BROWDER'S CONVERGENCE FOR UNIFORMLY ASYMPTOTICALLY REGULAR NONEXPANSIVE SEMIGROUPS IN BANACH SPACES

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1. INTRODUCTION

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ and let C be a nonempty closed convex subset of H. Then, a mapping $T : C \to C$ is called nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. We denote by $F(T)$ the set of fixed points of T. Let x be an element of C and for each t with $0 < t < 1$, let x_t be a unique element of C satisfying $x_t = tx + (1 - t)Tx_t$. In 1967, Browder [4] proved the following strong convergence theorem.

Theorem 1.1. *Let H be a Hilbert space, let C be a nonempty bounded closed convex subset of H and let T be a nonexpansive mapping of C into itself. Let x be an element of C and for each t with $0 < t < 1$, let x_t be a unique element of C satisfying $x_t = tx + (1 - t)Tx_t$. Then, $\{x_t\}$ converges strongly to the element of $F(T)$ nearest to x as $t \downarrow 0$.*

Reich [17] and Takahashi and Ueda [30] extended Browder's result to those of a Banach space. Using the idea of Shimizu and Takahashi [18, 19] and the notion of sequence of means, Shioji and Takahashi [20] proved the strong convergence of Browder's type sequences for nonexpansive semigroups (see also [21, 22, 23]). On the other hand, Domingues Benavides, Acedo and Xu [9] proved Browder's type strong convergence theorems for uniformly asymptotically regular one-parameter nonexpansive semigroups. Acedo and Suzuki [13] generalized Domingues Benavides, Acedo and Xu's results concerning the condition of the sequences in real numbers. Recently, the author [2] studied Browder's type iterations for nonexpansive semigroups and proved strong convergence theorems for uniformly asymptotically regular nonexpansive semigroups in Hilbert spaces by using the idea of [4, 9, 13, 28, 29]. Furthermore, the author [2] proved strong convergence theorems for the nonexpansive semigroups by the viscosity approximation method.

In this paper, we study Browder's type iterations for nonexpansive semigroups in Banach spaces. Then, we give strong convergence theorems for uniformly asymptotically regular nonexpansive semigroups in Banach spaces by using the idea of [4, 9, 13, 28, 29]. Furthermore, we also give strong convergence theorems for the nonexpansive semigroups in Banach spaces by the viscosity approximation method.

2. PRELIMINARIES AND NOTATIONS

Throughout this paper, we denote by $\mathbb{N}$ and $\mathbb{R}$ the set of all positive integers and the set of all real numbers, respectively. We also denote by $\mathbb{Z}^+$ and $\mathbb{R}^+$ the set of all nonnegative integers and the set of all nonnegative real numbers, respectively.

Let E be a real Banach space with norm $\| \cdot \|$. We denote by B_r the set $\{x \in E : \|x\| \leq r\}$. Let E^* be the dual space of a Banach space E. The value of $x^* \in E^*$ at $x \in E$ will be denoted by $\langle x, x^* \rangle$. Let C be a closed subset of a Banach space and let T be a mapping of C into itself. We denote by $F(T)$ the set $\{x \in C : x = Tx\}$.

We denote by I the identity operator on E. The duality mapping J from E into 2^{E^*} is defined by

$$J(x) = \{y^* \in E^* : \langle x, y^* \rangle = \|x\|^2 = \|y^*\|^2\}, x \in E.$$

From the Hahn-Banach theorem, we see that $J(x) \neq \emptyset$ for all $x \in E$.

Let E be a smooth Banach space. Then, J is said to be weakly sequentially continuous at zero if for every sequence $\{x_n\}$ in E which converges weakly to $0 \in E$, $\{J(x_n)\}$ converges weakly* to $0 \in E^*$.

We say that a Banach space E satisfies *Opial's condition* [15] if for each sequence $\{x_n\}$ in E which converges weakly to x,

$$\varliminf_{n \to \infty} \|x_n - x\| < \varliminf_{n \to \infty} \|x_n - y\| \tag{1}$$

for each $y \in E$ with $y \neq x$. If E is reflexive Banach space with weakly sequentially continuous duality mapping, then E satisfies Opial's condition. Each Hilbert space and the sequence spaces ℓ^p with $1 < p < \infty$ satisfy Opial's condition (see [15]). Though an L^p-space with $p \neq 2$ does not usually satisfy Opial's condition, each separable Banach space can be equivalently renormed so that it satisfies Opial's condition (see [10, 15]). In a reflexive Banach space, this condition is equivalent to the analogous condition for a bounded net which has been introduced in [12]. It is well known that this condition is equivalent to the analogous condition of $\varlimsup$ (see [1]).

