数理解析研究所講究録
第 1844 巻 2013 年 50-59

CUBATURE FORMULA ON WIENER SPACE FROM THE VIEWPOINT OF SPLITTING METHODS

HIDEYUKI TANAKA

ABSTRACT. The author gives a brief survey on cubature formulas on Wiener space and studies their connection to splitting methods for noncommutative exponential maps. These formulas have many applications in the field of numerical solutions of stochastic differential equations.

1. INTRODUCTION

A cubature formula for a finite measure ν on $\mathbf{R}^d$ is defined as follows: If there exist positive weights λ_i and points $x_i \in \mathbf{R}^d$ $(1 \leq i \leq k)$ such that for any polynomial $\mathcal{P}$ with degree less than or equal to m

$$\int_{\mathbf{R}^d} \mathcal{P}(x)\nu(dx) = \sum_{i=1}^{k} \lambda_i \mathcal{P}(x_i),$$

then we say that the pair $(\lambda_i, x_i)_{1\leq i\leq k}$ defines a cubature formula with degree m. The existence and construction of the above finite d-dimensional cubature formulas has been well-studied (see e.g. Stroud [10]). One important application of the formula $(\lambda_i, x_i)_{1\leq i\leq k}$ is the numerical integration formula

$$\int_{\mathbf{R}^d} f(x)\nu(dx) \approx \sum_{i=1}^{k} \lambda_i f(x_i)$$

for smooth functions f. The reason why this method works is based on the Taylor expansion or polynomial approximation of f. Therefore the regularity of f is a sufficient condition for the method to work well.

The main objective of this article is to review how to construct cubature formulas on Wiener space [1] using splitting methods which have been applied to many research fields such as numerical partial differential equations (e.g. [1]). Cubature formulas on Wiener space play a similar role to that in finite dimensional space in the calculation of infinite dimensional integrals[2].

Wiener space is defined as the space of continuous functions $C([0,1];\mathbf{R}^d)$ equipped with the so-called Wiener measure [3], under which the mapping $B_t := \omega(t)$ for $\omega \in C([0,1];\mathbf{R}^d)$ is a standard Brownian motion [4]. On this space, multiple integrals with respect to the time variable $t \mapsto B_t$ have a similar importance to polynomials in finite dimension. For example, the Itô-Wiener chaos expansion theorem ([2]) shows that L^2-random variables on Wiener space can be expanded by series of the multiple integrals. As seen later, cubature formulas on Wiener space can be applied to numerical approximations of stochastic differential equations, which appear in finance, physics, filtering etc. In this case, the stochastic Taylor expansion gives the error estimation.

Key words and phrases. Cubature on Wiener space; splitting method, weak approximation of SDE; Ninomiya-Victoir scheme.

[1] an infinite dimensional space to be defined below.

[2] expectations.

[3] which replaces the Lebesgue measure in finite dimensions.

[4] also called Wiener process.

This paper is organized as follows. In Section 2, we formulate the cubature formula and prepare some basic tools to discuss algebraic properties of the formula. In Section 3, we introduce the idea of splitting methods for exponential maps and also give some results applicable to the construction of cubature formulas.

This article is based on the talk given in the workshop 'Designs, Codes, Graphs and Related Areas' at the Research Institute for Mathematical Sciences (RIMS), Kyoto University, July 2012.

2. Cubature on Wiener space

2.1. Definitions. Let $(B_t^1, \dots, B_t^d)_{t\in[0,1]}$ be a d-dimensional standard Brownian motion on a complete probability space $(\Omega, \mathcal{F}, P)$, and set the $(d+1)$-dimensional path $B = (B_t^0, B_t^1, \dots, B_t^d)_{t\in[0,1]}$ with $B_t^0 = t$. We use the following notation.

