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Certain characterizations of inner product spaces

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1 Introduction

It is well known that a normed linear space X is an inner product space if and only if the space satisfies the parallelogram law, that is,

$$\|x+y\|^{2}+\|x-y\|^{2}=2(\|x\|^{2}+\|y\|^{2})$$

for all $x, y\in X$. This was shown by Jordan and von Neumann in their 1935 paper [5]. The book of Amir [1] contains almost 300 characterizations of inner product spaces which are based on norm inequalities, various notions of orthogonality in normed linear spaces and so on.

In 1944, Ficken showed the following result in [3].

Theorem 1 (Ficken, 1944). *A normed linear space X is an inner product space if and only if the following holds:*

(A) *If $x, y\in X$ and $\|x\|=\|y\|$ then $\|\alpha x+\beta y\|=\|\beta x+\alpha y\|$ for all $\alpha, \beta\in \mathbb{R}$.*

It is clear that the condition (A) is equivalent to the following condition:

(A′) If $x, y\in X$ and $\|x\|=\|y\|$ then $\|x+\alpha y\|=\|\alpha x+y\|$ for all $\alpha>0$.

Using Theorem 1, Lorch obtained some conditions which characterize inner product spaces in [7].

Theorem 2 (Lorch, 1948). *A normed linear space X is an inner product space if and only if the following holds:*

(B) *There exists a fixed real number $c_{0}\neq 0, \pm 1$ such that $x, y\in X$ and $\|x+y\|=\|x-y\|$ imply $\|x+c_{0}y\|=\|x-c_{0}y\|$.*

To prove Theorem 2, we need the following lemmas.

Lemma 1. *Suppose that* (B) *holds. If $x, y\in X$ and $\|x+y\|=\|x-y\|$, then $\|x+c_{0}^{n}y\|=\|x-c_{0}^{n}y\|$ for all $n\in \mathbb{Z}$.*

From this lemma, we may assume that $0<c_{0}<1$.

Lemma 2. *Suppose that* (B) *holds. Let $x, y \in X$ such that $\|x+y\| = \|x-y\|$. Then the convex function $t \mapsto \|x+ty\|$ on $\mathbb{R}$ takes the minimum only at* 0.

Proof of Theorem 2. Suppose that (B) holds. Let $x, y \in X$ such that $\|x+y\| = \|x-y\|$. If $\|x+sy\| = \|x-ty\|$ for $s, t > 0$, then we have

$$\left\|x + \frac{s-t}{2}y\right\| = \min_{r \in \mathbb{R}} \|x+ry\|$$

by (B) and Lemma 1. However, by Lemmas 1 and 2, the function $t \mapsto \|x+ty\|$ takes the minimum only at 0. Thus we have $s = t$. This means that $\|x+ty\| = \|x-ty\|$ for all $t \in \mathbb{R}$, which implies (A). Hence X is an inner product space by Theorem 1. □

Lorch also proved the following results in the same paper.

Theorem 3 (Lorch, 1948). *Let X be a normed linear space. Then the following are equivalent:*

(I) *X is an inner product space.*

(C) *If $x, y \in X$ and $\|x\| = \|y\|$ then $\|\alpha x + \alpha^{-1} y\| \geq \|x+y\|$ for all $\alpha \in \mathbb{R} \setminus \{0\}$.*

(D) *If $x, y \in X$ and $\|\alpha x + \alpha^{-1} y\| \geq \|x+y\|$ for all $\alpha \in \mathbb{R} \setminus \{0\}$ then $\|x\| = \|y\|$.*

Proof. First, we note that (C) $\Leftrightarrow$ (D). Suppose that (C) and (D) holds. Let $x, y \in X$ such that $\|x\| = \|y\|$ and let $\alpha, \beta > 0$. Putting

$$a = \alpha\beta + \alpha^{-1}\beta^{-1} \quad \text{and} \quad b = \alpha\beta^{-1} + \alpha^{-1}\beta,$$

then we have

$$ab = \alpha^2 + \alpha^{-2} + \beta^2 + \beta^{-2} \geq (\alpha + \alpha^{-1})^2.$$