Proposition 2.1. *Let H be a Hilbert space. Let $\{x_n\}$ be a sequence in H converging weakly to $x \in H$. Then,*

$$\varliminf_{n \to \infty} \|x_n - x\| < \varliminf_{n \to \infty} \|x_n - y\| \tag{2}$$

for each $y \in E$ with $y \neq x$.

Banach space E is said to be smooth if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each x and y in S_1, where $S_1 = \{u \in E : \|u\| = 1\}$. The norm of E is said to be uniformly Gâteaux differentiable if for each y in S_1, the limit is attained uniformly for x in S_1. We know that if E is smooth, then the duality mapping is single-valued and norm to weak star continuous and that if the norm of E is uniformly Gâteaux differentiable, then the duality mapping is single-valued and norm to weak star, uniformly continuous on each bounded subset of E.

A closed convex subset C of a Banach space E is said to have normal structure if for each bounded closed convex subset K of C which contains at least two points, there exists an element of K which is not a diametral point of K. It is well-known that a closed convex subset of a uniformly convex Banach space has normal structure and a compact convex subset of a Banach space has normal structure (see [29]). We also know that uniformly smooth Banach space has normal structure (see [29]). Every weakly compact convex subset of a Banach space satisfying Opial's condition has normal structure (see [11]). We note that closed convex subset C of a Banach space E is said to have the fixed point property for nonexpansive mappings if for every bounded closed convex subset K of C, every nonexpansive mapping on K, has

a fixed point. We also know that every weakly compact convex subset with Opial property has fixed point property.

Let C be a nonempty closed convex subset of E and let K be a nonempty subset of C. A mapping P of C onto K is said to be sunny if $P(Px + t(x - Px)) = Px$ for each $x \in C$ and $t \geq 0$ with $Px + t(x - Px) \in C$. P is a retraction if $Px = x$ for each $x \in K$. We know from [6, Theorem 3] and [16, Lemma 2.7] the following lemma (see also [29]).

Lemma 2.2 ([6, 16]). *Let E be a smooth Banach space, g let C be a convex subset of E and let K be a subset of C. Then, a retraction P of C onto K is sunny and nonexpansive if and only if*

$$\langle x - Px, J(y - Px) \rangle \leq 0 \quad \text{for all} \quad x \in C \quad \text{and} \quad y \in K.$$

Hence, there is at most one sunny nonexpansive retraction of C onto K.

If there is a sunny nonexpansive retraction of C onto K, K is said to be a sunny nonexpansive retract of C. The following theorem related to the existence of nonexpansive retractions was proved in [7, 8].

Theorem 2.3 ([7, 8]). *Let E be a reflexive Banach space, let C be a nonempty closed convex subset of E and let T be a nonexpansive mapping of C into itself with $F(T) \neq \emptyset$. If T has a fixed point in every nonempty bounded closed convex subset of E such that T leaves invariant, then $F(T)$ is a nonexpansive retract of C.*

Let μ be a mean on positive integers $\mathbb{N}$, i.e., a continuous linear functional on l^∞ satisfying $\|\mu\| = 1 = \mu(1)$. We know that μ is a mean on $\mathbb{N}$ if and only if $\inf\{a_n : n \in \mathbb{N}\} \leq \mu(f) \leq \sup\{a_n : n \in \mathbb{N}\}$ for each $f = (a_1, a_2, \ldots) \in l^\infty$. Occasionally, we use $\mu_n(a_n)$ instead of $\mu(f)$. So, a Banach limit μ is a mean on $\mathbb{N}$ satisfying $\mu_n(a_n) = \mu_n(a_{n+1})$. Let $f = (a_1, a_2, \ldots) \in l^\infty$ and let μ be a Banach limit on $\mathbb{N}$. Then,

$$\varliminf_{n \to \infty} a_n \leq \mu(f) = \mu_n(a_n) \leq \varlimsup_{n \to \infty} a_n.$$

In particular, if $a_n \to a$, then $\mu(f) = \mu_n(a_n) = a$ (see [27, 29]). The following lemma was proved in [30] (see also [17, 27]).

