\sum_{i=1}^n f(y_i) \leq \sum_{i=1}^n f(x_i)$. Furthermore for every continuous convex function $f : I^n \to \mathbb{R}$ invariant under the permutation of coordinates (Schur's convexity), $f(y_1, \ldots, y_n) \leq f(x_1, \ldots, x_n)$.

Embedding $\mathbb{R}^n$ into the space of $n \times n$ diagonal matrices and applying the exponential function, we may restate these beautiful results equivalently for the $n \times n$ positive diagonal matrices, where we replace the arithmetic mean by the geometric mean and a convex function by a geodesically convex function, $f(a^{1-t}b^t) \leq (1-t)f(a) + tf(b), t \in [0, 1]$. For instance, Rado's theorem is equivalent to the statement that for a positive diagonal matrix $\mathrm{diag}(a_1, \ldots, a_n)$, the diagonal matrix whose ii-th entry is the $\omega^i := (w_{i1}, \ldots, w_{in})$-weighted geometric mean of positive reals $a_1, \ldots, a_n$ is the diagonal matrix whose ii-th entry is the μ-weighted geometric mean of the $n!$ positive real numbers $a_{\sigma_1(i)}, \ldots, a_{\sigma_{n!}(i)}$. The main purpose of this paper is to extend these results

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on diagonal matrices to the non-commutative setting of positive definite matrices by taking a multivariable weighted geometric mean of positive definite matrices, a multivariable extension of the weighted geometric mean $A\#_t B = A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2}$ of two positive definite definite matrices A and B, and a geodesically convex function which satisfies $f(A\#_t B) \leq (1-t)f(A) + tf(B)$.

2. Weighted geometric means

A *symmetric weighted geometric mean* of n positive definite matrices is a map $G : \Delta_n \times \mathbb{P}^n \to \mathbb{P}$ that satisfies the following properties: For $\mathbb{A} = (A_1, \dots, A_n), \mathbb{B} = (B_1, \dots, B_n) \in \mathbb{P}^n, \sigma \in S^n$ a permutation on n-letters, $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{R}^n_{++}$ ($\mathbb{R}_{++} = (0, \infty)$), these are

- (P1) (Consistency with scalars) $G(\omega; \mathbb{A}) = A_1^{w_1} \cdots A_n^{w_n}$ if the A_i's commute;
- (P2) (Joint homogeneity) $G(\omega; a_1 A_1, \dots, a_n A_n) = a_1^{w_1} \cdots a_n^{w_n} G(\omega; \mathbb{A})$;
- (P3) (Permutation invariance) $G(\omega_\sigma; \mathbb{A}_\sigma) = G(\omega; \mathbb{A})$, where $\omega_\sigma = (w_{\sigma(1)}, \dots, w_{\sigma(n)})$ and $\mathbb{A}_\sigma = (A_{\sigma(1)}, \dots, A_{\sigma(n)})$;
- (P4) (Monotonicity) If $B_i \leq A_i$ for all $1 \leq i \leq n$, then $G(\omega; \mathbb{B}) \leq G(\omega; \mathbb{A})$;
- (P5) (Continuity) The map $G(\omega; \cdot)$ is continuous;
- (P6) (Congruence invariance) $G(\omega; M^*\mathbb{A}M) = M^*G(\omega; \mathbb{A})M$ for any invertible matrix M, where $M(A_1, \dots, A_n)M^* = (MA_1M^*, \dots, MA_nM^*)$;
- (P7) (Joint concavity) $G(\omega; \lambda\mathbb{A} + (1-\lambda)\mathbb{B}) \geq \lambda G(\omega; \mathbb{A}) + (1-\lambda)G(\omega; \mathbb{B})$ for $0 \leq \lambda \leq 1$;
- (P8) (Self-duality) $G(\omega; \mathbb{A}^{-1})^{-1} = G(\omega; \mathbb{A})$, where $\mathbb{A}^{-1} = (A_1^{-1}, \dots, A_n^{-1})$;
- (P9) (Determinental identity) $\mathrm{Det}G(\omega; A_1, \dots, A_n) = \prod_{i=1}^n (\mathrm{Det}A_i)^{w_i}$; and
- (P10) (AGH weighted mean inequalities) $(\sum_{i=1}^n w_i A_i^{-1})^{-1} \leq G(\omega; A_1, \dots, A_n) \leq \sum_{i=1}^n w_i A_i$.