- Let $\alpha \in \mathcal{I} := \{\emptyset \cup (\cup_{k\in\mathbf{N}}\{0,1,\dots,d\}^k)\}$ be an multi-index and then define the degree of α by
$$\|\alpha\| := \begin{cases} k + \#\{\alpha_j = 0\}, & \alpha = (\alpha_1, \dots, \alpha_k) \in \{0,1,\dots,d\}^k, k \geq 1 \\ 0, & \alpha = \emptyset. \end{cases}$$
- $C_{0,BV}([0,t];\mathbf{R}^{d+1})$: the set of all $\mathbf{R}^{d+1}$-valued continuous functions $g = (g_s^0, \dots, g_s^d)_{s\in[0,t]}$ of bounded variation in $[0,t]$ and which start at zero.
- For $\alpha = (\alpha_1, \dots, \alpha_k) \in \{0,1,\dots,d\}^k$, $k \geq 1$, we define the multiple Fisk-Stratonovich integral as follows.
$$I(t,\alpha,\circ dB) := \int_{0<t_k<\cdots<t_1<t} \circ dB_{t_k}^{\alpha_k} \cdots \circ dB_{t_1}^{\alpha_1}.$$
- Similarly, for $g = (g_t^0, \dots, g_t^d)_{t\in[0,1]} \in C_{0,BV}([0,1];\mathbf{R}^{d+1})$, we define
$$I(t,\alpha,dg) := \int_{0<t_k<\cdots<t_1<t} dg_{t_k}^{\alpha_k} \cdots dg_{t_1}^{\alpha_1}.$$
- Throughout the paper, we say that a measurable application
$$\omega = (\omega_t)_{t\in[0,1]} : \Omega \to C_{0,BV}([0,1];\mathbf{R}^{d+1})$$
is a "random path" if it satisfies the moment condition
$$I(1,\alpha,d|\omega|) \in L^1(\Omega,\mathcal{F},P)$$
for any $\alpha \in \mathcal{I}$ and almost all $\omega \in \Omega$. Here $|\cdot|$ denotes the total variation path, i.e., $|\omega|_t^i := \sup_{0=t_0<t_1<\cdots<t_k=t} \sum_{j=1}^k |\omega_{t_j}^i - \omega_{t_{j-1}}^i|$.
- A random path ω has finite mass if there exist finite functions $(g_i)_{1\leq i\leq L} \subset C_{0,BV}([0,1];\mathbf{R}^{d+1})$ and positive weights $(p_i)_{1\leq i\leq L}$ such that $P(\omega = g_i) = p_i$ and $\sum_{i=1}^L p_i = 1$.

Definition 2.1. A random path ω defines a cubature formula with degree m if ω has finite mass and satisfies for every $\|\alpha\| \leq m$

$$E[I(1,\alpha,\circ dB)] = E[I(1,\alpha,d\omega)] \ \Big(= \sum_{i=1}^L p_i I(1,\alpha,dg_i)\Big). \tag{2.1}$$

We denote the space of all random paths which define cubature formulas with degree m by $(\mathrm{Cub})_m$.

Remark 2.2. The original paper by Lyons and Victoir [5] assumes $\omega^0(t) = t$. However the above generalization is straightforward.

We extend $(\mathrm{Cub})_m$ to a more general class which includes Ninomiya-Victoir and Ninomiya-Ninomiya schemes (degree 5 formulas). The random path ω for these two schemes already appeared in Kusuoka's papers [3, 4].

Definition 2.3. A random path ω defines a moment matching formula with degree m if for every $\|\alpha\| \leq m$

$$E[I(1,\alpha,\circ dB)] = E[I(1,\alpha,d\omega)]. \tag{2.2}$$

We denote the space of all random paths satisfying (2.2) by $(\mathrm{M})_m$. Clearly, $(\mathrm{Cub})_m \subseteq (\mathrm{M})_m$.

Example 2.4. Here, we give two examples of random paths with moment matching of degrees 3 and 5. That is, elements of $(\mathrm{M})_m$ with $m = 3,\ 5$.

(Degree 3) For each $0 \leq i \leq d$, define

$$d\omega_t^i := B_1^i dt.$$

Then this ω defines a degree 3 formula. Indeed, due to the symmetry of the Gaussian law of B_1, if $\|\alpha\| = 1$ or 3, $E[I(1,\alpha,\circ dB)] = E[I(1,\alpha,d\omega)] = 0$. Thus it is enough to check the case $\|\alpha\| = 2$. If $\alpha_1 \neq \alpha_2$, clearly $E[I(1,\alpha,\circ dB)] = E[I(1,\alpha,d\omega)] = 0$. If $\alpha_1 = \alpha_2 = i \geq 1$, we have by Itô's formula[5], $I(1,\alpha,\circ dB) = (B_1^i)^2/2 = I(1,\alpha,d\omega)$.

(Degree 5) (Ninomiya-Victoir scheme) Let Λ be a random variable with probability

$$P(\Lambda = 1) = P(\Lambda = -1) = 1/2$$

and which is independent of (B_t). Then we define a piecewise smooth path ω by

$$d\omega_t^i := \begin{cases} (d+2)dt, & \text{if } i = 0, t \in [0, \frac{1}{d+2}) \cup [\frac{d+1}{d+2}, 1) \\ (d+2)B_1^i dt, & \text{if } 1 \leq i \leq d, \Lambda = 1, t \in [\frac{i}{d+2}, \frac{i+1}{d+2}) \\ (d+2)B_1^i dt, & \text{if } 1 \leq i \leq d, \Lambda = -1, t \in [\frac{d+1-i}{d+2}, \frac{d+2-i}{d+2}) \\ 0, & \text{otherwise.} \end{cases}$$

We remark $\omega_t^0 \neq t$.

