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A new approach to the existence of harmonic maps

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§1 Introduction

Throughout this article, let (M, g) and (N, h) be closed Riemannian manifolds of dimension m and n, respectively. A map $u:(M, g)\rightarrow(N, h)$ of class C^{∞} is said to be *harmonic* if it is a critical point of the so-called Dirichlet energy functional

$$\mathrm{E}(u):=\int_{M}|du|^{2}d\mu_{g}$$

with respect to a smooth variation of the image of u. Here $|du|$ stands for the Hilbert-Schmidt norm of the differential $du:TM\rightarrow TN$ of u and $d\mu_{g}$ for the volume element of (M, g). u is harmonic if and only if it satisfies the Euler-Lagrange equation

$$\tau(u)=\mathrm{div}_{g}(du)=0,$$

where div_{g} stands for the divergence with respect to g.

The aim of this article is to introduce a new approach to the existence theorem of harmonic maps into a manifold with nonpositive sectional curvature.

Given $\varepsilon>0$, we consider the energy functional $\mathbb{E}_{\varepsilon}$ defined as

$$\mathbb{E}_{\varepsilon}(u):=\int_{M}\frac{e^{\varepsilon|du|^{2}}-1}{\varepsilon}d\mu_{g}$$

for maps $u:(M, g)\rightarrow(N, h)$. A map $u:(M, g)\rightarrow(N, h)$ of class C^{∞} which extremizes $\mathbb{E}_{\varepsilon}$ is said to be ε-*exponentially harmonic*. Since $\mathbb{E}_{\varepsilon}\rightarrow \mathrm{E}$ as $\varepsilon\rightarrow 0$ formally, a sequence $\{u_{\varepsilon}\}_{\varepsilon>0}$ of ε-exponentially harmonic maps is expected to approximate a harmonic map from (M, g) to (N, h). We then actually have the following theorem.

Main Theorem. *Let (M,g) and (N,h) be closed Riemannian manifolds and assume that the sectional curvature of (N,h) is nonpositive. Let*

$$\{u_\varepsilon : (M,g) \to (N,h)\ ;\ \varepsilon\text{-exponentially harmonic map},\ \mathbb{E}_\varepsilon(u_\varepsilon) \le E_0\}_{\varepsilon>0}$$

be a given sequence. Then there exists a subsequence $\{u_{\varepsilon(k)}\}_{k=1}^{\infty} \subseteq \{u_\varepsilon\}_{\varepsilon>0}$, $\varepsilon(k) \to 0$ as $k \to \infty$, which uniformly converges to some harmonic map $u : (M,g) \to (N,h)$:

$$u_{\varepsilon(k)} \to u \ \ (k \to \infty) \quad \text{in } C^\infty(M,N).$$

As we shall mention later, it is known that, without any assumptions on the geometry of (M,g) nor (N,h), there always exists an ε-exponentially harmonic map for each $\varepsilon > 0$ in a given homotopy class. Therefore Main Theorem, combined with this fact, implies the following theorem due to Eells and Sampson.

Corollary 1 (Eells-Sampson [4]). *If* $\mathrm{sect}^N \le 0$, *then any homotopy class of continuous maps from M to N admits a harmonic map.*

§ 2 Exponentially harmonic maps

Definition. We say that a C^∞ map $u : (M,g) \to (N,h)$ is *exponentially harmonic* if it is a critical point of

$$\mathbb{E}(u) = \int_M e^{|du|^2} d\mu_g$$

with respect to a smooth variation of the image of u.

The Euler-Lagrange equation for an exponentially harmonic map $u : (M,g) \to (N,h)$ is given as follows:

$$\mathrm{div}_g(e^{|du|^2} du) = e^{|du|^2}\left\{\tau(u) + \langle \nabla |du|^2, du\rangle\right\} = 0, \tag{2.1}$$

where $\tau(u) = \mathrm{div}_g(du)$ stands for the tension field of u and $\langle \cdot, \cdot \rangle$ for the inner product with respect to g.

