数理解析研究所講究録
第 1841 巻 2013 年 17-25



# Fixed Point Theorems and Convergence Theorems for Non-self Mappings in Hilbert Spaces
# (ヒルベルト空間における非自己写像の不動点定理と収束定理)

新潟大学大学院自然科学研究科 北條 真弓 (Mayumi Hojo)
Graduate School of Science and Technology, Niigata University, Japan

**Abstract.** In this article, we first prove fixed point theorems for nonlinear non-self mappings in a Hilbert space. Next, we deal with weak and strong convergence theorems for nonlinear mappings in a Hilbert space. Using these results, we obtain new and well-known fixed point and convergence theorems. For example, we generalizes Hojo and Takahashi's mean strong convergence theorem [11] for generalized hybrid mappings.

## 1 Introduction

Let $H$ be a real Hilbert space and let $C$ be a nonempty subset of $H$. Kocourek, Takahashi and Yao [19] introduced a broad class of nonlinear mappings in a Hilbert space which covers nonexpansive mappings, nonspreading mappings [21] and hybrid mappings [30]. A mapping $T : C \to H$ is said to be *generalized hybrid* [19] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx-Ty\|^2+(1-\alpha)\|x-Ty\|^2 \leq \beta\|Tx-y\|^2+(1-\beta)\|x-y\|^2 \tag{1.1}$$

for all $x, y \in C$, where $\mathbb{R}$ is the set of real numbers. We call such $T$ an $(\alpha, \beta)$-*generalized hybrid* mapping. An $(\alpha, \beta)$-generalized hybrid mapping is nonexpansive for $\alpha = 1$ and $\beta = 0$, i.e., $\|Tx - Ty\| \leq \|Tx - Ty\|$ for all $x, y \in C$. It is nonspreading for $\alpha = 2$ and $\beta = 1$, i.e., $2\|Tx-Ty\|^2 \leq \|x-Ty\|^2 + \|y-Tx\|^2$ for all $x, y \in C$. Furthermore, it is hybrid for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$, i.e., $3\|Tx-Ty\|^2 \leq \|x-Ty\|^2+\|y-Tx\|^2+\|y-x\|^2$ for all $x, y \in C$. They proved fixed point theorems and nonlinear ergodic theorems of Baillon's type [3] for generalized hybrid mappings in a Hilbert space; see also Kohsaka and Takahashi [20] and Iemoto and Takahashi [15]. Putting $x = u$ with $u = Tu$ in (1.1), we have that for any $y \in C$,

$$\alpha\|u-Ty\|^2+(1-\alpha)\|u-Ty\|^2 \leq \beta\|u-y\|^2+(1-\beta)\|u-y\|^2$$

and hence $\|u - Ty\| \leq \|u - y\|$. This means that an $(\alpha, \beta)$-generalized hybrid mapping with a fixed point is quasi-nonexpansive. Kocourek, Takahashi and Yao [19] also introduced a more broad class of nonlinear mappings which covers generalized hybrid mappings. A mapping

$S : C \to H$ is called *super hybrid* [19, 34] if there exist $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$\begin{aligned} \alpha\|Sx - Sy\|^2 &+ (1 - \alpha + \gamma)\|x - Sy\|^2 \\ &\leq \big(\beta + (\beta - \alpha)\gamma\big)\|Sx - y\|^2 + \big(1 - \beta - (\beta - \alpha - 1)\gamma\big)\|x - y\|^2 \\ &\quad + (\alpha - \beta)\gamma\|x - Sx\|^2 + \gamma\|y - Sy\|^2 \end{aligned} \tag{1.2}$$