Let ω be a random path and then define the time-scaled path $(\omega_s[t])_{s\in[0,t]}$ by

$$\omega_s^i[t] := \begin{cases} t\omega_{s/t}^0, & i = 0 \\ \sqrt{t}\omega_{s/t}^i, & 1 \leq i \leq d. \end{cases}$$

Under $(\mathrm{Cub})_m$ or $(\mathrm{M})_m$, the scaling property for the Brownian motion (i.e. $B_t \stackrel{d}{=} \sqrt{t}B_1$ where $\stackrel{d}{=}$ denotes equality in law, also called equality in distribution) implies

$$E[I(t,\alpha,\circ dB)] = t^{\|\alpha\|}E[I(1,\alpha,\circ dB)] = t^{\|\alpha\|}E[I(1,\alpha,d\omega)] = E[I(t,\alpha,d\omega[t])]$$

for every $t > 0$. Therefore, it is enough to reduce our attention to the case $t = 1$ for the construction of cubature formulas.

2.2. **Application: random ODE and stochastic Taylor expansion.** Let $X_t = X_t^x$ be the unique solution to the stochastic differential equation (SDE)

$$X_t^x = x + \sum_{i=0}^{d} \int_0^t V_i(X_s^x) \circ dB_s^i \tag{2.3}$$

where $V_i \in C_b^\infty(\mathbf{R}^N;\mathbf{R}^N)$. We also define a random ordinary differential equation (ODE)

$$\bar{X}_t^x = x + \sum_{i=0}^{d} \int_0^t V_i(\bar{X}_s^x) d\omega_s^i.$$

We denote by $\bar{X}_t^x(d\omega)$ the solution.

[5]For more on this, see below.

The well-known Itô's formula in stochastic calculus is a fundamental theorem of calculus (or change of variable formula) as follows:

$$f(X_t^x) = f(x) + \sum_{i=0}^{d} \int_0^t (V_i f)(X_s^x) \circ dB_s^i$$

for a smooth function f, where V_i acts on f as a vector field $\sum_{j=1}^N V_i^j \frac{\partial}{\partial x_j}$ on $\mathbf{R}^N$. We can apply this formula to the integrands of the (stochastic) integrals. Then we get the so-called stochastic Taylor expansion

$$f(X_t^x) = \sum_{\|\alpha\| \le m} I(t, \alpha, \circ dB)\Big(V_{\alpha_k} \cdots V_{\alpha_1} f\Big)(x) + (\text{remainder}) .$$

Of course, we can also apply the fundamental of calculus to the bounded variation function ω, and then we have a similar formula

$$f(\bar{X}_t^x(d\omega[t])) = \sum_{\|\alpha\| \le m} I(t, \alpha, d\omega[t])\Big(V_{\alpha_k} \cdots V_{\alpha_1} f\Big)(x) + (\text{remainder}) .$$

We can show the following error estimates by using stochastic Taylor expansions.

Theorem 2.5. *Let a random path ω satisfy the condition* $(\mathrm{M})_m$. *Then for any $f \in C_b^\infty(\mathbf{R}^N; \mathbf{R})$, there exists a constant $C = C(m, f)$ such that*

$$|E[f(X_t^x)] - E[f(\bar{X}_t^x(d\omega[t]))]| \le C t^{(m+1)/2}.$$

Sketch of proof. The fundamental theorem of stochastic calculus (i.e. Itô's formula) can be applied to ω and B;

$$f(X_t^x) - \sum_{\|\alpha\| \le m} I(t, \alpha, \circ dB)\Big(V_{\alpha_k} \cdots V_{\alpha_1} f\Big)(x) =: R_m^X(t, x),$$

$$f(\bar{X}_t^x(d\omega[t])) - \sum_{\|\alpha\| \le m} I(t, \alpha, d\omega[t])\Big(V_{\alpha_k} \cdots V_{\alpha_1} f\Big)(x) =: R_m^{\bar{X}}(t, x).$$

We obtain from the assumption $(\mathrm{M})_m$

$$E\Big[\sum_{\|\alpha\| \le m} I(t, \alpha, \circ dB)\Big(V_{\alpha_k} \cdots V_{\alpha_1} f\Big)(x)\Big] = E\Big[\sum_{\|\alpha\| \le m} I(t, \alpha, d\omega[t])\Big(V_{\alpha_k} \cdots V_{\alpha_1} f\Big)(x)\Big].$$

One can easily check that the remainders R_m^X and $R_m^{\bar{X}}$ consist of the multiple integrals of B and ω with degree $m+1$ and $m+2$. Therefore the result follows from the time-scaling property for B_t and $\omega[t]$. □

If t is not small, we can use a Markov chain type approximation as follows.