One of the reasons why we are interested in studying the functional $\mathbb{E}$ is that the existence of its minima in a given homotopy class is always guaranteed without any special assumptions on (M,g) nor (N,h).

Proposition (Eells-Lemaire [3]). *Any homotopy class $\mathcal{H} \in [M,N]$ of continuous maps from M to N contains an $\mathbb{E}$-minimizer u in $\mathcal{H}$, which is necessarily α-Hölder continuous for any exponent $0 < \alpha < 1$.*

The proof is very simple and follows only from the following inequality

$$\frac{1}{k!}\int_M |\nabla u|^{2k} d\mu_g \leq \int_M e^{|\nabla u|^2} d\mu_g.$$

Indeed, a minimizing sequence for $\mathbb{E}$ is bounded in the Sobolev space $W^{1,2k}(M,N)$ for any $k \geq 1$ for the only reason that each of them has uniformly bounded $\mathbb{E}$-energy. From the proof in [3] of this proposition, however, it is not immediately followed that u has further regularity, even is Lipschitz continuous, or it satisfies the Euler-Lagrange equation (2.1), even in a weak sense.

However, the rapider the growth of a functional is, the higher regularity of its minima we can expect. Indeed, in the case of $N = \mathbb{R}$, Duc-Eells [2] showed that an $\mathbb{E}$-minimizer $u : (M,g) \to \mathbb{R}$ of the Dirichlet problem is of class C^∞ in the interior of M, where (M,g) is a compact Riemannian manifold with boundary, and Lieberman [6] showed the global regularity for $u : \Omega \to \mathbb{R}$, where $\Omega \subseteq \mathbb{R}^m$ is a domain. Also, for $n \geq 2$, Naito [7] showed that an $\mathbb{E}$-minimizer $u : \Omega \to \mathbb{R}^n$, where $\Omega \subseteq \mathbb{R}^m$ is a bounded domain, is of class C^∞ in the interior of Ω. Thereafter Duc [1] at last showed the following strongest regularity theorem for $\mathbb{E}$-minimizer.

Theorem (Duc [1]). *Any homotopy class $\mathcal{H} \in [M,N]$ of continuous maps from M to N contains an $\mathbb{E}$-minimizer u in $\mathcal{H}$, which is necessarily of class C^∞.*

§ 3 A gradient estimate for exponentially harmonic maps

In this section, we shall give an outline of the proof in [8] of the following gradient estimate for exponentially harmonic maps, which is a key ingredient for the proofs of Main Theorem.

Lemma 1 ([8, Lemma 3.1]). *If the sectional curvature of (N,h) is nonpositive, then any exponentially harmonic map u from (M,g) to (N,h) satisfies the following gradient estimate:*

$$\sup_M |du|^2 \leq C_0 \int_M \left(e^{|du|^2} - 1\right) d\mu_g,$$

where the constant $C_0 > 0$ depends only on the dimension $m = \dim M$ of M, the Ricci curvature Ric^M of (M,g), and the exponential energy $\mathbb{E}(u)$.

In this article, only some essential parts of the proof of Lemma 1 are provided. For a complete proof, see [8].

By means of J. Nash's isometric embedding $\iota : (N,h) \hookrightarrow \mathbb{R}^d$, we identify $\iota \circ u$ with u for a map $u : M \to N$. We mean by du the derivative of $u : M \to N$, while by ∇u the gradient of the function $\iota \circ u : M \to \iota(N) \subseteq \mathbb{R}^d$. Let $B_r = B_r(x) \subseteq M$ stand for the ball of radius $r > 0$ centered at a point $x \in M$.

If $u : (M,g) \to (N,h)$ satisfies the Euler-Lagrange equation for $\mathbb{E}$, then

$$0 = \sum_{A=1}^{d} \int_{B_r} \nabla_i u^A \nabla^i \varphi^A e^{|\nabla u|^2} d\mu_g + \sum_{A=1}^{d} \int_{B_r} \nabla d\Pi^A(u)(\nabla^i u, \nabla_i u)\varphi^A e^{|\nabla u|^2} d\mu_g \tag{3.1}$$

for any test function $\varphi \in C_0^\infty(B_r, \mathbb{R}^d)$. Here $\Pi : U_\delta(N) \to N$ is the nearest projection from a tubular neighborhood $U_\delta(N)$ of N onto N. Also, we use the Einstein summation convention, namely, when an index occurs more than once in the same expression, the expression is implicitly summed over all possible values for that index.

