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Stability and singularities of harmonic maps.

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1 Instability of smooth harmonic maps.

Let $\mathbb{S}^k$ be a k-dimensionla Euclidean sphere and N be an n-dimensional compact Riemannian manifold without boundary. We denote by $C^\infty(\mathbb{S}^k, N)$ the set of smooth maps from $\mathbb{S}^k$ to N. We define the Dirichlet energy functional $\mathbf{E}: C^\infty(\mathbb{S}^k, N) \to \mathbb{R}$ to be

$$\mathbf{E}(f) = \frac{1}{2}\int_{\mathbb{S}^k} |df|^2 \, d\mu$$

where$|df|$ is the Hilbert-Schmidt norm of a linear map $(df)_x \in T_x^*\mathbb{S}^k \otimes T_{f(x)}N$ and μ is the canonical measure on $\mathbb{S}^k$ induced by the Riemannian metric. A map $f \in C^\infty(\mathbb{S}^k, N)$ is said to be a harmonic map if it is a critical point of $\mathbf{E}$.
Let $f^{-1}TN$ be the pull-back bundle of TN by f and $C^\infty(f^{-1}TN)$ be the vector space of smooth sections of $f^{-1}TN$. If f_t is a smooth homotopy with $f_0 = f$,

$$V(x) = \frac{d}{dt} f_t(x)\Big|_{t=0}$$

is called a *variation vector field* of f_t. The second variation formula is [14]

$$\delta_f^2\mathbf{E}(V) = \frac{d^2}{dt^2}\mathbf{E}(f_t)\Big|_{t=0} = -\int_{\mathbb{S}^k} \langle V,\ \mathrm{Tr}(\tilde{\nabla}^2 V + R^N(V, df)df)\rangle \, d\mu$$

where $\tilde{\nabla}$ is the induced connection on $C^\infty(f^{-1}TN)$, R^N is a Riemannian curvature of N and Tr is the trace. An operator $J_f : C^\infty(f^{-1}TN) \to C^\infty(f^{-1}TN)$ defined by

$$J_f V = -\ \mathrm{Tr}(\tilde{\nabla}^2 V + R^N(V, df)df)$$

is called the *Jacobi operator* along f. This is a linear elliptic differential operator and its spectrum consists of a discrete sequence of real eigenvalues. We denote by $\lambda_1(J_f)$ the least eigenvalue of J_f. If $\lambda_1(J_f)$ is negative, f is called *unstable.*
It is known that every non-constant harmonic map $f \in C^\infty(\mathbb{S}^k, N)$ is unstable when $k \geq 3$. More precisely the following theorem holds.

Theorem 1.1. [15] *For a non-constant harmonic map* $f \in C^\infty(\mathbb{S}^k, N)$*, we have*

$$\lambda_1(J_f) \leq 2 - k.$$

A simple example shows that this estimate is sharp.

Theorem 1.2. [14] *For the identity map* $\mathrm{id} \in C^\infty(\mathbb{S}^k, \mathbb{S}^k)$*, we have*

$$\lambda_1(J_{\mathrm{id}}) = 2 - k.$$

In some sense, the converse is aloso true.

Theorem 1.3. [6] *Assume k is greater tham two. If a harmonic map $f \in C^\infty(\mathbb{S}^k, \mathbb{S}^k)$ satisfies*

$$\lambda_1(J_f) = 2 - k,$$

then there exists a $(k+1) \times (k+1)$ orthogonal matrix R such that

$$f(x) = Rx \quad x \in \mathbb{S}^k.$$

Remark 1.1. Assume k is greater than two and less than eight. Let d be an integer whose absolute value is greater than one. There exists a harmonic map $f_d \in C^\infty(\mathbb{S}^k, \mathbb{S}^k)$, the mapping degree of which is d [13]. From Theorem 1.3, we have

$$\lambda_1(J_{f_d}) < 2 - k.$$

Remark 1.2. For every harmonic map $f \in C^\infty(\mathbb{S}^2, \mathbb{S}^2)$, we have

$$\lambda_1(J_f) = 0$$

because it minimizes the Dirichlet energy in its homotopy class and the Dirichlet energy is conformally invariant in this case.

2 Singularities of energy minimizing maps.

Let Ω be a bounded domain with smooth boundary in n-dimensional Euclidean space $\mathbb{R}^n$. We will employ the space $H^1(\Omega, \mathbb{S}^k)$ of L^2 maps $u : \Omega \to \mathbb{R}^{k+1}$ with distribution gradient $\nabla u \in L^2$ and $u(x) \in \mathbb{S}^k$ for almost every $x \in \Omega$.