Theorem 2.6. *Let $\omega(i)$ $(1 \le i \le n)$ be i.i.d. random paths satisfying the condition defining* $(\mathrm{M})_m$. *Let us define a new random path $\bar{\omega}$ in $[0, 1]$ by*

$$\bar{\omega}_t := (\omega(i))_{t-(i-1)/n}[n^{-1}]$$

for $t \in [\frac{i-1}{n}, \frac{i}{n})$. Then for any $f \in C_b^\infty(\mathbf{R}^N; \mathbf{R})$, there exists a constant $C = C(m, f)$ such that

$$|E[f(X_1^x)] - E[f(\bar{X}_1^x(d\bar{\omega}))]| \le \frac{C}{n^{(m-1)/2}}.$$

2.3. **Formal series and expansion of SDEs.** When we investigate the short time asymptotics of the map $t \mapsto E[f(\bar{X}_t^x(d\omega[t]))]$, the vector fields $V_0, \ldots, V_d$ are considered just as the coefficients of the series. On the other hand, we notice that the coefficient of t^k is spanned by $\{V_{\alpha_k} \cdots V_{\alpha_1} f; \|\alpha\| = k\}$. Hence it is natural to regard it as formal power series with respect to the variables $V_0, \ldots, V_d$, and to forget the time parameter t with $t = 1$.

To discuss formal power series with variables $V_0, \ldots, V_d$, we use the following notation.

- $A = \{v_0, v_1, \cdots, v_d\}$: Alphabets.
- Powers (words) of $v = (v_0, \ldots, v_d)$:

$$v^\alpha := \begin{cases} v_{\alpha_k} \cdots v_{\alpha_1}, & \alpha = (\alpha_1; \ldots, \alpha_k) \in \{0, 1, \ldots, d\}^k, k \geq 1 \\ 1, & \alpha = \emptyset. \end{cases}$$

- $\mathbf{R}\langle A\rangle$: $\mathbf{R}$-algebra of noncommutative polynomials on A.
- $\mathbf{R}\langle\langle A\rangle\rangle$: $\mathbf{R}$-algebra of noncommutative formal power series on A with product topology. We regard $\mathbf{R}\langle\langle A\rangle\rangle$ as the space of $\mathbf{R}$-valued functions defined on all powers of A.
- J_m : The projection from $\mathbf{R}\langle\langle A\rangle\rangle$ to polynomials of degree less than or equal to m, i.e.

$$J_m(x) := \sum_{\|\alpha\| \leq m} a_\alpha v^\alpha \quad \text{for} \quad x = \sum_\alpha a_\alpha v^\alpha \in \mathbf{R}\langle\langle A\rangle\rangle, \ (a_\alpha) \subset \mathbf{R}.$$

- $\exp(x) := 1 + \sum_{k=1}^\infty \frac{x^k}{k!}$ for $x = \sum_{\|\alpha\|>0} a_\alpha v^\alpha$ and $(a_\alpha) \subset \mathbf{R}$. We note that this mapping is well-defined since $a_\emptyset = 0$ and so $J_{k-1}(x^k) = 0$ for every k.
- $\Gamma\langle\cdot\rangle$: The linear map from $\mathbf{R}\langle A\rangle$ to differential operators defined by $\Gamma\langle v^\alpha\rangle := V_{\alpha_k} \cdots V_{\alpha_1}$.
- $\Gamma\langle t, \cdot\rangle$: The linear map from $\mathbf{R}\langle A\rangle$ to differential operators defined by $\Gamma\langle t, v^\alpha\rangle := t^{\|\alpha\|} V_{\alpha_k} \cdots V_{\alpha_1}$.

Remark 2.7. In Lyons-Victoir [5], instead of polynomials on A, they consider the expansion with respect to Lie polynomials generated by

$$[v_{i_1}, [v_{i_2}, [\cdots [v_{i_{k-1}}, v_{i_k}] \cdots]]]$$

($[v_i, v_j] := v_i v_j - v_j v_i$). It is shown that the existence of the function in $C_{0,BV}([0,t]; \mathbf{R}^{d+1})$ corresponding to the exponential map $\exp(\mathcal{L})$ with arbitrary Lie polynomial $\mathcal{L}$ [6]. Their approach for constructing cubature formulas consists of two parts: The first is to find a pair of weights (p_i) and Lie polynomials $(\mathcal{L}_i)$ such that $\sum_{i=1}^k p_i J_m(\exp(\mathcal{L}_i)) = J_m(\exp(v_0 + \frac{1}{2}\sum_{i=1}^d v_i^2))$. The second is to construct bounded variation functions that come from $\exp(\mathcal{L}_i)$.

Let us define the $\mathbf{R}\langle\langle A\rangle\rangle$-valued SDE:

$$\mathbf{X}_t = 1 + \sum_{i=0}^d \int_0^t \mathbf{X}_s v_i \circ dB_s^i$$

which has the unique solution given by

$$\mathbf{X}_t = 1 + \sum_{\|\alpha\|>0} I(t, \alpha, \circ dB) v^\alpha.$$

The following result is well-known (e.g. [5]).