As in the proof of [7, Proposition 2.10], choose

$$\varphi^A = \nabla^k(\eta^2 \nabla_k u^A) \tag{3.2}$$

as a test function in (3.1), where $\eta : B_r \to \mathbb{R}$ is a cut-off function satisfying

$$0 \le \eta \le 1, \quad \eta = 1 \text{ on } B_{r/2}, \quad \operatorname{supp}\eta \subseteq B_r, \quad \text{and} \quad |\nabla \eta| \le \frac{2}{r}.$$

First we note that it follows from the Ricci identity that

$$\begin{aligned}
\nabla^i \varphi^A &= \nabla^i \nabla^k(\eta^2 \nabla_k u^A) \\
&= \nabla^k \nabla^i(\eta^2 \nabla_k u^A) - g^{ij} g^{kl} R^{M\,s}_{\ jlk}(\eta^2 \nabla_s u^A),
\end{aligned}$$

where $R^{M\,l}_{\ ijk}\partial_l = \nabla_{\partial_i}\nabla_{\partial_j}\partial_k - \nabla_{\partial_j}\nabla_{\partial_i}\partial_k$ is the curvature tensor of (M,g). Then after the integration by parts with respect to ∇^k, (3.1) becomes

$$\begin{aligned}
0 &= \sum_{A=1}^{d} \int_{B_r} (\nabla^k \nabla_i u^A + \nabla_i u^A \nabla^k |\nabla u|^2) \nabla^i(\eta^2 \nabla_k u^A) e^{|\nabla u|^2} d\mu_g \\
&\quad + \int_{B_r} \sum_{i,j=1}^{m} \left\langle du\left(\mathrm{Ric}^M(e_i,e_j)e_j\right), du(e_i)\right\rangle e^{|\nabla u|^2} \eta^2 d\mu_g \\
&\quad - \sum_{A=1}^{d} \int_{B_r} \nabla d\Pi^A(u)(\nabla^i u, \nabla_i u) \nabla^k(\eta^2 \nabla_k u^A) e^{|\nabla u|^2} d\mu_g \\
&= \sum_{A=1}^{d} \int_{B_r} (\nabla^k \nabla_i u^A + \nabla_i u^A \nabla^k |\nabla u|^2) \nabla^i \nabla_k u^A e^{|\nabla u|^2} \eta^2 d\mu_g \\
&\quad + 2\sum_{A=1}^{d} \int_{B_r} (\nabla^k \nabla_i u^A + \nabla_i u^A \nabla^k |\nabla u|^2) \nabla_k u^A e^{|\nabla u|^2} \eta \nabla^i \eta \, d\mu_g \\
&\quad + \int_{B_r} \sum_{i,j=1}^{m} \left\langle du\left(\mathrm{Ric}^M(e_i,e_j)e_j\right), du(e_i)\right\rangle e^{|\nabla u|^2} \eta^2 d\mu_g \\
&\quad - \sum_{A=1}^{d} \int_{B_r} \nabla d\Pi^A(u)(\nabla^i u, \nabla_i u) \nabla^k(\eta^2 \nabla_k u^A) e^{|\nabla u|^2} d\mu_g,
\end{aligned}$$

where $\{e_i\}_{i=1}^m$ is a local orthonormal frame of (M,g). Since $\nabla d\Pi(u)(\nabla^i u, \nabla_i u)$ is the vertical part of Δu to N, the last term becomes

$$-\int_{B_r} |\nabla d\Pi(u)(\nabla^i u, \nabla_i u)|^2 e^{|\nabla u|^2}\eta^2 d\mu_g.$$