It follows from (C) that

$$\begin{aligned}
\|\beta(\alpha x + \alpha^{-1} y) + \beta^{-1}(\alpha^{-1} x + \alpha y)\| &= \|ax + by\| \\
&= \sqrt{ab}\left\|\sqrt{\frac{a}{b}}x + \sqrt{\frac{b}{a}}y\right\| \\
&\geq (\alpha + \alpha^{-1})\|x+y\| \\
&= \|(\alpha x + \alpha^{-1} y) + (\alpha^{-1} x + \alpha y)\|,
\end{aligned}$$

which implies that

$$\|\beta(\alpha x + \alpha^{-1} y) + \beta^{-1}(\alpha^{-1} x + \alpha y)\| \geq \|(\alpha x + \alpha^{-1} y) + (\alpha^{-1} x + \alpha y)\|$$

for all $\beta \in \mathbb{R} \setminus \{0\}$. Thus, we have $\|\alpha x + \alpha^{-1} y\| = \|\alpha^{-1} x + \alpha y\|$ by (D). This shows (C) & (D) $\Rightarrow$ (A′), and so X is inner product space by Theorem 1. □

We shall consider the next characterization. If X is an inner product space, then the following holds:

(E) If $x, y \in X$ and $\|x\| = \|y\| = 1$ then

$$\left\|\frac{x+y}{2}\right\| \leq \|(1-t)x + ty\|$$

for all $t \in [0,1]$.

However, even in the space ℓ_p $(1 < p < \infty,\ p \neq 2)$, this does not hold.

So it is natural to ask whether the condition (E) characterizes inner product spaces. In 1970, Gurarii and Sozonov settled this problem affirmatively in their paper [4].

Theorem 4 (Gurarii and Sozonov, 1970). *A normed linear space X is an inner product space if and only if the following holds:*

(E) *If $x, y \in X$ and $\|x\| = \|y\| = 1$ then*

$$\left\|\frac{x+y}{2}\right\| \leq \|(1-t)x + ty\|$$

for all $t \in [0,1]$.

Proof. Suppose that (E) holds. Let $x, y \in X$ such that $\|x\| = \|y\| = 1$. Then, we have

$$\begin{aligned}\|\alpha x + \alpha^{-1} y\| &= (\alpha + \alpha^{-1}) \left\| \frac{\alpha}{\alpha + \alpha^{-1}}\, x + \frac{\alpha^{-1}}{\alpha + \alpha^{-1}}\, y \right\| \\ &\geq (\alpha + \alpha^{-1}) \left\| \frac{x+y}{2} \right\| \\ &\geq \|x + y\|.\end{aligned}$$

for all $\alpha > 0$. This is the essential case of (C), and so we obtain (E) $\Rightarrow$ (C). Thus X is an inner product space by Theorem 3. $\square$

Remark. This proof is based on an idea due to Kirk and Smiley.

As is easily seen that the condition (E) is equivalent to the following condition:

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{2\|x-y\|}{\|x\| + \|y\|}$$

for all $x, y \in X$. This means that the Dunkl-Williams inequality holds with constant 2. Kirk and Smiley [6] showed that this characterizes inner product spaces.

Therefore, in fact, the problem had been solved by Kirk and Smiley earlier than Gurarii and Sozonov.

We conclude this section with some remarks on Theorem 2.

Theorem 2. *A normed linear space X is an inner product space if and only if the following holds:*

(B) *There exists a fixed real number $c_0 \neq 0, \pm 1$ such that $x, y \in X$ and $\|x+y\| = \|x-y\|$ imply $\|x + c_0 y\| = \|x - c_0 y\|$.*

It is obvious that (B) is equivalent to the following two conditions:

(B′) There exists a fixed real number $c_0 > 1$ such that $\|c_0x + y\| = \|x + c_0y\|$ for all $x, y \in X$ with $\|x\| = \|y\| = 1$.

(B″) There exists a fixed real number $t_0 \in (0, 1/2)$ such that $\|(1 - t_0)x + t_0y\| = \|t_0x + (1 - t_0)y\|$ for all $x, y \in X$ with $\|x\| = \|y\| = 1$.