Proposition 2.8.

$$E[\mathbf{X}_1] = \exp\Big(v_0 + \frac{1}{2}\sum_{i=1}^d v_i^2\Big).$$

[6] Chen's theorem on Wiener space

Remark 2.9. The above result corresponds to the expansion (for $f \in C_b^\infty$)

$$E[f(X_t^x)] = f(x) + \sum_{j=1}^{k} \frac{t^j}{j!}\Big(V_0 + \frac{1}{2}\sum_{i=1}^{d} V_i^2\Big)^j f(x) + O(t^{k+1})$$
$$= \Gamma\Big\langle t, J_k\Big(\exp\Big(v_i + \frac{1}{2}\sum_{i=1}^{d} v_i^2\Big)\Big)\Big\rangle f(x) + O(t^{k+1}).$$

That is to say, the operation $f(\cdot) \mapsto E[f(X_t)]$ has the structure of the form $\exp(t\mathcal{L})$ where $\mathcal{L}$ is the generator of the Markov process X_t and is given by $\mathcal{L} = V_0 + \frac{1}{2}\sum_{i=1}^{d} V_i^2$.

We next consider the $\mathbf{R}\langle\langle A\rangle\rangle$-valued random ODE defined as

$$\bar{\mathbf{X}}_t = 1 + \sum_{i=0}^{d} \int_0^t \bar{\mathbf{X}}_s v_i d\omega_s^i. \tag{2.4}$$

The solution is denoted by by $\bar{\mathbf{X}}_t(d\omega)$. As in the case of $\mathbf{X}_t$, we can solve $\bar{\mathbf{X}}_t(d\omega)$ as

$$\bar{\mathbf{X}}_t(d\omega) = 1 + \sum_{\|\alpha\|>0} I(t, \alpha, d\omega)v^\alpha.$$

To be more precise, we define the solution of (2.4) rigorously. Let $g \in C_{0,BV}([0,1]; \mathbf{R}^{d+1})$ and $a \in \mathbf{R}\langle\langle A\rangle\rangle$ be fixed and consider the $\mathbf{R}\langle\langle A\rangle\rangle$-valued linear ODE

$$\bar{\mathbf{Y}}_t = a + \sum_{i=0}^{d} \int_0^t \bar{\mathbf{Y}}_s v_i dg_s^i. \tag{2.5}$$

We say that $(\bar{\mathbf{Y}}_t)_{t\geq 0}$ is a solution of (2.5) if the coefficients of $\bar{\mathbf{Y}}_t$ (as a formal series) are continuous function with respect to t and $\bar{\mathbf{Y}}_t$ satisfies the equation (2.5).

Lemma 2.10. *The equation (2.5) has the unique solution given by*

$$\bar{\mathbf{Y}}_t = a\left(1 + \sum_{\|\alpha\|>0} I(t, \alpha, dg)v^\alpha\right).$$

Proof. We can check that the function $t \mapsto a(1 + \sum_{\|\alpha\|>0} I(t, \alpha, dg)v^\alpha)$ is a solution of (2.5). If (2.5) has another solution $\tilde{\mathbf{Y}}_t$, then using the Taylor expansion we can derive that $J_m(\bar{\mathbf{Y}}_t - \tilde{\mathbf{Y}}_t) = 0$ for every m. Therefore the uniqueness of solutions holds. $\square$

We can define the solution of (2.4) pathwisely by means of the above lemma. By Proposition 2.8, we obtain the equivalent condition for cubature formulas.

Theorem 2.11. *Let ω be a random path. Then we have the followings.*
(i) For each $m \in \mathbf{N}$, ω satisfies (M)$_m$ *if and only if*

$$E[J_m(\bar{\mathbf{X}}_1(d\omega))] = J_m\Big(\exp\Big(v_0 + \frac{1}{2}\sum_{i=1}^{d} v_i^2\Big)\Big).$$

(ii) Assume that ω has finite mass. Then for each $m \in \mathbf{N}$, ω satisfies (Cub)$_m$ *if and only if the above equality holds.*

Proof. Notice that (M)$_m$ holds if and only if

$$E[J_m(\bar{\mathbf{X}}_1(d\omega))] - E[J_m(\mathbf{X}_1)] = \sum_{\|\alpha\|\leq m} \Big(E[I(t, \alpha, d\omega)] - E[I(t, \alpha, \circ dB)]\Big)v^\alpha = 0.$$

By using Proposition 2.8, this condition holds if and only if

$$E[J_m(\bar{\mathbf{X}}_1(d\omega))] - J_m\Big(\exp\Big(v_0 + \frac{1}{2}\sum_{i=1}^d v_i^2\Big)\Big) = 0.$$

$\square$

3. Splitting methods and construction of cubature formulas

We now review the idea of splitting methods (or exponential product formulas) that have been applied to approximations of ODEs, PDEs and more general exponential maps (e.g. [8], [9], [11], [12], [1], [13]). For simplicity, we consider two matrices $A, B \in \mathbf{R}^{k\times k}$ such that $AB \neq BA$. We can easily show by the Taylor expansion

$$\begin{aligned} \exp(tA)\exp(tB) &= \exp(t(A+B)) + O(t^2), \\ \exp(\frac{t}{2}A)\exp(tB)\exp(\frac{t}{2}A) &= \exp(t(A+B)) + O(t^3), \\ \frac{1}{2}\exp(tA)\exp(tB) + \frac{1}{2}\exp(tB)\exp(tA) &= \exp(t(A+B)) + O(t^3). \end{aligned}$$

The above computation and basic ideas are applicable to more general (unbounded) operator A, B. As mentioned in Remark 2.9, our interest is the case where $A, B(, C, D\dots)$ are generators of some Markov processes.