for all $x, y \in C$. We call such a mapping an $(\alpha, \beta, \gamma)$-*super hybrid* mapping. An $(\alpha, \beta, 0)$-super hybrid mapping is $(\alpha, \beta)$-generalized hybrid. So, the class of super hybrid mappings contains generalized hybrid mappings. On the other hand, Hojo, Takahashi and Yao [12] defined the following class of nonlinear mappings which contains generalized hybrid mappings. A mapping $U : C \to H$ is called *extended hybrid* if there exist $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$\begin{aligned} \alpha(1 + \gamma)\|Ux - Uy\|^2 &+ (1 - \alpha(1 + \gamma))\|x - Uy\|^2 \\ &\leq (\beta + \alpha\gamma)\|Ux - y\|^2 + (1 - (\beta + \alpha\gamma))\|x - y\|^2 \\ &\quad - (\alpha - \beta)\gamma\|x - Ux\|^2 - \gamma\|y - Uy\|^2 \end{aligned} \tag{1.3}$$

for all $x, y \in C$. We note that super hybrid mappings and extended hybrid mappings are not quasi-nonexpansive generally. We also know the following relation between generalized hybrid mappings and extended hybrid mappings

**Theorem 1.1.** *Let $C$ be a nonempty closed convex subset of a Hilbert space $H$ and let $\alpha$, $\beta$ and $\gamma$ be real numbers with $\gamma \neq -1$. Let $T$ and $U$ be mappings of $C$ into $H$ such that $U = \frac{1}{1+\gamma}T + \frac{\gamma}{1+\gamma}I$, where $Ix = x$ for all $x \in H$. Then, for $1 + \gamma > 0$, $T : C \to H$ is an $(\alpha, \beta)$-generalized hybrid mapping if and only if $U : C \to H$ is an $(\alpha, \beta, \gamma)$- extended hybrid mapping.*

In this article, motivated by these mappings and results, we first prove fixed point theorems for nonlinear non-self mappings in a Hilbert space. Next, we deal with weak and strong convergence theorems for nonlinear mappings in a Hilbert space. Using these results, we obtain new and well-known fixed point and convergence theorems. For example, we generalizes Hojo and Takahashi's mean strong convergence theorem [11] for generalized hybrid mappings.

## 2 Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers. Let $H$ be a (real) Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. We denote the strong convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. From [29], we know the following basic equality: For any $x, y \in H$ and $\lambda \in \mathbb{R}$, we have

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2. \tag{2.1}$$

Furthermore, we know that for any $x, y, u, v \in H$

$$2\langle x - y, u - v\rangle = \|x - v\|^2 + \|y - u\|^2 - \|x - u\|^2 - \|y - v\|^2. \tag{2.2}$$

Let $C$ be a nonempty closed convex subset of $H$ and let $T$ be a mapping from $C$ into itself. Then, we denote by $F(T)$ the set of fixed points of $T$. A mapping $T : C \to H$ is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A mapping $T : C \to H$ with $F(T) \neq \emptyset$

is called *quasi-nonexpansive* if $\|x - Ty\| \le \|x - y\|$ for all $x \in F(T)$ and $y \in C$. Let $C$ be a nonempty closed convex subset of $H$ and $x \in H$. Then, we know that there exists a unique nearest point $z \in C$ such that $\|x - z\| = \inf_{y \in C} \|x - y\|$. We denote such a correspondence by $z = P_C x$. The mapping $P_C$ is called the *metric projection* of $H$ onto $C$. It is known that $P_C$ is nonexpansive and $\langle x - P_C x, P_C x - u\rangle \ge 0$ for all $x \in H$ and $u \in C$. Furthermore, we know that

$$\|P_C x - P_C y\|^2 \le \langle x - y, P_C x - P_C y\rangle \tag{2.3}$$

for all $x, y \in H$; see [29] for more details. For proving main results in this paper, we also need the following lemmas proved in [31] and [2].

**Lemma 2.1** ([31]). *Let $D$ be a nonempty closed convex subset of $H$. Let $P$ be the metric projection from $H$ onto $D$. Let $\{u_n\}$ be a sequence in $H$. If $\|u_{n+1} - u\| \le \|u_n - u\|$ for all $u \in D$ and $n \in \mathbb{N}$, then $\{Pu_n\}$ converges strongly to some $u_0 \in D$.*