Also, by the Leibniz rule and the Gauss formula,

$$\begin{aligned}
&|\nabla\nabla(\iota\circ u)|^2 - |\nabla d\Pi(u)(\nabla^i u, \nabla_i u)|^2 \\
&\quad= |\nabla du|^2 + \langle \nabla d\Pi(u)(\nabla^i u, \nabla^j u), \nabla d\Pi(u)(\nabla_i u, \nabla_j u)\rangle - |\nabla d\Pi(u)(\nabla^i u, \nabla_i u)|^2 \\
&\quad= |\nabla du|^2 - \sum_{i,j=1}^m \left\langle R^N\left(du(e_i), du(e_j)\right) du(e_j), du(e_i)\right\rangle.
\end{aligned}$$

Substituting this into the above equation then yields

$$\begin{aligned}
0 &= \int_{B_r} |\nabla du|^2 e^{|\nabla u|^2}\eta^2 d\mu_g + \frac{1}{2}\int_{B_r} \left|\nabla|\nabla u|^2\right|^2 e^{|\nabla u|^2}\eta^2 d\mu_g \\
&\quad+ \int_{B_r} \left\{ \langle \nabla|\nabla u|^2, \nabla\eta\rangle + 2\sum_{A=1}^d \langle \nabla|\nabla u|^2, \nabla u^A\rangle\langle \nabla u^A, \nabla\eta\rangle \right\} e^{|\nabla u|^2}\eta\, d\mu_g \\
&\quad+ \int_{B_r} \sum_{i,j=1}^m \left\langle du\left(\mathrm{Ric}^M(e_i, e_j)e_j\right), du(e_i)\right\rangle e^{|\nabla u|^2}\eta^2 d\mu_g \\
&\quad- \int_{B_r} \sum_{i,j=1}^m \left\langle R^N\left(du(e_i), du(e_j)\right) du(e_j), du(e_i)\right\rangle e^{|\nabla u|^2}\eta^2 d\mu_g.
\end{aligned}$$

The last integral is nonpositive because (N,h) has nonpositive sectional curvature. By using the inequality $xe^x \le \delta^{-1}e^{(1+\delta)x}$ for any $\delta > 0$ and $x \ge 0$, the third and the fourth integrals are respectively estimated as

$$\begin{aligned}
&\int_{B_r} \left\{ \langle \nabla|\nabla u|^2, \nabla\eta\rangle + 2\sum_{A=1}^d \langle \nabla|\nabla u|^2, \nabla u^A\rangle\langle \nabla u^A, \nabla\eta\rangle \right\} e^{|\nabla u|^2}\eta\, d\mu_g \\
&\qquad\le C(m)\int_{B_r} \left|\nabla|\nabla u|^2\right| (1+|\nabla u|^2) e^{|\nabla u|^2}|\nabla\eta|\eta\, d\mu_g \\
&\qquad\le \frac{C(m)}{\delta}\int_{B_r} \left|\nabla|\nabla u|^2\right| e^{(1+\delta)|\nabla u|^2}|\nabla\eta|\eta\, d\mu_g, \\
&\int_{B_r} \sum_{i,j=1}^m \left\langle du\left(\mathrm{Ric}^M(e_i, e_j)e_j\right), du(e_i)\right\rangle e^{|\nabla u|^2}\eta^2 d\mu_g \\
&\qquad\le \|\mathrm{Ric}^M\|_{L^\infty}\int_{B_r} |\nabla u|^2 e^{|\nabla u|^2}\eta^2 d\mu_g \\
&\qquad\le \frac{1}{\delta}\|\mathrm{Ric}^M\|_{L^\infty}\int_{B_r} e^{(1+\delta)|\nabla u|^2}\eta^2 d\mu_g.
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
&\frac{1}{2}\int_{B_r} \left|\nabla|\nabla u|^2\right|^2 e^{|\nabla u|^2}\eta^2 d\mu_g \\
&\le C(m,\delta)\left(\int_{B_r} \left|\nabla|\nabla u|^2\right|^2 e^{|\nabla u|^2}\eta^2 d\mu_g\right)^{1/2}\left(\int_{B_r} e^{(1+2\delta)|\nabla u|^2}|\nabla\eta|^2 d\mu_g\right)^{1/2} \\
&\quad + C(\mathrm{Ric}^M,\delta)\int_{B_r} e^{(1+\delta)|\nabla u|^2}\eta^2 d\mu_g.
\end{aligned}$$