3.1. **Splitting method for $\mathbf{R}\langle\langle A\rangle\rangle$-valued SDEs.** Let us define $(d+1)$ SDEs considered as the splitting of $\mathbf{X}_t$ in each direction of $(B_t^0, B_t^1, \dots, B_t^d)$. For $i = 0, 1, \dots, d$, define the $\mathbf{R}\langle\langle A\rangle\rangle$-valued SDE

$$\mathbf{X}_t^{(i)} = 1 + \int_0^t \mathbf{X}_s^{(i)} v_i \circ dB_s^i.$$

We can immediately solve the above equations.

Lemma 3.1. *(a)*

$$\mathbf{X}_1^{(0)} = \exp(v_0).$$

(b) For $i \geq 1$,

$$\mathbf{X}_1^{(i)} = \exp(B_1^i v_i),$$

$$E[\mathbf{X}_1^{(i)}] = \exp\Big(\frac{v_i^2}{2}\Big).$$

Remark 3.2. The lemma is an algebraic version of the following probabilistic consideration: Let B_t be a one dimensional Brownian motion, W be a $C_b^\infty(\mathbf{R}^N;\mathbf{R}^N)$-vector field, and $\exp(sW)x$ be the solution to the ODE

$$Y_s^x = x + \int_0^s W(Y_r^x)dr.$$

Then using Itô's formula, we can show that $\exp(B_tW)x$ is the solution to the SDE

$$X_t^x = x + \int_0^t W(X_r^x) \circ dB_r.$$

This equation has much better analytical tractability than the original SDE (2.3) which is driven by a multidimensional Brownian motion. In other words, the splitting methods help us to avoid simulations of "Lévy areas" defined as $I^{ij}(t) := \int_0^t\int_0^s \circ dB_r^i \circ dB_s^j - \int_0^t\int_0^s \circ dB_r^j \circ dB_s^i$ for $i \neq j$. It is important to point out that (i) Levy areas naturally appear in the stochastic Taylor expansion via $\int_0^t\int_0^s \circ dB_r^i \circ dB_s^j =$

$\frac{1}{2}(B_t^iB_t^j + I^{ij}(t))$. (ii) The exact distribution of Lévy areas is not known. It is even difficult to know its moments.

We introduce some formulas of splitting methods with degree 5.

Theorem 3.3. [Ninomiya-Victoir scheme]: *For $Z = (Z^1, \ldots, Z^d) \sim N(0, I_d)$,*

$$E\Big[J_5\Big(\frac{1}{2}\big(\exp(v_0/2)\exp(Z^1v_1)\cdots\exp(Z^dv_d)\exp(v_0/2) + \exp(v_0/2)\exp(Z^dv_d)\cdots\exp(Z^1v_1)\exp(v_0/2)\big)\Big)\Big]$$

$$= E\Big[J_5\Big(\frac{1}{2}\big(\exp(v_0)\exp(Z^1v_1)\cdots\exp(Z^dv_d) + \exp(Z^dv_d)\cdots\exp(Z^1v_1)\exp(v_0)\big)\Big)\Big]$$

$$= J_5\Big(\exp\Big(v_0 + \frac{1}{2}\sum_{i=1}^d v_i^2\Big)\Big).$$

[Strang's splitting type scheme]: *For $Z = (Z^1, \ldots, Z^{2d-1}) \sim N(0, I_{2d-1})$,*

$$E\Big[J_5\Big(\exp(v_0/2)\exp(Z^1v_1/\sqrt{2})\cdots\exp(Z^{d-1}v_{d-1}/\sqrt{2})\exp(Z^dv_d)$$
$$\exp(Z^{d+(d-1)}v_{d-1}/\sqrt{2})\cdots\exp(Z^{d+1}v_1/\sqrt{2})\exp(v_0/2)\Big)\Big]$$

$$= J_5\Big(\exp\Big(v_0 + \frac{1}{2}\sum_{i=1}^d v_i^2\Big)\Big).$$

Proof. Using the independence of (Z^i), we can derive

$$E[\exp(Z^iv_i)\exp(Z^jv_j)] = E[\exp(Z^iv_i)]E[\exp(Z^jv_j)] = \exp\Big(\frac{v_i^2}{2}\Big)\exp\Big(\frac{v_j^2}{2}\Big)$$

for $i, j \geq 1$, $i \neq j$. Therefore we can obtain the desired results from formal computation of the Taylor series for exponential maps such as we have seen in previous for the matrices A, B. □

Remark 3.4. The weight "$\frac{1}{2}$" corresponds to the probability weight of a Bernoulli random variable independent of Z (recall Example 2.4).