**Lemma 2.2** ([2]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers, let $\{\alpha_n\}$ be a sequence of $[0, 1]$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$, let $\{\beta_n\}$ be a sequence of nonnegative real numbers with $\sum_{n=1}^{\infty} \beta_n < \infty$, and let $\{\gamma_n\}$ be a sequence of real numbers with $\limsup_{n\to\infty} \gamma_n \le 0$. Suppose that*

$$s_{n+1} \le (1 - \alpha_n)s_n + \alpha_n\gamma_n + \beta_n$$

*for all $n = 1, 2, \ldots$. Then $\lim_{n\to\infty} s_n = 0$.*

Let $l^\infty$ be the Banach space of bounded sequences with supremum norm. Let $\mu$ be an element of $(l^\infty)^*$ (the dual space of $l^\infty$). Then we denote by $\mu(f)$ the value of $\mu$ at $f = (x_1, x_2, x_3, \ldots) \in l^\infty$. Sometimes, we denote by $\mu_n(x_n)$ the value $\mu(f)$. A linear functional $\mu$ on $l^\infty$ is called a mean if $\mu(e) = \|\mu\| = 1$, where $e = (1, 1, 1, \ldots)$. A mean $\mu$ is called a Banach limit on $l^\infty$ if $\mu_n(x_{n+1}) = \mu_n(x_n)$. We know that there exists a Banach limit on $l^\infty$. If $\mu$ is a Banach limit on $l^\infty$, then for $f = (x_1, x_2, x_3, \ldots) \in l^\infty$,

$$\liminf_{n\to\infty} x_n \le \mu_n(x_n) \le \limsup_{n\to\infty} x_n.$$

In particular, if $f = (x_1, x_2, x_3, \ldots) \in l^\infty$ and $x_n \to a \in \mathbb{R}$, then we have $\mu(f) = \mu_n(x_n) = a$. See [27] for the proof of existence of a Banach limit and its other elementary properties. Using Banach limits, Kocourek, Takahashi and Yao [19] proved the following fixed point theorem for generalized hybrid mappings in a Hilbert space.

**Theorem 2.3** ([19]). *Let $C$ be a nonempty closed convex subset of a Hilbert space $H$ and let $T : C \to C$ be a generalized hybrid mapping. Then $T$ has a fixed point in $C$ if and only if $\{T^n z\}$ is bounded for some $z \in C$.*

## 3 Fixed Point Theorem for Non-Self Mappings

In this section, we first prove a fixed point theorem for generalized hybrid non-self mappings in a Hilbert space. For proving it, we need the following lemmas.

**Lemma 3.1.** *Let $H$ be a Hilbert space and let $C$ be a nonempty subset of $H$. Let $\alpha$ and $\beta$ be in $\mathbb{R}$. Then, a non-self mapping $T : C \to H$ is $(\alpha, \beta)$-generalized hybrid if and only if it satisfies that*

$$\|Tx - Ty\|^2 \le (\alpha - \beta)\|x - y\|^2 + 2(\alpha - 1)\langle x - Tx, y - Ty\rangle - (\alpha - \beta - 1)\|y - Tx\|^2$$

*for all* $x, y \in C$.

Using Lemma 3.1, we have the following result.

**Lemma 3.2.** *Let* $H$ *be a Hilbert space and let* $C$ *be a nonempty bounded subset of* $H$. *If a non-self mapping* $T : C \to H$ *is generalized hybrid, then* $TC$ *is bounded.*

The following is a fixed point theorem for non-self generalized hybrid mappings in a Hilbert space.

**Theorem 3.3** ([12]). *Let* $C$ *be a nonempty bounded closed convex subset of a Hilbert space* $H$ *and let* $\alpha$ *and* $\beta$ *be real numbers. Let* $T$ *be an* $(\alpha, \beta)$*-generalized hybrid mapping with* $\alpha - \beta \geq 0$ *of* $C$ *into* $H$. *Suppose that there exists* $m > 1$ *such that for any* $x \in C$, $Tx = x + t(y - x)$ *for some* $y \in C$ *and* $t$ *with* $1 \leq t \leq m$. *Then,* $T$ *has a fixed point in* $C$.

Recently, Hojo, Suzuki and Takahashi [10] also proved a more general fixed point theorem for nonlinear non-self mappings in a Hilbert space.

