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Some reduced expressions of the classical Weyl groups and the Weyl groupoids of the Lie superalgebras osp$(2m|2n)$

Hiroyuki Yamane[†]

Abstract

We give some reduced expressions of the classical Weyl groups $W(\mathrm{A}_{N-1})$, $W(\mathrm{B}_{N})=W(\mathrm{C}_{N})$, $W(\mathrm{D}_{N})$ and the Weyl groupoid of the Lie superalgebra osp$(2m|2(N-m))$.

1 Some reduced expressions of the classical Weyl groups

For m, $n\in \mathbb{Z}$, let $J_{n,m}:=\{k\in \mathbb{Z}\,|\,m\leq k\leq n\}$.

Let $N\in \mathbb{N}$. Let $\mathrm{M}_{N}(\mathbb{R})$ be the $\mathbb{R}$-algebra of $N\times N$-matrices. For k, $r\in J_{1,N}$, let $E_{k,r}:=[\delta_{k,k'}\delta_{r,r'}]_{k',r'\in J_{1,N}}\in \mathrm{M}_{N}(\mathbb{R})$, that is $E_{k,r}$ is the matrix unite such that its (k, r)-component is 1 and the other components is 0. Then $\mathrm{M}_{N}(\mathbb{R})=\oplus_{k,r\in J_{1,N}}\mathbb{R}E_{k,r}$. Let $\mathbb{R}^{N}$ denote the $\mathbb{R}$-linear space of $N\times 1$-matrices. For $k\in J_{1,N}$, let e_{k} is the element of $\mathbb{R}^{N}$ such that its $(k, 1)$-component is 1 and the other components is 0. That is $\{e_{k}|k\in J_{1,N}\}$ is the standard basis of $\mathbb{R}^{N}$. The $\mathbb{R}$-algebra $\mathrm{M}_{N}(\mathbb{R})$ acts on $\mathbb{R}^{N}$ in the ordinal way, that is $E_{k,r}e_{p}=\delta_{r,p}e_{r}$. Let $\mathrm{GL}_{N}(\mathbb{R})$ be the group of invertible $N\times N$-matrices, that is $\mathrm{GL}_{N}(\mathbb{R})=\{X\in \mathrm{M}_{N}(\mathbb{R})|\det X\neq 0\}$. Let $(\ ,\ ):\mathbb{R}^{N}\times \mathbb{R}^{N}\rightarrow \mathbb{R}$ be the $\mathbb{R}$-bilinear map defined by $(e_{k}, e_{r}):=\delta_{kr}$.

[†]Department of Pure and Applied Mathematics, Graduate School of Information Science and Technology, Osaka University, Toyonaka, Osaka, 560-0043, Japan, E-mail: yamane@ist.osaka-u.ac.jp

Definition 1.1. *For $v \in \mathbb{R}^N \setminus \{0\}$, define $s_v \in \mathrm{GL}_N(\mathbb{R})$ by $s_v(u) := u - \frac{2(u,v)}{(v,v)} v$ ($u \in \mathbb{R}^N$), that is s_v is the reflection with respect to v.*

Note that

$$s_v^2 = 1. \tag{1.1}$$

We say that a subset R of $\mathbb{R}^N \setminus \{0\}$ is a *root system* (in $\mathbb{R}^N$) if $|R| < \infty$, $s_v(R) = R$ and $\mathbb{R}v \cap R = \{v, -v\}$ for all $v \in R$, see [Hum, 1.1].

Let R be a root system in $\mathbb{R}^N$. We say that a subset Π of R is a *root basis* of Π if Π is a (set) basis of $\mathrm{Span}_{\mathbb{R}}(\Pi)$ as an $\mathbb{R}$-linear space and $R \subset \mathrm{Span}_{\mathbb{R}_{\geq 0}}(\Pi) \cup -\mathrm{Span}_{\mathbb{R}_{\geq 0}}(\Pi)$ (this is called a *simple system* in [Hum, 1.3]).

Let R be a root system in $\mathbb{R}^N$. Let Π be a root basis of R. Let $R^+(\Pi) := R \cap \mathrm{Span}_{\mathbb{R}_{\geq 0}}(\Pi)$. We call $R^+(\Pi)$ *a positive root system of R associated with* Π (this is called a *positive system* in [Hum, 1.3]).

Definition 1.2. (See [Hum, 2.10].) Let R be a root system in $\mathbb{R}^N$. Let Π be a root basis of R.

(1) Assume $N \geq 2$. We call R the A_{N-1}*-type root system* if

$$R = \{\, e_x - e_y \,|\, x, y \in J_{1,N},\ x \neq y \,\}.$$

We call Π the A_{N-1}*-type standard root basis* if

$$\Pi = \{\, e_x - e_{x+1} \,|\, x \in J_{1,N-1} \,\}.$$

(2) Assume $N \geq 2$. We call R the B_N*-type standard root system* if

$$R = \{\, ce_x + c'e_y \,|\, x, y \in J_{1,N},\ x < y,\ c,\ c' \in \{1, -1\} \,\} \cup \{\, c''e_z \,|\, c'' \in \{1, -1\} \,\}.$$

We call Π the B_N*-type standard root basis* if

$$\Pi = \{\, e_x - e_{x+1} \,|\, x \in J_{1,N-1} \,\} \cup \{\, e_N \,\}.$$

(3) Assume $N \geq 2$. We call R the C_N*-type root system* if

$$R = \{\, ce_x + c'e_y \,|\, x, y \in J_{1,N},\ x < y,\ c,\ c' \in \{1, -1\} \,\} \cup \{\, 2c''e_z \,|\, c'' \in \{1, -1\} \,\}.$$

We call Π the C_N*-type standard root basis* if

$$\Pi = \{\, e_x - e_{x+1} \,|\, x \in J_{1,N-1} \,\} \cup \{\, 2e_N \,\}.$$

(4) Assume $N \geq 4$. We call R the D_N*-type root system* if

$$R = \{\, ce_x + c'e_y \,|\, x, y \in J_{1,N},\ x < y,\ c,\ c' \in \{1, -1\}\,\}.$$

We call Π the D_N*-type standard root basis* if

$$\Pi = \{\, e_x - e_{x+1} \,|\, x \in J_{1,N-1}\,\} \cup \{\, e_{N-1} + e_N \,\}.$$