Lemma 2.4 ([30]). *Let C be a nonempty closed convex subset of a Banach space with a uniformly Gâteaux differentiable norm. Let $\{x_n\}$ be a bounded sequence in E and let μ be a Banach limit. Let $z \in C$. Then, $\mu_n\|x_n - z\|^2 = \min_{y \in C} \mu_n\|x_n - y\|^2$ if and only if $\mu_n\langle y - z, J(x_n - z) \rangle \leq 0$ for each $y \in C$, where J is the duality mapping of E.*

We write $x_n \to x$ (or $\lim_{n \to \infty} x_n = x$) to indicate that the sequence $\{x_n\}$ of vectors in H converges strongly to x. We also write $x_n \rightharpoonup x$ (or w-$\lim_{n \to \infty} x_n = x$) to indicate that the sequence $\{x_n\}$ of vectors in H converges weakly to x. In a Hilbert space, it is well known that $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ imply $x_n \to x$.

Let S be a semitopological semigroup. A semitopological semigroup S is called right (resp. left) reversible if any two closed left (resp. right) ideals of S have nonvoid intersection. If S is right reversible, $(S, \leq)$ is a directed system when the binary relation "$\leq$" on S is defined by $s \leq t$ if and only if $\{s\} \cup \overline{Ss} \supset \{t\} \cup \overline{St}$, $s, t \in S$, where $\overline{A}$ is the closure of A. A commutative semigroup S is a directed system when the binary relation is defined by $s \leq t$ if and only if $\{s\} \cup (S + s) \supset \{t\} \cup (S + t)$.

Let C be a nonempty closed convex subset of a Hilbert space H. A family $\mathcal{S} = \{T(t) : t \in S\}$ of mappings of C into itself is said to be a nonexpansive semigroup on C if it satisfies the following conditions:

(i) For each $t \in S$, $T(t)$ is nonexpansive;

(ii) $T(ts) = T(t)T(s)$ for each $t, s \in S$.

We denote by $F(\mathcal{S})$ the set of common fixed points of $\mathcal{S}$, i.e., $F(\mathcal{S}) = \bigcap_{t \in S} F(T(t))$.

3. STRONG CONVERGENCE THEOREMS

In this section, we prove strong convergence theorems for uniformly asymptotically regular nonexpansive semigroups in Banach spaces. Let C be a nonempty closed convex subset of a Banach space E, let S be a commutative semigroup and let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C. We say that a nonexpansive semigroup $\mathcal{S} = \{T(t) : t \in S\}$ is asymptotically regular if

$$\lim_{s \in S} \|T(h)T(s)x - T(s)x\| = 0$$

for all $h \in S$ and $x \in C$ (see also [28, 29]). The following lemma plays an important role in the proof of the main theorem (see [13, 2]):

Lemma 3.1 ([3]). *Let C be a nonempty closed convex subset of a Banach space E, and let S be a commutative semigroup. Let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$. Assume that $\mathcal{S} = \{T(t) : t \in S\}$ is asymptotically regular, that is,*

$$\lim_{t \in S} \|T(h)T(t)x - T(t)x\| = 0$$

for all $h \in S$ and $x \in C$. Then,

$$F(T(h)) = F(\mathcal{S})$$

for each $h \in S$.

We say that a nonexpansive semigroup $\mathcal{S} = \{T(t) : t \in S\}$ is uniformly asymptotically regular if for every $h \in S$ and for every bounded subset K of C,

$$\lim_{s \in S} \sup_{x \in K} \|T(h)T(s)x - T(s)x\| = 0.$$

holds. Several authors prove Browder's convergence theorems for uniformly asymptotically regular one-parameter nonexpansive semigroups (see [9, 13, 26]).

The following lemma is essential in the proof of the main theorem (see [13, 2]).