**Theorem 3.4** ([10]). *Let* $C$ *be a nonempty, bounded, closed and convex subset of a Hilbert space* $H$ *and let* $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. *Let* $T : C \to H$ *be an* $(\alpha, \beta, \gamma, \delta)$*-normal generalized hybrid mapping, i.e., there exist* $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ *such that*

$$\alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \leq 0$$

*for all* $x, y \in C$. *Suppose that it satisfies the following condition* (1) *or* (2)*:*

(1) $\alpha + \beta + \gamma + \delta \geq 0, \;\; \alpha + \gamma > 0$ *and* $\alpha + \beta \geq 0$;
(2) $\alpha + \beta + \gamma + \delta \geq 0, \;\; \alpha + \beta > 0$ *and* $\alpha + \gamma \geq 0$.

*Assume that there exists* $m > 1$ *such that for any* $x \in C$, $Tx = x + t(y - x)$ *for some* $y \in C$ *and* $t$ *with* $0 < t \leq m$. *Then* $T$ *has a fixed point in* $C$. *In particular, a fixed point of* $T$ *is unique in the case of* $\alpha + \beta + \gamma + \delta > 0$ *on the conditions* (1) *and* (2).

For proving this result, Hojo, Suzuki and Takahashi [10] used the following fixed point theorem obtained by Kawasaki and Takahashi [18].

**Theorem 3.5** ([18]). *Let* $H$ *be a Hilbert space, let* $C$ *be a nonempty, closed and convex subset of* $H$ *and let* $T$ *be an* $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$*-widely more generalized hybrid mapping from* $C$ *into itself, i.e., there exist* $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta \in \mathbb{R}$ *such that*

$$\begin{aligned}\alpha\|Tx - Ty\|^2 &+ \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \\ &+ \varepsilon\|x - Tx\|^2 + \zeta\|y - Ty\|^2 + \eta\|(x - Tx) - (y - Ty)\|^2 \leq 0\end{aligned}$$

*for all* $x, y \in C$. *Suppose that it satisfies the following condition* (1) *or* (2)*:*

(1) $\alpha + \beta + \gamma + \delta \geq 0, \;\; \alpha + \gamma + \varepsilon + \eta > 0$ *and* $\zeta + \eta \geq 0$;
(2) $\alpha + \beta + \gamma + \delta \geq 0, \;\; \alpha + \beta + \zeta + \eta > 0$ *and* $\varepsilon + \eta \geq 0$.

*Then* $T$ *has a fixed point if and only if there exists* $z \in C$ *such that* $\{T^n z : n = 0, 1, \ldots\}$ *is bounded. In particular, a fixed point of* $T$ *is unique in the case of* $\alpha + \beta + \gamma + \delta > 0$ *on the conditions* (1) *and* (2).

Let us give an example of mappings which is related to the conditions in Theorem 3.4. In the case of $H = \mathbb{R}$, consider a mapping $T : [0, 1] \to \mathbb{R}$:

$$Tx = (1 + 2x)\cos x - 2x^2, \quad \forall x \in [0, 1].$$

Then, we have

$$Tx = (1 + 2x)(\cos x - x) + x, \quad \forall x \in [0, 1].$$

Take $m = 3$. For any $x \in [0, 1]$, take $t = 1 + 2x$ and $y = \cos x$. Then, we have that $Tx = t(y - x) + x$, $y = \cos x \in [0, 1]$ and $0 < t = 1 + 2x \leq 3$.

## 4 Weak convergence theorems

In this section, using the technique developed by Takahashi [26], we first prove a mean convergence theorem of Baillon's type [3] for super hybrid mappings in a Hilbert space. For proving it, we need the following lemma.

**Lemma 4.1.** *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $T$ be a generalized hybrid mapping from $C$ into itself. Suppose that $\{T^n x\}$ is bounded for some $x \in C$. Define $S_n x = \frac{1}{n}\sum_{k=1}^{n} T^k x$. Then, $\lim_{n\to\infty} \|S_n x - TS_n x\| = 0$. In particular, if $C$ is bounded, then*

$$\lim_{n\to\infty} \sup_{x\in C} \|S_n x - TS_n x\| = 0.$$

Using Lemma 4.1, we obtain the the following mean convergence theorem.