Let R be a root system in $\mathbb{R}^N$. Let Π be a root basis of R. Let $W(\Pi)$ be the subgroup of $\mathrm{GL}_N(\mathbb{R})$ generated by all s_v with $v \in \Pi$. We call $W(\Pi)$ the *Coxeter group associated with* (R, Π). Let $S(\Pi) := \{\, s_v \in W(\Pi) \,|\, v \in \Pi\,\}$. We call $(W(\Pi), S(\Pi))$ the *Coxeter system associated with* (R, Π), see [Hum, 1.9 and Theorem 1.5]. Define the map $\ell : W(\Pi) \to \mathbb{Z}_{\geq 0}$ in the following way, see [Hum, 1.6]. Let $\ell(1) := 0$, where 1 is a unit of $W(\Pi)$. Note that an arbitrary $w \in W(\Pi)$ can be written as a product of finite s_v's with some $v \in \Pi$, say $w = \underbrace{s_{v_1} \cdots s_{v_r}}_{r}$ for some $r \in \mathbb{N}$ and some $v_x \in \Pi$ $(x \in J_{1,r})$. If $w \neq 1$, let $\ell(w)$ be the smallest r for which such an expression exists, and call the expression *reduced.* For $w \in W(\Pi)$, we call $\ell(w)$ the *length of* w. Let

$$\mathfrak{L}(w) := \{\, v \in R^+(\Pi) \,|\, w(v) \in -R^+(\Pi)\}.$$

It is well-known that

$$\ell(w) = |\mathfrak{L}(w)| \tag{1.2}$$

(see [Hum, Corollary 1.7]). It is also well-known that for $v \in \Pi$,

$$s_v(R^+(\Pi) \setminus \{v\}) = R^+(\Pi) \setminus \{v\} \tag{1.3}$$

(see [Hum, Propsoition 1.4]), and

$$\ell(ws_v) = \begin{cases} \ell(w) + 1 & \text{if } w(v) \in R^+(\Pi), \\ \ell(w) - 1 & \text{if } w(v) \in -R^+(\Pi) \end{cases} \tag{1.4}$$

(see [Hum, Lemma 1.6 and Corollary 1.7]). Assume that $|R| < \infty$. By the above properties, we can see that there exists a unique $w_\circ \in W(\Pi)$ such that $w_\circ(\Pi) = -\Pi$, see [Hum, 1.8]. It is well-known that

$$\ell(w_\circ) = |R^+(\Pi)|, \tag{1.5}$$

which can easily be proved by (1.2), (1.3) and (1.4). Note that $w_\circ$ is the only element $W(\Pi)$ that $\ell(w) \leq \ell(w_\circ)$ for all $w \in W(\Pi)$, and $\ell(w) = \ell(w_\circ) - \ell(w_\circ w^{-1})$ for all $w \in W(\Pi)$. We call $w_\circ$ *the longest element of the Coxeter system* of $(W(\Pi), S(\Pi))$.

Let k, $r \in J_{1,N}$ be such that $k \leq r$. For $z_p \in J_{k,r} \cup (-J_{k,r})$ ($p \in J_{k,r}$) with $|u_p| \neq |u_t|$ ($p \neq t$), let

$$\left\{ \begin{array}{cccc} k & k+1 & \dots & r \\ z_k & z_{k+1} & \dots & z_r \end{array} \right\} := \sum_{p \in J_{k,r}} \frac{z_p}{|z_p|} E_{|z_p|,p} + \sum_{t \in J_{1,N} \setminus J_{k,r}} E_{t,t} \in \mathrm{GL}_N(\mathbb{R}).$$

We have

$$s_{e_k} = \left\{ \begin{array}{c} k \\ -k \end{array} \right\} \qquad (k \in J_{1,N}), \tag{1.6}$$

$$s_{e_k - e_{k+1}} = \left\{ \begin{array}{cc} k & k+1 \\ k+1 & k \end{array} \right\} \qquad (k \in J_{1,N-1}), \tag{1.7}$$

and

$$s_{e_k + e_{k+1}} = \left\{ \begin{array}{cc} k & k+1 \\ -(k+1) & -k \end{array} \right\} \qquad (k \in J_{1,N-1}). \tag{1.8}$$

Let k, p, $r \in J_{k,r}$ with $k < r$ and $k \leq p \leq r$, let

$$\left\{ \begin{array}{ccccccc} k & \dots & p & ; & p+1 & \dots & r \\ z_k & \dots & z_p & ; & z_{p+1} & \dots & z_r \end{array} \right\} := \left\{ \begin{array}{ccc} k & \dots & p \\ z_k & \dots & z_p \end{array} \right\} \left\{ \begin{array}{ccc} p+1 & \dots & r \\ z_{p+1} & \dots & z_r \end{array} \right\}.$$

Let k, $r \in J_{1,N-1}$ with $k \leq r$. Define $s_{(k,r)}$ inductively by

$$s_{(k,r)} := \left\{ \begin{array}{ll} 1 & \text{if } k = r \\ s_{(k,r-1)} s_{e_{r-1} - e_r} & \text{if } k < r. \end{array} \right. \tag{1.9}$$

Then, if $r > k$, we have

$$s_{(k,r)} = \left\{ \begin{array}{cccccccc} k & \dots & p & \dots & r-1 & ; & r \\ k+1 & \dots & p+1 & \dots & r & ; & k \end{array} \right\}, \tag{1.10}$$

since (if $r \geq k+2$)

$$\begin{aligned} s_{(k,r)} &= s_{(k,r-1)} s_{e_{r-1} - e_r} \\ &= \left\{ \begin{array}{ccccccc} k & \dots & p & \dots & r-2 & ; & r-1 \\ k+1 & \dots & p+1 & \dots & r-1 & ; & k \end{array} \right\} \left\{ \begin{array}{cc} r-1 & r \\ r & r-1 \end{array} \right\} \\ &\qquad \text{(by (1.7) and an induction)} \\ &= \left\{ \begin{array}{ccccccc} k & \dots & p & \dots & r-1 & ; & r \\ k+1 & \dots & p+1 & \dots & r & ; & k \end{array} \right\}. \end{aligned} \tag{1.11}$$