Since the first integral of the first term in the right hand side can be absorbed into the left hand side and since

$$\int_{B_r} \left|\nabla|\nabla u|^2\right|^2 e^{|\nabla u|^2}\eta^2 d\mu_g = 4\int_{B_r} \left|\nabla(e^{\frac{1}{2}|\nabla u|^2})\right|^2 \eta^2 d\mu_g,$$

by using the Sobolev embedding theorem, we infer

$$\left(\int_{B_{r/2}} e^{\frac{m}{m-2}|\nabla u|^2} d\mu_g\right)^{\frac{m-2}{m}} \le C_1\int_{B_r}\left|\nabla(e^{\frac{1}{2}|\nabla u|^2}\eta)\right|^2 d\mu_g \le \frac{C_1C_2}{r^2}\int_{B_r} e^{(1+\delta)|\nabla u|^2} d\mu_g,$$

where $C_1 > 0$ is the Sobolev constant and depends only on (M,g), while $C_2 > 0$ is a constant depending only on $m = \dim M$, Ric^M, and $\delta > 0$.

This inequality is actually a priori estimate because we can take $\delta > 0$ small enough so that it satisfies, for example, $1 + \delta < \frac{m}{m-2}$.

This is a key ingredient of the proof of Lemma 1. In [8], we can actually prove

$$\int_{B_{r/2}} e^{(1+\delta)|\nabla u|^2} d\mu_g \le C\int_{B_r} e^{|\nabla u|^2} d\mu_g.$$

We can then apply the Moser iteration method to obtain

$$\sup_M |\nabla u| \le C = C(m, \mathrm{Ric}^M, \mathbb{E}(u)). \tag{3.3}$$

To obtain the inequality in Lemma 1, we then need the following identity of Bochner-Weitzenböck type

$$\begin{aligned}
S^{ij}\nabla_i\nabla_j e^{|du|^2} &= 2e^{|du|^2}|\nabla du|^2 + 2e^{|du|^2}|\tau(u)|^2 \\
&\quad + 2e^{|du|^2}\sum_{i,j=1}^{m}\left\langle du(\mathrm{Ric}^M(e_i,e_j)e_j), du(e_i)\right\rangle \\
&\quad - 2e^{|du|^2}\sum_{i,j=1}^{m}\left\langle R^N(du(e_i),du(e_j))du(e_j), du(e_i)\right\rangle,
\end{aligned}$$

where the tensor $S \in \Gamma(TM \otimes TM)$ is given by

$$S^{ij} := g^{ij} + 2\langle du(e_i), du(e_j)\rangle \quad (i,j = 1,2,\dots,m). \tag{3.4}$$

This inequality and (3.3), combined with the assumption on the sectional curvature of (N,h), imply

$$\begin{aligned} S^{ij}\nabla_i\nabla_j\left(e^{|\nabla u|^2} - 1\right) &= S^{ij}\nabla_i\nabla_j e^{|\nabla u|^2} \\ &\geq 2e^{|\nabla u|^2}\sum_{i,j=1}^{m}\langle du(\mathrm{Ric}^M(e_i,e_j)e_j), du(e_i)\rangle \\ &\geq -C(m, \|\mathrm{Ric}^M\|_{L^\infty})e^{|\nabla u|^2}|\nabla u|^2 \\ &\geq -C\left(m, \|\mathrm{Ric}^M\|_{L^\infty}, e^{\|\nabla u\|^2_{L^\infty}}\right)\left(e^{|\nabla u|^2} - 1\right). \end{aligned}$$

In the fourth line we have used the inequality $|\nabla u|^2 \leq e^{|\nabla u|^2} - 1$. Moreover (3.3) then guarantees that S^{ij} has the bounded eigenvalues both from above and from below by a constant depending only on m, Ric^M and $\mathbb{E}(u)$. This observation enables us to successfully apply the maximum principle [5, Theorem 9.20] to acquire

$$|\nabla u|^2 \leq e^{|\nabla u|^2} - 1 \leq C_0(M, \mathbb{E}(u)) \int_M \left(e^{|\nabla u|^2} - 1\right) d\mu_g,$$

proving Lemma 1.