The ω-weighted *Karcher mean*, also called the *least squares mean*, of n positive definite matrices $A_1, \dots, A_n$ and $\omega = (w_1, \dots, w_n) \in \Delta_n$ is defined as the unique minimizer of the sum of squares of the Riemannian trace metric distances to each of the A_i, i.e.,

$$\Lambda(\omega; A_1, \dots, A_n) = \underset{X \in \mathbb{P}}{\arg\min} \sum_{i=1}^n w_i \delta^2(X, A_i). \tag{2.1}$$

The Karcher mean coincides with the unique positive definite solution of the *Karcher equation*

$$\sum_{i=1}^{n} w_i \log(X^{1/2}A_i^{-1}X^{1/2}) = 0. \tag{2.2}$$

Theorem 2.1 ([4]). *The Karcher mean is a symmetric weighted geometric mean.*

The weighted BMP mean is indeed a symmetric weighted geometric mean; satisfying all the properties $(P1)-(P10)$ and that the weighted BMP mean is constructed by induction and the following symmetrization procedure:

(1) For $n = 2$, $\mathrm{Bmp}_2(w_1, w_2; A_1, A_2) = A_1\#_{w_2}A_2$.

(2) Assume that $\mathrm{Bmp}_{n-1}(\cdot;\cdot) : \Delta_{n-1} \times \mathbb{P}^{n-1} \to \mathbb{P}$ is defined. Let $\{A_i^{(r)}\}_{r=0}^{\infty}$ be the sequence defined by; $A_i^{(0)} = A_i$ and

$$A_i^{(r+1)} = A_i^{(r)}\#_{1-w_i}\mathrm{Bmp}_{n-1}\left((\frac{w_j}{1-w_i})_{j\neq i}; (A_j^{(r)})_{j\neq i}\right), \quad 1 \le i \le n, \tag{2.3}$$

where $(a_j)_{j\neq i} := (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)$. Then $\lim_{r\to\infty} A_i^{(r)}$ exists and has the same value for every i; we denote the common limit by $\lim_{r\to\infty} A_i^{(r)} = \mathrm{Bmp}_n(\omega; A_1, \cdots, A_n)$.

See [6, 5] for the weighted BMP mean in a general setting of metric spaces.

Definition 2.2. A subset $C \subset \mathbb{P}$ is called geodesically convex (occasionally, in context, convex) if $A\#_tB \in C$ for all $t \in [0,1]$ whenever $A, B \in C$. A function $f : C \to \mathbb{R}$ on a geodesically convex set C is called geodesically convex if for any $A, B \in C$ and $t \in [0,1]$, $f(A\#_tB) \le (1-t)f(A) + tf(B)$.

Definition 2.3. A weighted geometric mean $G : \Delta_n \times \mathbb{P}^n \to \mathbb{P}$ is called *convex* if

$$f(G(\omega; A_1, \ldots, A_n)) \le \sum_{i=1}^{n} w_i f(A_i)$$

for all $\omega = (w_1, \ldots, w_n) \in \Delta_n$, $(A_1, \ldots, A_n) \in \mathbb{P}^n$ and continuous geodesically convex functions $f : \mathbb{P} \to \mathbb{R}$. We denote by $\mathcal{C}_n$ the set of all weighted convex geometric n-means.