Define $s_{(r,k)}$ inductively by $s_{(r,k)} := s_{e_{r-1}-e_r} s_{(r-1,k)}$ if $r \geq k+1$. Clearly (if $r > k$) we have

$$(1.12) \qquad s_{(r,k)} = s_{(k,r)}^{-1} = \left\{ \begin{array}{cccccccc} k & ; & k+1 & \ldots & p & \ldots & r \\ r & ; & k & \ldots & p-1 & \ldots & r-1 \end{array} \right\}.$$

Lemma 1.3. *Let Π be the A_{N-1}-type standard root basis. Let $w_\circ$ be the longest element of $(W(\Pi), S(\Pi))$. Let $s_k := s_{e_k - e_{k+1}} \in S(\Pi)$ for $k \in J_{1,N-1}$.*

(1) *We have*

$$(1.13) \qquad w_\circ = \left\{ \begin{array}{ccccc} 1 & \ldots & p & \ldots & N \\ N & \ldots & N-p+1 & \ldots & 1 \end{array} \right\}.$$

Moreover

$$(1.14) \qquad w_\circ = \underbrace{(s_1 s_2 \cdots s_{N-1})}_{N-1} \underbrace{(s_1 s_2 \cdots s_{N-2})}_{N-2} \cdots \underbrace{(s_1 s_2)}_{2} \underbrace{s_1}_{1}.$$

Furthermore RHS of (1.14) *is the reduced expression of $w_\circ$.*

(2) *Let $m \in J_{2,N-1}$. Then*

(1.15)

$$\begin{aligned} w_\circ \;=\; & \underbrace{(s_1 s_2 \cdots s_{m-1})}_{m-1} \underbrace{(s_1 s_2 \cdots s_{m-2})}_{m-2} \cdots \underbrace{(s_1 s_2)}_{2} \underbrace{s_1}_{1} \\ & \cdot \underbrace{(s_{m+1} s_{m+2} \cdots s_{N-1})}_{N-m-1} \underbrace{(s_{m+1} s_{m+2} \cdots s_{N-1})}_{N-m-2} \cdots \underbrace{(s_{m+1} s_{m+2})}_{2} \underbrace{s_{m+1}}_{1} \\ & \cdot \underbrace{(s_m s_{m+1} \cdots s_{N-1})}_{N-m} \underbrace{(s_{m-1} s_m \cdots s_{N-2})}_{N-m} \cdots \underbrace{(s_1 s_2 \cdots s_{N-m})}_{N-m}, \end{aligned}$$

and RHS of (1.15) *is a reduced expression of $w_\circ$.*

Proof. By (1.5), we have

$$(1.16) \qquad \ell(w) = \frac{N(N-1)}{2}.$$

Let k, $r \in J_{1,n}$ with $k < r$. Let

$$x_{(k,r)} := \left\{ \begin{array}{ccccc} k & \ldots & p & \ldots & r \\ r & \ldots & r-p+k & \ldots & k \end{array} \right\}.$$

Then

$$s_{(k,r)}s_{(k,r-1)}\cdots s_{(k,k+1)} = x_{(k,r)}, \tag{1.17}$$

since, if $r \geq k+2$, we have

$$\begin{aligned} & s_{(k,r)}\big(s_{(k,r-1)}\cdots s_{(k,k+1)}\big) \\ & = \left\{ \begin{matrix} k & \ldots & p & \ldots & r-1 & ; & r \\ k+1 & \ldots & p+1 & \ldots & r & ; & k \end{matrix} \right\} \cdot x_{(k,r-1)} \\ & \qquad \text{(by (1.11) and an induction)} \\ & = x_{(k,r)}. \end{aligned}$$

We have

$$x_{(k,r)} \in W(\Pi) \quad \text{and} \quad \ell(x_{(k,r)}) = \frac{(k-r+1)(k-r)}{2}, \tag{1.18}$$

where the first claim follows from (1.17) and the second claim follows from by (1.2), since $\mathfrak{L}(x_{(k,r)}) = \{e_x - e_y | k \leq x < y \leq r\}$.

We obtain the claim (1) from (1.16). (1.17) and (1.18) for $k = 1$ and $r = N$.

For k, r, $t \in J_{1,N-1}$ with $k < r \leq t$, let

(1.19)

$$\begin{aligned} & y_{(k,r-1;r,t)} \\ & := \left\{ \begin{matrix} k & \ldots & x & \ldots & r-1 & ; & r & \ldots & y & \ldots & t \\ k+t-r+1 & \ldots & x+t-r+1 & \ldots & t & ; & k & \ldots & y+k-r & \ldots & t+k-r \end{matrix} \right\} \end{aligned}$$

We have

$$s_{(k+t-r,t)}s_{(k+t-r-1,t-1)}\cdots s_{(k+1,r+1)}s_{(k,r)} = y_{(k,r-1;r,t)} \tag{1.20}$$

since, if $t > r$,

$$\begin{aligned} & \big(s_{(k+t-r,t)}s_{(k+t-r-1,t-1)}\cdots s_{(k+1,r+1)}\big)s_{(k,r)} \\ & = y_{(k+1,r;r+1,t)} \cdot \left\{ \begin{matrix} k & \ldots & p & \ldots & r-1 & ; & r \\ k+1 & \ldots & p+1 & \ldots & r & ; & k \end{matrix} \right\} \\ & \qquad \text{(by (1.11) and an induction)} \\ & = y_{(k,r-1;r,t)}. \end{aligned}$$

We have

$$y_{(k,r-1;r,t)} \in W(\Pi) \quad \text{and} \quad \ell(y_{(k,r-1;r,t)}) = (t-r+1)(r-k), \tag{1.21}$$

where the first claim follows from (1.20) and the second claim follows from by (1.2), since $\mathfrak{L}(x_{(k,r)}) = \{e_x - e_y | x \in J_{k,r-1},\ x \in J_{r,t}\}$.