**Theorem 4.2** ([12]). *Let $H$ be a Hilbert space and let $C$ be a nonempty closed convex subset of $H$. Let $\alpha$, $\beta$ and $\gamma$ be real numbers with $\gamma \geq 0$ and let $S : C \to C$ be an $(\alpha, \beta, \gamma)$-super hybrid mapping with $F(S) \neq \emptyset$ and let $P$ be the mertic projection of $H$ onto $F(T)$. Then, for any $x \in C$,*

$$S_n x = \frac{1}{n}\sum_{k=1}^{n}(\frac{1}{1+\gamma}S + \frac{\gamma}{1+\gamma}I)^k x$$

*converges weakly to $z \in F(S)$, where $z = \lim_{n\to\infty} PT^n x$ and $T = \frac{1}{1+\gamma}S + \frac{\gamma}{1+\gamma}I$.*

Next, we prove a weak convergence theorem of Mann's type [23] for nonlinear non-self mappings in a Hilbert space. For proving the result, we need the following two lemmas.

**Lemma 4.3.** *Let $C$ be a nonempty, closed and convex subset of a Hilbert space $H$ and let $T$ be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from $C$ into $H$ with $F(T) \neq \emptyset$ which satisfies the condition* (1) *or* (2)*:*

(1) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta > 0$ *and* $\zeta + \eta \geq 0$;
(2) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \gamma > 0$ *and* $\varepsilon + \eta \geq 0$.

*Then $T$ is quasi-nonexpansive.*

We remark that if $T : C \to H$ is quasi-nonexpansive, then $F(T)$ is closed and convex; see Itoh and Takahashi [16]. It is not difficult to prove such a result in a Hilbert space. In fact, for proving that $F(T)$ is closed, take a sequence $\{z_n\} \subset F(T)$ with $z_n \to z$. Since $C$ is weakly closed, we have $z \in C$. Furthermore, from $\|z - Tz\| \leq \|z - z_n\| + \|z_n - Tz\| \leq 2\|z - z_n\| \to 0$, we have that $z$ is a fixed point of $T$ and so $F(T)$ is closed. Let us show that $F(T)$ is convex.

For $x, y \in F(T)$ and $\alpha \in [0,1]$, put $z = \alpha x + (1-\alpha)y$. Then we have from (2.1) that

$$\begin{aligned}\|z - Tz\|^2 &= \|\alpha x + (1-\alpha)y - Tz\|^2 \\ &= \alpha\|x - Tz\|^2 + (1-\alpha)\|y - Tz\|^2 - \alpha(1-\alpha)\|x-y\|^2 \\ &\leq \alpha\|x - z\|^2 + (1-\alpha)\|y - z\|^2 - \alpha(1-\alpha)\|x-y\|^2 \\ &= \alpha(1-\alpha)^2\|x-y\|^2 + (1-\alpha)\alpha^2\|x-y\|^2 - \alpha(1-\alpha)\|x-y\|^2 \\ &= \alpha(1-\alpha)(1-\alpha+\alpha-1)\|x-y\|^2 = 0\end{aligned}$$

and hence $Tz = z$. This implies that $F(T)$ is convex.

**Lemma 4.4.** *Let $H$ be a Hilbert space and let $C$ be a nonempty, closed and convex subset of $H$. Let $T : C \to H$ be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping. Suppose that it satisfies the following condition* (1) *or* (2)*:*

(1) $\alpha + \beta + \gamma + \delta \geq 0$ *and* $\alpha + \gamma + \varepsilon + \eta > 0$*;*
(2) $\alpha + \beta + \gamma + \delta \geq 0$ *and* $\alpha + \beta + \zeta + \eta > 0$.

*If $x_n \rightharpoonup z$ and $x_n - Tx_n \to 0$, then $z \in F(T)$.*

Using Lemmas 4.3, 4.4 and the technique developed by Ibaraki and Takahashi [13, 14], we can prove the following weak convergence theorem.