§ 4 Proof of Main Theorem

The complete proof of Main Theorem is given in this section. All we need are the gradient estimate in Lemma 1 and Lemma 2 stated below.

Lemma 2. *For any $\varepsilon > 0$, $u : (M,g) \to (N,h)$ is ε-exponentially harmonic if and only if $u : (M,g) \to (N,h_\varepsilon)$ is 1-exponentially harmonic, where $h_\varepsilon := \varepsilon h$.*

Proof of Main Theorem. If we consider the homothetic transformation $h_\varepsilon = \varepsilon h$, then the given u_ε is, by Lemma 2, a 1-exponentially harmonic map $u_\varepsilon : (M,g) \to (N,h_\varepsilon)$. Then it follows from Lemma 1 that

$$\sup_M |\nabla u_\varepsilon|^2_{h_\varepsilon} \leq C_\varepsilon \int_M \left(e^{|\nabla u_\varepsilon|^2_{h_\varepsilon}} - 1\right) d\mu_g.$$

Here the constant $C_\varepsilon > 0$ depends on m, Ric^M, and $\mathbb{E}^{h_\varepsilon}(u_\varepsilon)$, but not on $R^{(N,h_\varepsilon)}$ because (N,h) has nonpositive sectional curvature. Since $|u_\varepsilon|^2_{h_\varepsilon} = \varepsilon|\nabla u_\varepsilon|^2_h$,

$$\mathbb{E}^{h_\varepsilon}(u_\varepsilon) = \int_M e^{\varepsilon|\nabla u_\varepsilon|^2_h}\, d\mu_g$$

is bounded by a constant depending only on E_0 and $\mathrm{Vol}_g(M)$. Therefore, $C_\varepsilon > 0$ is uniformly bounded (by, say, $C_0 > 0$) in $\varepsilon > 0$ and thus

$$\sup_M \varepsilon |\nabla u_\varepsilon|_h^2 \leq C_0 \int_M \left(e^{\varepsilon |\nabla u_\varepsilon|_h^2} - 1\right) d\mu_g,$$

which yields, after divided by $\varepsilon > 0$, a gradient estimate of $u_\varepsilon : (M, g) \to (N, h)$:

$$\sup_M |\nabla u_\varepsilon|_h^2 \leq C_0 \int_M \frac{e^{\varepsilon |\nabla u_\varepsilon|_h^2} - 1}{\varepsilon} d\mu_g \leq C_0 E_0.$$

This proves the theorem. □

References

[1] D. M. Duc, *Variational problems of certain functionals*, Internat. J. Math. **6** (1995), no. 4, 503–535.

[2] D. M. Duc and J. Eells, *Regularity of exponentially harmonic functions*, Internat. J. Math. **2** (1991), no. 4, 395–408.

[3] J. Eells and L. Lemaire, *Some properties of exponentially harmonic maps*, Partial differential equations, Part 1, 2 (Warsaw, 1990), Banach Center Publ., 27, Part 1, vol. 2, Polish Acad. Sci., Warsaw, 1992, pp. 129–136.

[4] J. Eells and J. H. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. **86** (1964), 109–160.

[5] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[6] G. M. Lieberman, *On the regularity of the minimizer of a functional with exponential growth*, Comment. Math. Univ. Carolin. **33** (1992), no. 1, 45–49.

[7] H. Naito, *On a local Hölder continuity for a minimizer of the exponential energy functional*, Nagoya Math. J. **129** (1993), 97–113.

[8] T. Omori, *On Eells-Sampson's existence theorem for harmonic maps via exponentially harmonic maps,* to appear in Nagoya Math. J.

[9] J. Sacks and K. Uhlenbeck, *The existence of minimal immersions of* 2*-spheres*, Ann. of Math. (2) **113** (1981), no. 1, 1–24.