Let $m \in J_{2,N-1}$. By (1.13), we have

$$w_\circ = x_{(1,m)} x_{(m+1,N)} y_{(1,N-m;N-m+1,N)}. \tag{1.22}$$

Then we obtain the claim (2) from (1.16), (1.18), (1.21) and (1.22), since $\frac{m(m-1)}{2} + \frac{(N-m)(N-m-1)}{2} + (N-m)m = \frac{N(N-1)}{2}$. □

Let k, $r \in J_{1,N}$ with $k \leq r$. Let

$$b_{(k,r)} := \underbrace{s_{e_k} \cdots s_{e_r}}_{r-k+1} = \left\{ \begin{array}{ccccc} k & \dots & p & \dots & r \\ -k & \dots & -p & \dots & -r \end{array} \right\}, \tag{1.23}$$

see also (1.6). By (1.10), we have

$$(s_{(k,r)})^{r-k+1} = 1. \tag{1.24}$$

By (1.6) and (1.10), we have

$$s_{e_t} s_{(k,r)} = s_{(k,r)} s_{e_{t-1}} \tag{1.25}$$

By (1.23), (1.24) and (1.25), for $t \in J_{k+1,r}$, we have

$$(s_{(k,r)} s_{e_r})^{r-k+1} = (s_{(k,r)})^{r-k+1} s_{e_k} \cdots s_{e_r} = b_{(k,r)}. \tag{1.26}$$

By (1.6), (1.10) and (1.12), we have

$$\underbrace{s_{e_k - e_{k+1}} \cdots s_{e_{r-1} - e_r}}_{k-r} s_{e_r} \underbrace{s_{e_{r-1} - e_r} \cdots s_{e_k - e_{k+1}}}_{k-r} = s_{(k,r)} s_{e_r} s_{(r,k)} = s_{e_k}. \tag{1.27}$$

Lemma 1.4. *Let Π be the B_N-type standard root basis. Let $w_\circ$ be the longest element of $(W(\Pi), S(\Pi))$. Let $s_k := s_{e_k - e_{k+1}} \in S(\Pi)$ for $k \in J_{1,N-1}$ and let $s_N := s_{e_N} \in S(\Pi)$.*

(1) *We have*

$$w_\circ = b_{(1,N)} = (\underbrace{s_1 s_2 \cdots s_N}_{N})^N. \tag{1.28}$$

Moreover the rightmost hand side of (1.28) *is a reduced expression of* $w_\circ$.

(2) *Let* $k,\ r \in J_{1,N}$ *with* $k \leq r$. *Then*

$$b_{(k,r)} = (\underbrace{s_k s_{k+1} \cdots s_{N-1} s_N s_{N-1} \cdots s_{r+1} s_r}_{2N-k-r+1})^{r-k+1}. \tag{1.29}$$

Moreover RHS of (1.29) *is a reduced expression of* $b_{(k,r)}$.

(3) *Let* $k_1,\ k_2, \ldots, k_{r-1} \in J_{1,N}$ *with* $k_1 < k_2 < \ldots < k_{r-1}$. *Let* $b'_y := b_{(k_{y-1},k_y-1)}$ $(y \in J_{1,r})$, *where let* $k_0 := 1$ *and* $k_r := N+1$. *Then we have* $w_\circ = b'_1 b'_2 \cdots b'_r$ *and* $\ell(w_\circ) = \sum_{y=1}^r \ell(b'_y)$. *Moreover* $b'_y b'_z = b'_z b'_y$ *for* $y,\ z \in J_{1,r}$.

(4) *Let* $m \in J_{1,N-1}$. *Then*

$$\begin{aligned} w_\circ \ &= \ (\underbrace{s_{N-m+1} s_{N-m+2} \cdots s_N}_{m})^m \\ &\quad \cdot(\underbrace{s_1 s_2 \cdots s_{N-1} s_N s_{N-1} \cdots s_{N-m+1} s_{N-m}}_{N+m})^{N-m}. \end{aligned} \tag{1.30}$$

Moreover RHS of (1.30) *is a reduced expression of* $w_\circ$.

Proof. We can easily show (1.29) by (1.26) and (1.27).

Let $k,\ r \in J_{1,N}$ be such that $k \leq r$. Note that

$$\mathfrak{L}(b_{(k,r)}) = \{\, e_t \,|\, t \in J_{k,r} \} \cup \{\, e_t + c e_{t'} \,|\, c \in \{-1,1\},\ t \in J_{k,r},\ t' \in J_{t',N} \,\}.$$

Hence by (1.2), we have

$$\begin{aligned} \ell(b_{(k,r)}) &= (r-k+1) + 2\textstyle\sum_{t=k}^r (N-t) \\ &= (r-k+1) + 2N(r-k+1) - 2(\tfrac{r(r+1)}{2} - \tfrac{k(k-1)}{2}) \\ &= (r-k+1)(1+2N-(r+k)) \\ &= (2N-k-r+1)(r-k+1). \end{aligned} \tag{1.31}$$

Hence we obtain the second claim of the claim (2). We also obtain the claim (1) since $|R^+(\Pi)| = N^2$.

Let $k,\ t,\ r \in J_{1,N}$ be such that $k \leq t < r$. By (1.23), we have

$$b_{(k,t)} b_{(t+1,r)} = b_{(k,r)}. \tag{1.32}$$

By (1.31), we have

$$
\begin{aligned}
&\ell(b_{(k,t)}) + \ell(b_{(t+1,r)}) \\
&= (2N-k-t+1)(t-k+1) + (2N-t-r)(r-t) \\
&= 2N(r-k+1) - (k+t-1)(t-k+1) - (t+r)(r-t) \\
&= 2N(r-k+1) - (-k^2+t^2+2k-1) - (r^2-t^2) \\
&= 2N(r-k+1) + (k^2-r^2-2k+1) \\
&= 2N(r-k+1) + (k-1+r)(k-1-r) \\
&= (2N-r-k-1)(r-k+1) \\
&= \ell(b_{(k,r)}).
\end{aligned} \tag{1.33}
$$

By (1.32), (1.32) and the claim (1), we get the claim (3).