Theorem 2.4 ([7]). *There are infinitely many convex geometric means. In particular, the weighted BMP mean* Bmp_n *and the Karcher mean* Λ_n *are convex.*

3. Geometric mean majorizations

Definition 3.1. Let G be a weighted geometric n-mean and let $\mathbb{A} = (A_1, \dots, A_n), \mathbb{B} = (B_1, \dots, B_n) \in \mathbb{P}^n$. We say that $\mathbb{A}$ is G-majorized by $\mathbb{B}$ (abbreviated, $\mathbb{A} \prec^G \mathbb{B}$) if there exists a positive doubly stochastic matrix $W = (w_{ij})_{n\times n}$ such that for all $i = 1, \dots, n$,

$$A_i = G(w_{i1}, \dots, w_{in}; \mathbb{B}).$$

Remark 3.2. $[n = 2]$ Since the map $(t, A, B) \mapsto A\#_t B$ is the unique weighted geometric mean for $n = 2$, $(A_1, A_2) \prec (B_1, B_2)$ if and only if $A_1 = B_1\#_t B_2$ and $A_2 = B_1\#_{1-t}B_2 = B_2\#_t B_1$ for some $t \in [0, 1]$.

Example 3.3. We have $(G(\mathbb{A}), \dots, G(\mathbb{A})) \prec^G \mathbb{A}$ for all $\mathbb{A} \in \mathbb{P}^n$. Use

$$W = \frac{1}{n}\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ & & \ddots & \\ 1 & 1 & \cdots & 1 \end{pmatrix}.$$

The following are some basic properties of the geometric mean majorization.

Proposition 3.4 ([7]). *Suppose that* $\mathbb{A} \prec^G \mathbb{B}$.

(1) $M\mathbb{A}M^* \prec^G M\mathbb{B}M^*$ *for any non-singular* M.
(2) $\mathbb{A}^{-1} \prec^G \mathbb{B}^{-1}$.
(3) $\log \det\mathbb{A} \prec \log \det\mathbb{B}$, *where* $\det(\mathbb{A}) = (\det A_1, \dots, \det A_n)$.
(4) *If* G *is symmetric, then* $\mathbb{A}_\sigma \prec^G \mathbb{B}_\sigma$ *for any permutation* σ.

The next theorem is a partial extension of Hardy-Littlewood-Póyla Theorem to convex geometric means of positive definite matrices.

Theorem 3.5 ([7]). *Let* G *be a convex geometric mean. If* $\mathbb{A} \prec^G \mathbb{B}$, *then* $\sum_{i=1}^n f(A_i^{\pm 1}) \leq \sum_{i=1}^n f(B_i^{\pm 1})$ *for any continuous geodesically convex function* f.

Let

$$S^n = \{\sigma_i : 1 \leq i \leq n!\},$$

the set of all permutations of n-letters. For $\sigma \in S^n$ and $\mathbb{A} = (A_1, \dots, A_n) \in \mathbb{P}^n$, we denote

$$\mathbb{A}_\sigma = (A_{\sigma(1)}, \dots, A_{\sigma(n)}) \in \mathbb{P}^n.$$

The Karcher mean plays an important role for our geometric mean majorization.

Theorem 3.6 (Rado's theorem, [7]). *$\mathbb{A} \prec^{\Lambda_n} \mathbb{B}$ if and only if $\mathbb{A} = \Lambda_{n!}(\omega; \mathbb{B}_{\sigma_1}, \ldots, \mathbb{B}_{\sigma_{n!}})$ for some $\omega \in \overline{\Delta_{n!}}$.*

Corollary 3.7 (Schur's convexity, [7]). *If $\mathbb{A} \prec^{\Lambda_n} \mathbb{B}$, then*

(1) *$f(\mathbb{A}) \leq f(\mathbb{B})$ for any continuous geodesically convex function $f : \mathbb{P}^n \to \mathbb{R}$ invariant under the permutation of coordinates;*

(2) *$\mathbb{A}$ lies in the convex hull of the $n!$ permutations of $\mathbb{B}$ in the product space $\mathbb{P}^n$.*

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