**Theorem 4.5** ([10]). *Let $H$ be a Hilbert space and let $C$ be a nonempty, closed and convex subset of $H$. Let $T : C \to H$ be a widely more generalized hybrid mapping with $F(T) \neq \emptyset$ which satisfies the condition* (1) *or* (2)*:*

(1) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \gamma > 0$ *and* $\varepsilon + \eta \geq 0$*;*
(2) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta > 0$ *and* $\zeta + \eta \geq 0$.

*Let $P$ be the mertic projection of $H$ onto $F(T)$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n \leq 1$ and $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. Suppose that $\{x_n\}$ is the sequence generated by $x_1 = x \in C$ and*

$$x_{n+1} = P_C\left(\alpha_n x_n + (1-\alpha_n)Tx_n\right), \quad n \in \mathbb{N}.$$

*Then $\{x_n\}$ converges weakly to $v \in F(T)$, where $v = \lim_{n\to\infty} Px_n$.*

Using Theorem 4.5, we can show the following weak convergence theorem of Mann's type for generalized hybrid mappings in a Hilbert space.

**Theorem 4.6** ([19]). *Let $H$ be a Hilbert space and let $C$ be a nonempty, closed and convex subset of $H$. Let $T : C \to C$ be a generalized hybrid mapping with $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n \leq 1$ and $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. Suppose that $\{x_n\}$ is the sequence generated by $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \quad n \in \mathbb{N}.$$

*Then the sequence $\{x_n\}$ converges weakly to an element $v \in F(T)$.*

*Proof.* Since $T : C \to C$ is a generalized hybrid mapping, there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \leq \beta\|Tx - Ty\|^2 + (1-\beta)\|x - Ty\|^2$$

for all $x, y \in C$. We have that this mapping is an $(\alpha, 1-\alpha, -\beta, -(1-\beta), 0, 0, 0)$-widely more generalized hybrid mapping which satisfies the condition (2) in Theorem 4.5. Therefore, we have the desired result from Theorem 4.5. □

## 5 Strong Convergence Theorem

In this section, using an idea of mean convergence by Shimizu and Takahashi [24] and [25], we prove a strong convergence theorem of Halpern's type for super hybrid mappings in a Hilbert space.

**Theorem 5.1** ([12]). *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$ and let $\alpha$, $\beta$ and $\gamma$ be real numbers with $\gamma \geq 0$. Let $S : C \to C$ be a $(\alpha, \beta, \gamma)$-super hybrid mapping with $F(S) \neq \emptyset$ and let $P$ be the metric projection of $H$ onto $F(S)$. Suppose that $\{x_n\}$ is a sequence generated by $x_1 = x \in C$, $u \in C$ and*

$$\begin{cases} x_{n+1} = \alpha_n u + (1-\alpha_n) z_n, \\ z_n = \dfrac{1}{n} \displaystyle\sum_{k=1}^{n} (\frac{1}{1+\gamma} S + \frac{\gamma}{1+\gamma} I)^k x_n \end{cases}$$

*for all $n = 1, 2, \ldots$, where $0 \leq \alpha_n \leq 1$, $\alpha_n \to 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to $Pu$.*

Recently, Hojo, Suzuki and Takahashi [10] also proved the following strong convergence theorem for widely more generalized hybrid mappings in a Hilbert space.

**Theorem 5.2** ([10]). *Let $C$ be a nonempty, closed and convex subset of a real Hilbert space $H$. Let $T$ be a widely more generalized hybrid mapping of $C$ into itself which satisfies the following condition (1) or (2):*

(1) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \gamma > 0$, $\varepsilon + \eta \geq 0$ *and* $\zeta + \eta \geq 0$;
(2) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta > 0$, $\zeta + \eta \geq 0$ *and* $\varepsilon + \eta \geq 0$.