The claim (4) follows immediately from the claims (1) and (2). □

Using Lemma 1.4, we have

Lemma 1.5. *Let* Π *be the* D_N*-type standard root basis. Let* $w_\circ$ *be the longest element of* $(W(\Pi), S(\Pi))$. *Let* $s_k := s_{e_k-e_{k+1}} \in S(\Pi)$ *for* $k \in J_{1,N-1}$ *and let* $s_N := s_{e_k+e_{k+1}} \in S(\Pi)$. *For* $k \in J_{1,N-1}$, *let*

$$d_{(k)} := (\underbrace{s_k \cdots s_{N-2}s_{N-1}s_N}_{N-k+1})^{N-k}. \tag{1.34}$$

Then

$$\ell(d_{(k)}) = (N-k)(N-k+1) \tag{1.35}$$

and

$$d_{(k)} = \begin{cases} b_{(k,N)} & \text{if } N-k \text{ is odd,} \\ b_{(k,N-1)} & \text{if } N-k \text{ is even.} \end{cases} \tag{1.36}$$

In particular,

$$w_\circ = d_{(1)}. \tag{1.37}$$

Proof. By (1.6), (1.7) and (1.8), we have

$$s_{N-1}s_N = \left\{ \begin{matrix} N-1 & N \\ -(N-1) & -N \end{matrix} \right\} = s_{e_{N-1}}s_{e_N}. \tag{1.38}$$

Then we have

$$\begin{aligned} &\text{RHS of (1.34)} \\ &= (s_{(k,N-1)}s_{e_{N-1}}s_{e_N})^{N-k} \quad \text{(by (1.38))} \\ &= (s_{(k,N-1)}s_{e_{N-1}})^{N-k}s_{e_N}^{N-k} \quad \text{(by (1.6) and (1.10))} \\ &= b_{(k,N-1)}s_{e_N}^{N-k} \quad \text{(by (1.26))} \\ &= \text{RHS of (1.36)} \end{aligned} \tag{1.39}$$

By (1.36), we have

$$\mathfrak{L}(d_{(k)}) = \{\, e_t + ce_{t'} \mid c \in \{-1,1\},\ t \in J_{k,r},\ t' \in J_{t',N} \,\}.$$

Hence by (1.2), we have (1.35) and (1.37). This completes the proof. □

2 Weyl groupoids of super CD-type

Let $m \in J_{1,N-1}$. Let $\mathcal{D}_{m|N-m}$ be the set of maps $a : J_{1,n} \to J_{0,1}$ with $|a^{-1}(\{0\})| = m$.

Let $a \in \mathcal{D}_{m|N-m}$. Let $(\,,\,)^a : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be the $\mathbb{R}$-bilinear map defined by $(e_i, e_j)^a := \delta_{ij} \cdot (-1)^{a(i)}$. For $v \in \mathbb{R}^N$ with $(v,v)^a \neq 0$, define $s_v \in \mathrm{GL}_N(\mathbb{R})$ by $s_v^a(u) := u - \frac{2(u,v)^a}{(v,v)^a}v$ $(u \in \mathbb{R}^N)$,

Let

$$\dot{\mathcal{D}}_{m|N-m} := \{\, (a,d) \in \mathcal{D}_{m|N-m} \times J_{0,1} \mid d \in J_{0,a(N)} \,\}.$$

For $i \in J_{1,N}$, define the bijection $\tau_i : \dot{\mathcal{D}}_{m|N-m} \to \dot{\mathcal{D}}_{m|N-m}$ by

$$\tau_i(a,d) := \begin{cases} (a \circ s_{e_i - e_{i+1}}, d) & \text{if } i \in J_{1,N-2} \text{ and } a(i) \neq a(i+1), \\ (a \circ s_{e_{N-1} - e_N}, d) & \text{if } i \in N-1,\ d=0 \text{ and } a(N-1) \neq b(N), \\ (a \circ s_{e_{N-1} - e_N}, 1) & \text{if } i = N,\ a(N-1)=1,\ a(N)=0, \\ (a \circ s_{e_{N-1} - e_N}, 0) & \text{if } i = N,\ a(N-1)=0,\ a(N)=1 \text{ and } d=1, \\ (a,d) & \text{otherwise.} \end{cases}$$

Then $\tau_i^2 = \mathrm{id}_{\mathbb{R}^N}$.

Let $(a,d) \in \dot{\mathcal{D}}_{m|N-m}$. Let

$$\begin{aligned} R_+^{(a,d)} &:= \{\, e_x + te_y \,|\, x,y \in J_{1,N},\ x<y,\ t \in \{1,-1\}\,\} \\ &\quad \cup \{\, 2e_z \,|\, z \in J_{1,N},\ a(z) = 1\,\}, \end{aligned}$$

and $R^{(a,d)} := R_+^{(a,d)} \cup -R_+^{(a,d)}$. Then

$$|R_+^{(a,d)}| = N(N-1) + (N-m) = N^2 - m. \tag{2.1}$$

For $i \in J_{1,N}$, let

$$\alpha_i^{(a,d)} := \begin{cases} e_i - e_{i+1} & \text{if } i \in J_{1,N-2}, \\ e_{N-1} - e_N & \text{if } i = N-1 \text{ and } d = 0, \\ 2e_N & \text{if } i = N-1 \text{ and } d=1, \\ e_{N-1} + e_N & \text{if } i = N,\ a(N)=0 \text{ and } d=0, \\ 2e_N & \text{if } i = N,\ a(N) = 1 \text{ and } d = 0, \\ e_{N-1} - e_N & \text{if } i = N,\ d=1. \end{cases}$$

Let $\Pi^{(a,d)} := \{\alpha_i^{(a,d)} | i \in J_{1,N}\}$. Then $\Pi^{(a,d)}$ is an $\mathbb{R}$-basis of $\mathbb{R}^N$. Moreover

$$\Pi^{(a,d)} \subset R_+^{(a,d)} \subset (\bigoplus_{i=1}^{N} \mathbb{Z}_{\geq 0} \alpha_i^{(a,d)}) \setminus \{0\}.$$

Note that

$$\tau_i(a,d) = (a,d) \quad \text{if and only if} \quad (\alpha_i^{(a,d)}, \alpha_i^{(a,d)})^a \neq 0.$$