For a map $u \in H^1(\Omega, \mathbb{S}^k)$, the Dirichlet energy $\mathbf{E}(u)$ of u is defined by

$$\mathbf{E}(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx$$

We consider the Dirichlet problem of $\mathbf{E}$. For any $\phi \in C^\infty(\partial\Omega, \mathbb{S}^k)$, we define $H_\phi(\Omega, \mathbb{S}^k)$ by

$$H^1_\phi(\Omega, \mathbb{S}^k) = \{u \in H^1(\Omega, \mathbb{S}^k) \mid u = \phi \quad \text{on } \partial\Omega\}.$$

A map $u \in H^1_\phi(\Omega, \mathbb{S}^k)$ is called an *energy minimizing map* if it satisfies

$$\mathbf{E}(u) = \inf\{\mathbf{E}(v) \mid v \in H^1_\phi(\Omega, \mathbb{S}^k)\}.$$

This is a natural extension of harmonic functions. In contrast to harmonic functions, energy minimiing maps may have discontinuous points. In accordance with custom, we use the word *singular* when we discuss the discontinuity of energy minimizing maps. The following theorem shows the existence of energy minimizing maps with singular points.

Theorem 2.1. [3] *Let n be an integer greater than two and ϕ be the identity map of $\mathbb{S}^{n-1}$. The map*

$$x/|x| \in H^1_\phi(\mathbb{B}^n, \mathbb{S}^{n-1})$$

is an energy minimizing map, where $\mathbb{B}^n$ is the unit open ball centered at the origin.

In 1987, Brezis-Coron-Lieb [2] investigated the behavior of energy minimizing maps from domains in $\mathbb{R}^3$ to $\mathbb{S}^2$.

Theorem 2.2. [2, 9, 10, 12] *Let Ω be a bounded domain with a smoooth boundary in $\mathbb{R}^3$ and $\phi \in C^\infty(\partial\Omega, \mathbb{S}^2)$. If $u \in H^1_\phi(\Omega, \mathbb{S}^2)$ is an energy minimizing map, u has at most finitely many interior singular points. If $p \in \Omega$ is a singular point of u, it is an isolated singular point and there exists a 3×3 orthogonal matrix R such that for any small positive number ϵ,*

$$\sup_{\epsilon<|x|<1} \left| u(p+rx) - R\frac{x-p}{|x-p|} \right|$$

converges to zero as r tends to zero.

In the case of energy minimizing maps from four-dimensional domains to $\mathbb{S}^3$, the following theorem holds.

Theorem 2.3. [4, 5, 7, 10, 11] *Let Ω be a bounded domain with a smoooth boundary in $\mathbb{R}^4$ and $\phi \in C^\infty(\partial\Omega, \mathbb{S}^3)$. If $u \in H^1(\Omega, \mathbb{S}^3)$ is an energy minimizing map, u has at most finitely many interior singular points. If $p \in \Omega$ is a singular point of u, it is an isolated singular point and there exists a 4×4 orthogonal matrix R such that for any small positive number ϵ,*

$$\sup_{\epsilon<|x|<1} \left| u(p+rx) - R\frac{x-p}{|x-p|} \right|$$

converges to zero as r tends to zero.

Though these two theorems look similar, they imply quite different results. For any energy minimizing map $u \in H^1(\Omega, \mathbb{S}^k)$, we denote by $N(u)$ the number of singular points of u.

Theorem 2.4. [1] *For any bounded domain $\Omega \subset \mathbb{R}^3$, there exists a constant $C > 0$ satisfying the following.*

For any $\phi \in C^\infty(\partial\Omega, \mathbb{S}^2)$ and any energy minimizing map $u \in H^1_\phi(\Omega, \mathbb{S}^2)$, we have

$$N(u) \leq C \int_{\partial\Omega} |\nabla\phi|^2 \, d\mathcal{H}^2 ,$$

where $\mathcal{H}^2$ is the two-dimensional Hausdorff measure.

On the other hand, combining Theorem 2.3 and Lemma 2 in [8], we have the following.

Theorem 2.5. *For any small positive number ϵ and a natural number D, there exits a map $\phi \in C^\infty(\partial\mathbb{B}^4, \mathbb{S}^3)$ with*

$$\int_{\partial\mathbb{B}^4} |\nabla\phi|^2 \, d\mathcal{H}^3 < \epsilon$$

such that every energy minimizing map $u \in H^1_\phi(\mathbb{B}^4, \mathbb{S}^3)$ satisfies $N(u) \geq D$.

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