*Let $u \in C$ and define sequences $\{x_n\}$ and $\{z_n\}$ in $C$ as follows: $x_1 = x \in C$ and*

$$\begin{cases} x_{n+1} = \alpha_n u + (1-\alpha_n) z_n, \\ z_n = \dfrac{1}{n} \displaystyle\sum_{k=0}^{n-1} T^k x_n \end{cases}$$

*for all $n = 1, 2, \ldots$, where $0 \leq \alpha_n \leq 1$, $\alpha_n \to 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. If $F(T) \neq \emptyset$, then $\{x_n\}$ and $\{z_n\}$ converge strongly to $Pu$, where $P$ is the metric projection of $H$ onto $F(T)$.*

Using Theorem 5.2, we can show the following result obtained by Hojo and Takahashi [11].

**Theorem 5.3** ([11]). *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $T$ be a generalized hybrid mapping of $C$ into itself. Let $u \in C$ and define two sequences $\{x_n\}$ and $\{z_n\}$ in $C$ as follows: $x_1 = x \in C$ and*

$$\begin{cases} x_{n+1} = \alpha_n u + (1-\alpha_n) z_n, \\ z_n = \dfrac{1}{n} \displaystyle\sum_{k=0}^{n-1} T^k x_n \end{cases}$$

*for all $n = 1, 2, \ldots$, where $0 \leq \alpha_n \leq 1$, $\alpha_n \to 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. If $F(T)$ is nonempty, then $\{x_n\}$ and $\{z_n\}$ converge strongly to $Pu \in F(T)$, where $P$ is the metric projection of $H$ onto $F(T)$.*

*Proof.* As in the proof of Theorem 4.6, a generalized hybrid mapping is a widely more generalized hybrid mapping. Therefore, we have the desired result from Theorem 5.2. □

## References


[1] K. Aoyama, S. Iemoto, F. Kohsaka and W. Takahashi, *Fixed point and ergodic theorems for λ-hybrid mappings in Hilbert spaces*, J. Nonlinear Convex Anal. **11** (2010), 335-343.

[2] K. Aoyama, Y. Kimura, W. Takahashi and M. Toyoda, *Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space*, Nonlinear Anal. **67** (2007), 2350–2360.

[3] J.-B. Baillon, *Un theoreme de type ergodique pour les contractions non lineaires dans un espace de Hilbert,* C.R. Acad. Sci. Paris Ser. A-B **280** (1975), 1511-1514.

[4] E. Blum and W. Oettli, *From optimization and variational inequalities to equilibrium problems*, Math. Student **63** (1994), 123–145.

[5] F. E. Browder, *Convergence theorems for sequences of nonlinear operators in Banach spaces*, Math. Z. **100** (1967), 201–225.

[6] F. E. Browder and W. V. Petryshyn, *Construction of fixed points of nonlinear mappings in Hilbert spaces*, J. Math. Anal. Appl. **20** (1967), 197–228.

[7] P. L. Combettes and A. Hirstoaga, *Equilibrium problems in Hilbert spaces*, J. Nonlinear Convex Anal. **6** (2005), 117–136.

[8] K. Goebel and W. A. Kirk, *Topics in Metric Fixed Point Theory,* Cambridge University Press, Cambridge, 1990.

[9] B. Halpern, *Fixed points of nonexpanding maps*, Bull. Amer. Math. Soc. **73** (1967), 957–961.

[10] M. Hojo, M. Suzuki and W. Takahashi, *Fixed point theorems and convergence theorems for generalized hybrid non-self mappings in Hilbert spaces*, to appear.

[11] M. Hojo and W. Takahashi, *Weak and strong convergence theorems for generalized hybrid mappings in Hilbert spaces*, Sci. Math. Jpn. **73** (2011), 31–40.

[12] M. Hojo, W. Takahashi and J.-C. Yao, *Weak and strong mean convergence theorems for super hybrid mappings in Hilbert spaces*, Fixed Point Theory **12** (2011), 113–126.

[13] T. Ibaraki and W. Takahashi, *Weak convergence theorem for new nonexpansive mappings in Banach spaces and its applications*, Taiwanese J. Math. **11** (2007), 929–944.