For $i \in J_{1,N}$, define $s_i^{(a,d)} \in \mathrm{GL}_N(\mathbb{R})$ by

$$s_i^{(a,d)}(\alpha_i^{(a,d)}) := \begin{cases} -\alpha_i^{\tau_i(a,d)} & \text{if } i = j, \\ s^a_{\alpha_i^{\tau_i(a,d)}}(\alpha_j^{\tau_i(a,d)}) & \text{if } i \neq j \text{ and } (\alpha_i^{(a,d)}, \alpha_i^{(a,d)})^a \neq 0, \\ \alpha_j^{\tau_i(a,d)} & \text{if } i \neq j \text{ and } (\alpha_i^{(a,d)}, \alpha_i^{(a,d)})^a = (\alpha_i^{(a,d)}, \alpha_j^{(a,d)})^a = 0, \\ \alpha_j^{\tau_i(a,d)} + \alpha_i^{\tau_i(a,d)} & \text{if } i \neq j,\ (\alpha_i^{(a,d)}, \alpha_i^{(a,d)})^a = 0 \text{ and } (\alpha_i^{(a,d)}, \alpha_j^{(a,d)})^a \neq 0. \end{cases}$$

We can directly see

Lemma 2.1. *Let $(a,d) \in \dot{\mathcal{D}}_{m|N-m}$, and $i \in J_{1,N}$. Assume that $d = 0$. Assume that $i \in J_{1,N-1}$ if $a(N-1) = 1$ and $a(N) = 0$. Then $s_i^{(a,d)} = s_{\alpha_i^{(a,d)}}$, where $s_{\alpha_i^{(a,d)}}$ is the one of Definition* 1.1.

Notation. Let $(a,d) \in \dot{\mathcal{D}}_{m|N-m}$. Let Map_0^N be a set with $|\mathrm{Map}_0^N| = 1$. For $r \in \mathbb{N}$, let Map_r^N be the set of all maps from $J_{1,r}$ to $J_{1,N}$. Let Map_∞^N be the set of all maps from $\mathbb{N}$ to $J_{1,N}$. For $r \in \mathbb{Z}_{\geq 0}$, $f \in \mathrm{Map}_r^N \cup \mathrm{Map}_\infty^N$ and $t \in J_{1,r}$, let

$$(a,d)_{f,0} := (a,d), \quad 1^{(a,d)}s_{f,0} := \mathrm{id}_{\mathbb{R}^N}$$
$$(a,d)_{f,t} := \tau_i((a,d)_{f,t-1}), \quad 1^{(a,d)}s_{f,t} := 1^{(a,d)}s_{f,t-1}s_{f(t)}^{(a,d)_{f,t}}.$$

Proposition 2.2. *Let $(a,d) \in \dot{\mathcal{D}}_{m|N-m}$ be such that $d = 0$, $b(z) = 1$ ($z \in J_{1,N-m}$) and $b(z') = 0$ ($z' \in J_{N-m+1,N}$). Let $n := |R_+^{(a,d)}|$. Define $f \in \mathrm{Map}_n^N$ by*

$$(2.2) \qquad f(t) := \begin{cases} N - m + t & (\text{if } t \in J_{1,m}), \\ f(t-m) & (\text{if } t \in J_{m+1,m(m-1)}), \\ t - m(m-1) & (\text{if } t \in J_{m(m-1)+1,m(m-1)+N}), \\ 2N + m(m-1) - t & (\text{if } t \in J_{m(m-1)+N+1,m^2+N}), \\ f(t-(N+m)) & (\text{if } t \in J_{m^2+N+1,n}). \end{cases}$$

Then

$$1^{(a,d)}s_{f,n} = \begin{cases} b_{(1,N)} & \text{if } m \text{ is odd,} \\ b_{(1,N-1)} & \text{if } m \text{ is even.} \end{cases} \tag{2.3}$$

Proof. For $y \in J_{1,m}$, define $a^{\langle y\rangle} \in \mathcal{D}_{m|N-m}$ by

$$a^{\langle y\rangle}(z) := \begin{cases} 1 & \text{if } z \in J_{1,N-m-1} \cup \{N-m+y\}, \\ 0 & \text{if } z \in J_{N-m,N-m+y-1} \cup J_{N-m+y+1,N}. \end{cases}$$

Then we can directly see that for $t \in J_{1,n}$,

$$(a,d)_{f,t} = \begin{cases} (a,d) & \text{if } t \in J_{1,m(m-1)+N-m-1}, \\ (a^{\langle t-(N-m-1)\rangle},0) & \text{if } t \in J_{m(m-1)+N-m,m(m-1)+N-1}, \\ (a^{\langle m-(t-(m(m-1)+N))\rangle},0) & \text{if } t \in J_{m(m-1)+N,m(m-1)+N+m}, \\ (a,d)_{f,t-(N+m)} & \text{if } t \in J_{m^2+N+1,n}. \end{cases}$$

So we see that for $t \in J_{1,n}$,

$$s_{f(t)}^{(a,d)_{f,t}} = \begin{cases} s_{e_{f(t)}-e_{f(t)+1}} & \text{if } f(t) \in J_{1,N-1}, \\ s_{e_{N-1}+e_N} & \text{if } t \in J_{1,m(m-1)} \text{ and } f(t) = N, \\ s_{2e_N}(=s_{e_N}) & \text{if } t \in J_{m(m-1)+1,n} \text{ and } f(t) = N. \end{cases} \tag{2.4}$$

Define $f' \in \mathrm{Map}^N_{n-m(m-1)}$ by $f'(t) := f(t+m(m-1))$, so

$$1^{(a,d)}s_{f,n} = 1^{(a,d)}s_{f,m(m-1)}1^{(a,d)}_{f',n-m(m-1)}. \tag{2.5}$$

By (1.29) and (1.36), $1^{(a,d)}s_{f,m(m-1)}$ equals $b_{(N-m+1,N)}$ (resp. $b_{(N-m+1,N-1)}$) if m is odd (resp. even). By (1.29) and (2.4), $1^{(a,d)}s_{f',n-m(m-1)} = b_{(1,N-m)}$. Hence by (1.22) and (2.5), we have (2.3), as desired. □

For $(a,d) \in \dot{\mathcal{D}}_{m|N-m}$ and i, $j \in J_{1,N}$, define $C^{(a,d)} = [c_{ij}^{(a,d)}]_{i,j\in J_{1,N}} \in \mathrm{M}_N(\mathbb{Z})$ by

$$s_i^{(a,d)}(\alpha_j^{(a,d)}) = \alpha_j^{\tau_i(a,d)} - c_{ij}^{(a,d)}\alpha_i^{\tau_i(a,d)}.$$

Then $C^{(a,d)}$ is a *generalized Cartan matrix*, i.e., (M1) and (M2) below hold.