Lemma 3.2 ([3]). *Let E be a Banach space, let C be a locally weakly compact convex subset of E, and let S be a commutative semigroup. Let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$. Let $\{m_n\}$ be a sequence in $\mathbb{Z}^+$ such that $m_n \to \infty$ or $m_n \to N$ for some $N \in \mathbb{Z}^+$. Let $\{\alpha_n\}$ be a sequence in $\mathbb{R}$ such that $0 < \alpha_n < 1$, and $\alpha_n \to 0$. Let $u \in C$, let $t \in S$, and let $\{x_n\}$ be the sequence defined by*

$$x_n = \alpha_n u + (1 - \alpha_n)(T(t))^{m_n} x_n$$

for each $n \in \mathbb{N}$. Assume that E is smooth, the normalized duality mapping J of E is weakly sequentially continuous at zero and C has the Opial property. Assume also that $\{x_n\}$ converges weakly to some $x \in F(\mathcal{S})$. Then, $\{x_n\}$ converges strongly.

We prove strong convergence theorems for uniformly asymptotically regular nonexpansive semigroups in Banach spaces by using the idea of [2, 4, 9, 26, 28, 29].

Theorem 3.3 ([3]). *Let E be a Banach space, let C be a locally weakly compact convex subset of E, and let S be a commutative semigroup. Let $\mathcal{S} = \{T(t) : t \in S\}$ be a uniformly asymptotically regular nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$. Let $\{m_n\}$ be a sequence in $\mathbb{Z}^+$ such that $m_n \to \infty$ or $m_n \to N$ for some $N \in \mathbb{Z}^+$. Let $\{\alpha_n\}$ be a sequence*

in $\mathbb{R}$ *such that* $0 < \alpha_n < 1$, *and* $\alpha_n \to 0$. *Let* $u \in C$, *let* $t \in S$, *and let* $\{x_n\}$ *be the sequence defined by*

$$x_n = \alpha_n u + (1 - \alpha_n)(T(t))^{m_n} x_n$$

for each $n \in \mathbb{N}$. *Assume that* E *is smooth, the normalized duality mapping* J *of* E *is weakly sequentially continuous at zero and* C *has the Opial property. Then,* $\{x_n\}$ *converges strongly to* Pu, *where* P *is the unique sunny nonexpansive retraction from* C *onto* $F(\mathcal{S})$.

4. Deduced results

In this section, using Theorem 3.3, we obtain some strong convergence theorems for families of nonexpansive mappings. In the case of Hilbert space setting, we have the following strong convergence theorem for a nonexpansive semigroup in a Hilbert space by Theorem 3.3 (see [2]):

Theorem 4.1 ([2]). *Let* H *be a Hilbert space, let* C *be a closed convex subset of* H, *and let* S *be a commutative semigroup. Let* $\mathcal{S} = \{T(t) : t \in S\}$ *be a uniformly asymptotically regular nonexpansive semigroup on* C *such that* $F(\mathcal{S}) \neq \emptyset$. *Let* $\{m_n\}$ *be a sequence in* $\mathbb{Z}^+$ *such that* $m_n \to \infty$ *or* $m_n \to N$ *for some* $N \in \mathbb{Z}^+$. *Let* $\{\alpha_n\}$ *be a sequence in* $\mathbb{R}$ *such that* $0 < \alpha_n < 1$, *and* $\alpha_n \to 0$. *Let* $u \in C$, *let* $t \in S$, *and let* $\{x_n\}$ *be the sequence defined by*

$$x_n = \alpha_n u + (1 - \alpha_n)(T(t))^{m_n} x_n$$

for each $n \in \mathbb{N}$. *Then,* $\{x_n\}$ *converges strongly to* Pu, *where* P *is the metric projection from* C *onto* $F(\mathcal{S})$.

Let C be a nonempty closed convex subset of E. A family $\mathcal{S} = \{T(t) : t \in \mathbb{R}^+\}$ of mappings of C into itself satisfying the following conditions is said to be a one-parameter nonexpansive semigroup on C:

(i) For each $t \in \mathbb{R}^+$, $T(t)$ is nonexpansive;
(ii) $T(t + s) = T(t)T(s)$ for every $t, s \in \mathbb{R}^+$;
(iii) for each $x \in C$, $t \mapsto T(t)x$ is continuous.