[14] T. Ibaraki and W. Takahashi, *Fixed point theorems for nonlinear mappings of nonexpansive type in Banach spaces*, J. Nonlinear Convex Anal. **10** (2009), 21–32.

[15] S. Iemoto and W. Takahashi, *Approximating fixed points of nonexpansive mappings and nonspreading mappings in a Hilbert space*, Nonlinear Anal. **71** (2009), 2082–2089.

[16] S. Itoh and W. Takahashi, *The common fixed point theory of single-valued mappings and multi-valued mappings*, Pacific J. Math. **79** (1978), 493–508.

[17] T. Kawasaki and W. Takahashi, *Fixed point and nonlinear ergodic theorems for new nonlinear mappings in Hilbert spaces*, J. Nonlinear Convex Anal. **13** (2012), 529–540.

[18] T. Kawasaki and W. Takahashi, *Existence and mean approximation of fixed points of generalized hybrid mappings in Hilbert spaces*, J. Nonlinear Convex Anal. **14** (2013), to appear.

[19] P. Kocourek, W. Takahashi and J. -C. Yao, *Fixed point theorems and weak convergence theorems for generalized hybrid mappings in Hilbert spaces*, Taiwanese J. Math. **14** (2010), 2497–2511.

[20] F. Kohsaka and W. Takahashi, *Existence and approximation of fixed points of firmly nonexpansive-type mappings in Banach spaces*, SIAM. J. Optim. **19** (2008), 824–835.

[21] F. Kohsaka and W. Takahashi, *Fixed point theorems for a class of nonlinear mappings related to maximal monotone operators in Banach spaces*, Arch. Math. (Basel) **91** (2008), 166–177.

[22] Y. Kurokawa and W. Takahashi, *Weak and strong convergence theorems for nonlspreading mappings in Hilbert spaces*, Nonlinear Anal. **73** (2010), 1562–1568.

[23] W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc. **4** (1953), 506–510.

[24] T. Shimizu and W. Takahashi, *Strong convergence theorem for asymptotically nonexpansive mappings*, Nonlinear Anal. **26** (1996), 265–272.

[25] T. Shimizu and W. Takahashi, *Strong convergence to common fixed points of families of nonexpansive mappings*, J. Math. Anal. Appl. **211** (1997), 71–83.

[26] W. Takahashi, *A nonlinear ergodic theorem for an amenable semigroup of nonexpansive mappings in a Hilbert space*, Proc. Amer. Math. Soc. **81** (1981), 253–256.

[27] W. Takahashi, *Nonlinear Functional Analysis,* Yokohoma Publishers, Yokohoma, 2000.

[28] W. Takahashi, *Convex Analysis and Approximation of Fixed Points*, Yokohama Publishers, Yokohama, 2000 (in Japanese).

[29] W. Takahashi, *Introduction to Nonlinear and Convex Analysis,* Yokohoma Publishers, Yokohoma, 2009.

[30] W. Takahashi, *Fixed point theorems for new nonlinear mappings in a Hilbert space*, J. Nonlinear Convex Anal. **11** (2010), 79–88.

[31] W. Takahashi and M. Toyoda, *Weak convergence theorems for nonexpansive mappings and monotone mappings*, J. Optim. Theory Appl. **118** (2003), 417–428.

[32] W. Takahashi, N.-C. Wong and J.-C. Yao, *Attractive point and weak convergence theorems for new generalized hybrid mappings in Hilbert spaces*, J. Nonlinear Convex Anal. **13** (2012), 745–757.

[33] W. Takahashi and J.-C. Yao, *Fixed point theorems and ergodic theorems for nonlinear mappings in Hilbert spaces*, Taiwanese J. Math. **15** (2011), 457–472.

[34] W. Takahashi, J.-C. Yao and P. Kocourek, *Weak and strong convergence theorems for generalized hybrid nonself-mappings in Hilbert spaces*, J. Nonlinear Convex Anal. **11** (2010), 567–586.

[35] R. Wittmann, *Approximation of fixed points of nonexpansive mappings*, Arch. Math. (Basel) **58** (1992), 486–491.