(M1) $c_{ii}^{(a,d)} = 2$ $(i \in J_{1,N})$.
(M2) $c_{jk}^{(a,d)} \leq 0$, $\delta_{c_{jk}^{(a,d)},0} = \delta_{c_{kj}^{(a,d)},0}$ $(j,\, k \in J_{1,N},\, j \neq k)$.

Then the data

$$\dot{\mathcal{C}}_{m|N-m} := \mathcal{C}(J_{1,N}, \dot{\mathcal{D}}_{m|N-m}, (\tau_i)_{i\in J_{1,N}}, (C^{(a,d)})_{(a,d)\in\dot{\mathcal{D}}_{m|N-m}})$$

a *(rank-N) Cartan scheme*, i.e., (C1) and (C2) below hold.

(C1) $\tau_i^2 = \mathrm{id}_{\dot{\mathcal{D}}_{m|N-m}}$ $(i \in J_{1,N})$.
(C2) $c_{ij}^{\tau_i((a,d))} = c_{ij}^{(a,d)}$ $(i \in J_{1,N})$.

Note that

$$-c_{ij}^{(a,d)} = |R_+^{(a,d)} \cap (\mathbb{Z}\alpha_i^{(a,d)} \oplus \mathbb{Z}\alpha_j^{(a,d)})| \quad (i,\, j \in J_{1,N},\; i \neq j).$$

The data

$$\dot{\mathcal{R}}_{m|N-m} := \mathcal{R}(\dot{\mathcal{C}}_{m|N-m}, (R_+^{(a,d)})_{(a,d)\in\dot{\mathcal{D}}_{m|N-m}}).$$

is a *generalized root system of type* $\mathcal{C}$, i.e., (R1)-(R4) below hold.

(R1) $R^{(a,d)} = R_+^{(a,d)} \cup -R_+^{(a,d)}$ $\quad((a,d) \in \dot{\mathcal{D}}_{m|N-m})$.
(R2) $R^{(a,d)} \cap \mathbb{Z}\alpha_i = \{\, \alpha_i,\, -\alpha_i \,\}$ $\quad((a,d) \in \dot{\mathcal{D}}_{m|N-m},\, i \in J_{1,N})$.
(R3) $s_i^{(a,d)}(R^{(a,d)}) = R^{\tau_i(a,d)}$ $\quad((a,d) \in \dot{\mathcal{D}}_{m|N-m},\, i \in J_{1,N})$.
(R4) $(\tau_i\tau_j)^{-c_{ij}^{(a,d)}}(a,d) = (a,d)$ $\quad((a,d) \in \dot{\mathcal{D}}_{m|N-m},\, i,\, j \in J_{1,N})$.

For $(a,d) \in \dot{\mathcal{D}}_{m|N-m}$, let

$$W^{(a,d)} := \{\, 1^{(a,d)}s_{f,r} \in \mathrm{GL}_N(\mathbb{R}) \,|\, r \in \mathbb{Z}_{\geq 0},\, f \in \mathrm{Map}_r^N \,\},$$

and define the map $\ell^{(a,d)} : W^{(a,d)} \to \mathbb{Z}_{\geq 0}$ by

$$\ell^{(a,d)}(w) := \min\{\, r \in \mathbb{Z}_{\geq 0} \,|\, \exists f \in \mathrm{Map}_r^N,\, w = 1^{(a,d)}s_{f,r} \,\}.$$

By [HY08, Lemma 8 (iii)], we see that

(2.6) $\qquad 1^{(a,d)}s_{f,r} = 1^{(a,d)}s_{f',r'}$ implies $(a,d)_{f,r} = (a,d)_{f',r'}$, and that

$$\ell^{(a,d)}(w) = |w^{-1}(R_+^{(a,d)}) \cap - \oplus_{i=1}^N \mathbb{Z}_{\geq 0}\alpha_i|. \tag{2.7}$$

For $(a,d) \in \dot{\mathcal{D}}_{m|N-m}$, $w \in W^{(a,d)}$ and $f \in \mathrm{Map}^N_{\ell^{(a,d)}(w)}$, if $w = 1^{(a,d)} s_{f,\ell^{(a,d)}(w)}$, we call f a *reduced word map* of w.

By (2.6) and (2.7), we have formulas for $W^{(a,d)}$ similar to (1.3) and (1.4). In particular, for each $(a,d) \in \dot{\mathcal{D}}_{m|N-m}$, there exists a unique $w_\circ^{(a,d)} \in W^{(a,d)}$ such that

$$\ell^{(a,d)}(w_\circ^{(a,d)}) = |R_+^{(a,d)}|,$$

and we call $w_\circ^{(a,d)}$ the *longest element* of $W^{(a,d)}$.

By Proposition 2.2, we have

Theorem 2.3. *Let $(a,d) \in \dot{\mathcal{D}}_{m|N-m}$ be such that $d = 0$, $a(z) = 1$ ($z \in J_{1,N-m}$) and $a(z') = 0$ ($z' \in J_{N-m+1,N}$). Then a reduced word map of $w_\circ^{(a,d)}$ is given by* (2.2). *Moreover,*

$$w_\circ^{(a,d)} = \begin{cases} b_{(1,N)} & \text{if } m \text{ is odd}, \\ b_{(1,N-1)} & \text{if } m \text{ is even}. \end{cases} \tag{2.8}$$