Another formula is given by Ninomiya and Ninomiya ([6]). They forcus on the number of solving or approximating ODEs. The proof differs from Theorem 3.3 due to the lack of independence.

Theorem 3.5. [Ninomiya-Ninomiya scheme]: *For $Z = (Z^1, \ldots, Z^{2d}) \sim N(0, I_{2d})$,*

$$E\Big[J_5\Big(\exp\Big(\frac{v_0}{2} + \sum_{i=1}^d\Big(\frac{1}{2}Z^i + \frac{1}{\sqrt{2}}Z^{d+i}\Big)v_i\Big)\exp\Big(\frac{v_0}{2} + \sum_{i=1}^d\Big(\frac{1}{2}Z^i - \frac{1}{\sqrt{2}}Z^{d+i}\Big)v_i\Big)\Big)\Big]$$

$$= J_5\Big(\exp\Big(v_0 + \frac{1}{2}\sum_{i=1}^d v_i^2\Big)\Big).$$

Proof. As in the proof of Theorem 3.3, it follows from the computation of the moments of correlated Gaussian random variables with degree 2 and 4. □

3.2. **Construction of paths of bounded variation.** We give here a hint for construction of moment matching or cubature formulas of degree 5.

Lemma 3.6. *Let $Z = (Z^0, \ldots, Z^d)$ be a random variable and for $0 \leq i \leq d$,*

$$d\omega_t^i := Z^i dt.$$

Then

$$\bar{\mathbf{X}}_1(d\omega) = \exp\Big(\sum_{i=0}^{d} Z^i v_i\Big).$$

Proof. The result follows from

$$\bar{\mathbf{X}}_1(d\omega) = 1 + \sum_{j=1}^{\infty}\Big(\sum_{i=0}^{d} Z^i v_i\Big)^j \int_{0<t_j<\cdots<t_1<1} dt_j \cdots dt_1 = 1 + \sum_{j=1}^{\infty}\Big(\sum_{i=0}^{d} Z^i v_i\Big)^j \frac{1}{j!}.$$

□

The above lemma shows the relationship between the exponential $\exp(\sum_{i=0}^{d} Z^i v_i)$ and the random ODE $\bar{\mathbf{X}}_1(d\omega)$. We now extend this lemma to discuss more general compositions of the exponential maps including Ninomiya-Ninomiya type schemes.

Theorem 3.7. (1) *Let $\ell \in \mathbf{N}$ be fixed and $Z = (Z^{ij})_{0\le i\le d, 1\le j\le \ell}$ be a $\mathbf{R}^{(d+1)\ell}$-valued random variable. Let us define for each $1 \le j \le \ell$*

$$d\omega_t^i := \ell Z^{ij} dt, \quad t \in \Big[\frac{j-1}{\ell}, \frac{j}{\ell}\Big). \tag{3.1}$$

Then we have

$$\bar{\mathbf{X}}_1(d\omega) = \exp\Big(\sum_{i=0}^{d} Z^{i1} v_i\Big) \cdots \exp\Big(\sum_{i=0}^{d} Z^{i\ell} v_i\Big).$$

(2) *Let $(Z^{0j})_{1\le j\le \ell}$ be non-negative constants such that $\sum_{j=1}^{\ell} Z^{0j} = 1$, and $(Z^{ij})_{1\le i\le d, 1\le j\le \ell}$ be Gaussian random variables (which need not to be independent). Assume that ω defined in (3.1) satisfies* $(\mathrm{M})_m$, *and $\bar{Z} = (\bar{Z}^{ij})_{1\le i\le d, 1\le j\le \ell}$ is a discrete $\mathbf{R}^{d\ell}$-valued random variable with probabilities $(p_l)_{1\le l\le L}$ so that*

$$E[\mathcal{P}((Z^{ij})_{1\le i\le d, 1\le j\le \ell})] = \sum_{l=1}^{L} p_l \mathcal{P}((\bar{Z}^{ij}(l))_{1\le i\le d, 1\le j\le \ell})$$

for any polynomial $\mathcal{P}$ on $\mathbf{R}^{d\ell}$ with degree less than or equal to m. Then a random path $\bar{\omega}$ given by

$$d\bar{\omega}_t^0 := \ell Z^{0j} dt, \quad t \in \Big[\frac{j-1}{\ell}, \frac{j}{\ell}\Big),$$
$$d\bar{\omega}_t^i := \ell \bar{Z}^{ij} dt, \quad t \in \Big[\frac{j-1}{\ell}, \frac{j}{\ell}\Big), i \ge 1$$

defines a cubature formula with degree m.