In the case when $S = \mathbb{R}^+$, that is, $\mathcal{S}$ is a uniformly asymptotically regular one-parameter nonexpansive semigroup, we have the following strong convergence theorem for a one-parameter nonexpansive semigroup by Theorem 3.3 (see [9, 13]):

Theorem 4.2 ([3]). *Let* E, C *and* $\{m_n\}$ *be as in Theorem 3.3. Let* $\mathcal{S} = \{T(t) : t \in \mathbb{R}^+\}$ *be a uniformly asymptotically regular one-parameter nonexpansive semigroup on* C *such that* $F(\mathcal{S}) \neq \emptyset$. *Let* $\{\alpha_n\}$ *be a sequence in* $\mathbb{R}$ *such that* $0 < \alpha_n < 1$, *and* $\alpha_n \to 0$. *Let* $u \in C$ *and let* $t \in (0, \infty)$. *Let* $\{x_n\}$ *be the sequence defined by*

$$x_n = \alpha_n u + (1 - \alpha_n)T(t^{m_n})x_n$$

for each $n \in \mathbb{N}$. *Then,* $\{x_n\}$ *converges strongly to* Pu, *where* P *is the unique sunny nonexpansive retraction from* C *onto* $F(\mathcal{S})$.

Theorem 4.3. *Let* E, C *and* $\{m_n\}$ *be as in Theorem 3.3. Let* T *be a nonexpansive mapping from* C *into itself such that* $F(T) \neq \emptyset$. *Assume that* T *is uniformly asymptotically regular. Let* $\{\alpha_n\}$ *be a sequence in* $\mathbb{R}$ *such that* $0 < \alpha_n < 1$, *and* $\alpha_n \to 0$. *Let* $u \in C$. *Let* $\{x_n\}$ *be the sequence defined by*

$$x_n = \alpha_n u + (1 - \alpha_n)T^{m_n} x_n$$

for each $n \in \mathbb{N}$. *Then,* $\{x_n\}$ *converges strongly to* Pu, *where* P *is the unique sunny nonexpansive retraction from* C *onto* $F(T)$.

We know that $f : C \to C$ is said to be a contraction on C if there exists $r \in (0,1)$ such that

$$\|f(x) - f(y)\| \leq r\|x - y\|$$

for each $x, y \in C$. Using [25] and Theorem 3.3, we obtain the following strong convergence theorem by the viscosity approximation method (see also [14, 2]).

Theorem 4.4 ([3]). *Let E, C, S, $\mathcal{S} = \{T(t) : t \in S\}$ and $\{m_n\}$ be as in Theorem 3.3. Let f be a contraction on C. Let $\{\alpha_n\}$ be a sequence in $\mathbb{R}$ such that $0 < \alpha_n < 1$, and $\alpha_n \to 0$. Let $u \in C$, let $t \in S$, and let $\{x_n\}$ be the sequence defined by*

$$x_n = \alpha_n f(x_n) + (1 - \alpha_n)(T(t))^{m_n} x_n$$

for each $n \in \mathbb{N}$. Then, $\{x_n\}$ converges strongly to Pu, where P is the unique sunny nonexpansive retraction from C onto $F(\mathcal{S})$.

In the case when $S = \mathbb{R}^+$, that is, $\mathcal{S}$ is a uniformly asymptotically regular one-parameter nonexpansive semigroup, we have the following strong convergence theorem for a one-parameter nonexpansive semigroup by Theorems 3.3 and 4.4 (see [2, 9, 13, 14, 25]):

Theorem 4.5 ([3]). *Let E, C and $\{m_n\}$ be as in Theorem 3.3. Let $\mathcal{S} = \{T(t) : t \in \mathbb{R}^+\}$ be a uniformly asymptotically regular one-parameter nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$. Let f be a contraction on C. Let $\{\alpha_n\}$ be a sequence in $\mathbb{R}$ such that $0 < \alpha_n < 1$, and $\alpha_n \to 0$. Let $u \in C$ and let $t \in (0, \infty)$, and let $\{x_n\}$ be the sequence defined by*

$$x_n = \alpha_n f(x_n) + (1 - \alpha_n)T(t^{m_n})x_n g$$

for each $n \in \mathbb{N}$. Then, $\{x_n\}$ converges strongly to Pu, where P is the unique sunny nonexpansive retraction from C onto $F(\mathcal{S})$.

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