Definition 2.4. For (a,d), $(a',d') \in \dot{\mathcal{D}}_{m|N-m}$, let $W^{(a,d)}_{(a',d')}$ be the subset of $W^{(a,d)}$ composed of all the elements $1^{(a,d)}s_{f,r}$ with $r \in \mathbb{Z}_{\geq 0}$, $f \in \mathrm{Map}^N_r$ and $(a,d)_{f,r} = (a',d')$, and $\mathcal{H}^{(a,d)}_{(a',d')} := \{(a,d)\} \times W^{(a,d)}_{(a',d')} \times \{(a',d')\} (\subset \dot{\mathcal{D}}_{m|N-m} \times \mathrm{GL}_N(\mathbb{R}) \times \dot{\mathcal{D}}_{m|N-m})$. Let

$$(\dot{\mathcal{W}}_{m|N-m})' := \bigcup_{(a,d),(a',d') \in \dot{\mathcal{D}}_{m|N-m}} \mathcal{H}^{(a,d)}_{(a',d')},$$

and $\dot{\mathcal{W}}_{m|N-m} := (\dot{\mathcal{W}}_{m|N-m})' \cup \{o\}$, where o is an element such that $o \notin (\dot{\mathcal{W}}_{m|N-m})'$. We regard $\dot{\mathcal{W}}_{m|N-m}$ as the semigroup by $o\omega := \omega o := o$ ($\omega \in \dot{\mathcal{W}}_{m|N-m}$) and

$$\begin{aligned} &((a_1,d_1),w_1,(a_2,d_2))((a_3,d_3),w_2,(a_4,d_4)) \\ &:= \begin{cases} ((a_1,d_1),w_1w_2,(a_4,d_4)) & \text{if } (a_2,d_2) = (a_3,d_3), \\ o & \text{if } (a_2,d_2) \neq (a_3,d_3). \end{cases} \end{aligned}$$

We call $\dot{\mathcal{W}}_{m|N-m}$ the *Weyl groupoid* of the Lie superalgebra $\mathrm{osp}(2m|2(N-m))$.

For $(a,d)\in\dot{\mathcal{D}}_{m|N-m}$, let $\varepsilon^{(a,d)}:=((a,d),\mathrm{id}_{\mathbb{R}^N},(a,d))\in\mathcal{H}^{(a,d)}_{(a,d)}$. For $(a,d)\in\dot{\mathcal{D}}_{m|N-m}$ and $i\in J_{1,N}$, let $\sigma_i^{(a,d)}:=(\tau_i(a,d),s_i^{(a,d)},(a,d))\in\mathcal{H}^{(a,d)}_{\tau_i(a,d)}$. For $r\in\mathbb{Z}_{\geq 0}$, $t\in J_{0,r}$ and $f\in\mathrm{Map}_r^N$, let $1^{(a,d)}\sigma_{f,r}:=((a,d),1^{(a,d)}s_{f,r},(a,d)_{f,r})\in\mathcal{H}^{(a,d)}_{(a,d)_{f,r}}$. For i, $j\in J_{1,N}$, define $f_{ij}\in\mathrm{Map}_\infty^N$ by $f_{ij}(2t-1):=i$, $f_{ij}(2t):=j$ $(t\in\mathbb{N})$.

By [HY08, Theorem 1], we have

Theorem 2.5. *The semigroup $\dot{\mathcal{W}}_{m|N-m}$ can also be defined by the generators*

$$o,\ \varepsilon^{(a,d)},\ \sigma_i^{(a,d)}\quad ((a,d)\in\dot{\mathcal{D}}_{m|N-m},\ i\in J_{1,N}),$$

and relations

$$o\omega=\omega o=o\quad(\omega\in\dot{\mathcal{W}}_{m|N-m}),$$
$$\varepsilon^{(a,d)}\varepsilon^{(a,d)}=\varepsilon^{(a,d)},\ \varepsilon^{(a,d)}\varepsilon^{(a',d')}=o\quad((a,d)\neq(a',d')),$$
$$\varepsilon^{\tau_i(a,d)}\sigma_i^{(a,d)}=\sigma_i^{(a,d)}\varepsilon^{(a,d)}=\sigma_i^{(a,d)},\ \sigma_i^{\tau_i(a,d)}\sigma_i^{(a,d)}=\varepsilon^{(a,d)},$$
$$1^{(a,d)}\sigma_{f_{ij},-2c_{ij}^{(a,d)}}=\varepsilon^{(a,d)}\quad(i\neq j).$$

Let $(a,d)\in\dot{\mathcal{D}}_{m|N-m}$, $r\in\mathbb{Z}_{\geq 0}$ and f, $f'\in\mathrm{Map}_r^N$. We write $f\dot{\sim}_r^{(a,d)}f'$ if there exist i, $j\in J_{1,N}$ and $t\in J_{0,r}$ such that $i\neq j$, $t-c_{ij}^{(a,d)_{f,k}}\leq r$, $f(k_1)=f'(k_1)$ $(k_1\in J_{1,t}\cup J_{t-c_{ij}^{(a,d)_{f,k}}+1,r})$, $f(k_2)=i$, $f'(k_2)=j$ $(k_2\in J_{t+1,t-c_{ij}^{(a,d)_{f,k}}}\cap 2\mathbb{N}-1)$ and $f(k_3)=j$, $f'(k_3)=i$ $(k_3\in J_{t+1,t-c_{ij}^{(a,d)_{f,k}}}\cap 2\mathbb{N})$. We write $f\sim_r^{(a,d)}f'$ if $f=f'$ or there exists $t\in\mathbb{N}$ and $f_k\in\mathrm{Map}_r^N$ $(k\in J_{1,t})$ such that $f\dot{\sim}_r^{(a,d)}f_1$, $f_k\dot{\sim}_r^{(a,d)}f_{k+1}$ $(k\in J_{1,t-1})$ and $f_t\dot{\sim}_r^{(a,d)}f'$.

By [HY08, Theorem 5, Corollary 6], we have

Theorem 2.6. *Let $(a,d)\in\dot{\mathcal{D}}_{m|N-m}$ and $w\in W^{(a,d)}$.*

(1) Let f, $f'\in\mathrm{Map}^N_{\ell^{(a,d)}(w)}$ be such that $1^{(a,d)}s_{f,\ell^{(a,d)}(w)}=1^{(a,d)}s_{f',\ell^{(a,d)}(w)}=w$. Then $f\sim^{(a,d)}_{\ell^{(a,d)}(w)}f'$.

(2) Let $r\in\mathbb{N}$ and $f\in\mathrm{Map}_r^N$ be such that $r>\ell^{(a,d)}(w)$ and $1^{(a,d)}s_{f,r}=w$. Then there exist $f'\in\mathrm{Map}_r^N$ and $t\in J_{1,r-1}$ such that $f\sim_r^{(a,d)}f'$ and $f'(t)=f'(t+1)$.

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