Proof. The result (1) is obtained from Lemma 3.6 and the uniqueness of solutions of $\bar{\mathbf{X}}_t(d\omega)$ (Lemma 2.10). Indeed, for $t \in [1/\ell, 2/\ell)$,

$$\begin{aligned}
&\exp\Big(\sum_{i=0}^{d} Z^{i1} v_i\Big) \exp\Big((t - 1/\ell)\sum_{i=0}^{d} Z^{i2} v_i\Big)\\
&= \bar{\mathbf{X}}_{1/\ell}\Big(1 + \sum_{i=0}^{d} \int_{1/\ell}^{t} \exp\Big((s - 1/\ell)\sum_{i=0}^{d} Z^{i2} v_i\Big) v_i d\omega_s^i\Big)\\
&= 1 + \sum_{i=0}^{d}\Big(\int_0^{1/\ell} \exp\Big(s\sum_{i=0}^{d} Z^{i1} v_i\Big) v_i d\omega_s^i + \int_{1/\ell}^{t} \exp\Big(\sum_{i=0}^{d} Z^{i1} v_i\Big) \exp\Big((s - 1/\ell)\sum_{i=0}^{d} Z^{i2} v_i\Big) v_i d\omega_s^i\Big).
\end{aligned}$$

This implies $\bar{\mathbf{X}}_{2/\ell}(d\omega) = \exp\Big(\sum_{i=0}^{d} Z^{i1} v_i\Big) \exp\Big(\sum_{i=0}^{d} Z^{i2} v_i\Big)$. We obtain the result for $\bar{\mathbf{X}}_1(d\omega)$ by induction.

Through the representation via exponential maps, we notice that the conditions $(\mathrm{M})_m$ and $(\mathrm{Cub})_m$ depend only on the polynomials of Z with degree less than or equal to m. Therefore the assertion (2) immediately follows. □

Remark 3.8. Theorem 3.7 lifts the original problem of cubature formula essentially in infinite dimension down the finite dimensional problem of Gaussian measure.

Remark 3.9. The condition for the covariance of Z so that (3.1) satisfies $(\mathrm{M})_m$ has been given in [6]. However nobody has shown the existence of such a covariance with $m \geq 7$.

Remark 3.10. Let a path of ω in Theorem 3.7 be fixed. Then the random ODE $\bar{X}_t(d\omega)$ becomes an ODE which has piecewise random coefficients. We can apply the Runge-Kutta method for the ODE in each interval ([6], [13]).

Acknowledgement

The author is supported by JSPS Research Fellowships for Young Scientists.

References

[1] I. Gyöngy, N.V. Krylov, Expansion of solutions of parameterized equations and acceleration of numerical methods, *Illinois J. Math.*, **50**, No. 2, Special Volume in Memory of Joseph Doob (2006) 473-514.

[2] K. Itô, Multiple Wiener integral, *Journal of the Mathematical Society of Japan*, **3** (1951) 157-169.

[3] S. Kusuoka, Malliavin calculus and numerical analysis (in Japanese), lecture note, Probability Summer School, 2005.

[4] S. Kusuoka, Gaussian K-scheme, *UTMS Preprint Series*, 2009.

[5] T. Lyons, N. Victoir, Cubature on Wiener space, *Proc. R. Soc. Lond. Ser. A Math.* **460** (2004) 169-198.

[6] M. Ninomiya, S. Ninomiya, A new weak approximation scheme of stochastic differential equations by using the Runge-Kutta method, *Finance Stoch.* **13** (2009) 415-443.

[7] S. Ninomiya, N. Victoir, Weak approximation of stochastic differential equations and application to derivative pricing, *Appl. Math. Finance* **15** (2008) 107-121.

[8] G. Strang, Accurate partial difference methods I: linear Cauchy problems, *Arch. Rat. Mech. Anal.* **12** (1963) 392-402.

[9] G. Strang, On the construction and comparison of difference schemes, *SIAM J. Numer. Anal.* **5** (1968) 506-517.

[10] A.H. Stroud, *Approximate calculation of multiple integrals*, Prentice-Hall, 1971.

[11] M. Suzuki, Fractal decomposition of exponential operators with applications to many-body theories and Monte Carlo simulations, *Phys. Lett. A* **146** (1990) 319-323.

[12] M. Suzuki, General theory of higher-order decomposition of exponential operators and symplectic integrators, *Phys. Lett. A* **165** (1992) 387-395.

[13] H. Tanaka, A. Kohatsu-Higa, An operator approach for Markov chain weak approximations with an application to infinite activity Lévy driven SDEs, *Ann. Appl. Probab.* **19** (2009) 1026-1062.

Graduate School of Science and Engineering, Ritsumeikan University, 1-1-1 Nojihigashi, Kusatsu, Shiga 525-8577, Japan

E-mail address: `hide.worldwide@gmail.com`