Thus, Theorem 2 can be restated as follows:

Theorem 2′. *Let X be a normed linear space and let S_X denotes its unit sphere. Then the following are equivalent:*

(i) *X is an inner product space.*

(ii) *There exists a fixed real number $c_0 > 1$ such that*

$$\|c_0x + y\| = \|x + c_0y\|$$

for all $x, y \in S_X$.

(iii) *There exists a fixed real number $t_0 \in (0, 1/2)$ such that*

$$\|(1 - t_0)x + t_0y\| = \|t_0x + (1 - t_0)y\|$$

for all $x, y \in S_X$.

2 A new characterization and its application

Weakening conditions (ii) and (iii) in Theorem 2′, we have the following result in [8].

Theorem 5. *Let X be a normed linear space and let S_X denotes its unit sphere. Then the following are equivalent:*

(i) *X is an inner product space.*

(ii) *For each $x, y \in S_X$, there exists a real number $c > 1$ such that*

$$\|cx + y\| = \|x + cy\|.$$

(iii) *For each $x, y \in S_X$, there exists a real number $t \in (0, 1/2)$ such that*

$$\|(1 - t)x + ty\| = \|tx + (1 - t)y\|.$$

In particular, real numbers c and t may depend on $x, y \in S_X$.

Proof. It is enough to prove that (iii) $\Rightarrow$ (i). To see this, we prove that (iii) $\Rightarrow$ (E), that is, we show that if $x, y \in X$ and $\|x\| = \|y\| = 1$ then

$$\left\|\frac{x+y}{2}\right\| \leq \|(1-t)x + ty\|$$

for all $t \in [0,1]$.

Let $x,\ y \in S_X$. We may assume that $\{x, y\}$ is linearly independent. Let A be the subset of $(0, 1/2)$ defined by

$$A = \{t \in (0,1/2) : \|(1-t)x + ty\| = \|tx + (1-t)y\|\}.$$

Then A is nonempty by (iii).

Put $t_0 := \sup A$. Once it has been proved that $t_0 = 1/2$, one can easily obtain

$$\left\|\frac{x+y}{2}\right\| \leq \|(1-t)x + ty\|$$

for all $t \in [0,1]$ by the convexity of the function $t \mapsto \|(1-t)x + ty\|$.

Assume that $t_0 < 1/2$. Then the continuity of the norm assures that $t_0 \in A$. Putting $u = (1-t_0)x + t_0 y$ and $v = t_0 x + (1-t_0)y$, we have $\|u\| = \|v\|$. Let $x_0 = \|u\|^{-1}u$ and $y_0 = \|v\|^{-1}v$, respectively. From the assumption, there exists a real number $s_0 \in (0, 1/2)$ such that

$$\|(1-s_0)x_0 + s_0 y_0\| = \|s_0 x_0 + (1-s_0)y_0\|.$$

Put $t_1 = (1-s_0)t_0 + s_0(1-t_0)$. Then we note that $t_0 < t_1 < 1/2$ and

$$\|(1-t_1)x + t_1 y\| = \|t_1 x + (1-t_1)y\|.$$

So we have $t_1 \in A$, but this contradicts to $t_0 = \max A$. Therefore we have $t_0 = 1/2$, as desired. □

As an application of Theorem 5, we have the following simple characterization of inner product spaces:

Corollary 1. *A normed linear space X is an inner product space if and only if*

$$\left\|x + \frac{x+y}{\|x+y\|}\right\| = \left\|y + \frac{x+y}{\|x+y\|}\right\|$$

for all $x, y \in S_X$ with $x + y \neq 0$.

Proof. We only prove the sufficiency of the condition. Let $x, y \in S_X$ with $x + y \neq 0$. Then, by the above equation, we have

$$\|(1 + \|x+y\|)x + y\| = \|x + (1 + \|x+y\|)y\|.$$

We note that $1 + \|x+y\| > 1$ since $x + y \neq 0$. Thus by Theorem 5 (ii), X is an inner product space. □

Remark. Clearly, the value $1 + \|x+y\|$ depends on $x, y \in S_X$. Hence, Corollary 1 can not be a consequence of Theorem 2$'$. This shows an advantage of Theorem 5.

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