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閉曲面上のグラフにおける matching extension について

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1 Introduction

We consider only simple graphs without loops or multiple edges. A set M of edges in a graph is said to be a matching if no two members of M share a vertex. A perfect matching of a graph G is a matching of G which covers all the vertices of G. If a matching M of G is a subset of a perfect matching in G, then M is said to be extendable in G, and a graph with at least $2m+2$ vertices in which every matching of size m is extendable is called m-extendable. For a matching with at most two edges, Plummer [10] proved the following.

Theorem 1. *Every* 5*-connected planar graph with an even order is* 2*-extendable.*

To show the similar result for any graph G on some other surface F^2, Kawarabayashi et al. added a condition on the representativity $\rho(G)$ (the minimum number r such that any non-contractible simple closed curve on F^2 meets G in at least r places), and proved the following.

Theorem 2 ([7]). *Every 5-connected graph G with an even order embedded on a closed surface F^2, except the sphere, is* 2*-extendable if $\rho(G) \geq 7-2\chi(F^2)$, where $\chi(F^2)$ is the Euler characteristic of F^2.*

Moreover, they proved that there are infinitely many 5-connected triangulations G on F^2 with $\rho(G)=3$ that are not 2-extendable, for any closed surface F^2 except the sphere and the projective plane. Therefore, the condition on the representativity is necessary in Theorem 2 when F^2 is not the projective plane.

For a matching with 3 or more edges, it is shown that no planar graph with at least 8 vertices is 3-extendable ([9]). Later, Aldred and Plummer ([1], [3], [4]) studied the distance between the

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edges in the matchings which are not extendable. A graph G with at least $2m+2$ edges is said to be distance d m-extendable if any matching M of G with m edges in which the edges lie pair-wise distance at least d is extendable.

Theorem 3 ([1]). *Every 5-connected triangulation on the plane with an even order is distance 2 3-extendable.*

It follows from Theorem 3 that, in any 5-connected triangulation on the plane with an even order, any non-extendable matching with three edges contains a pair of edges with distance 1.

It is shown in [3] that the conclusion of Theorem 3 can not be extended to "distance 2 4-extendable". However, if the pair-wise distance of the given matching is increased, then we can extend 4 or more edges.

Theorem 4 ([3], [4]). *Every 5-connected triangulation on the plane with an even order is distance 3 4-extendable. Moreover, there exist infinitely many 5-connected triangulations on the plane with an even order which are not distance 3 10-extendable.*

Theorem 5 ([4]). *Every 5-connected triangulation on the plane with an even order is distance 4 7-extendable.*

Note that, in distance two or three case, there exist a maximum value on the number of edges to extend. Though we do not know whether such a maximum exists or not in distance four case, it disappears in distance five case.

Theorem 6 ([4]). *Every 5-connected triangulation on the plane with an even order is distance 5 m-extendable for any m.*

It is also shown in [4] that the same thing as Theorem 6 holds for projective planar graphs. For the graphs on the toroidal or Klein bottle, it follows from Mizukai et al.'s result [8] that every 5-connected triangulation on the torus or the Klein bottle with an even order is distance 3 2-extendable for any m. This result is sharp in the following sense.

Proposition 7 (Aldred and Plummer, [5]). *There are 5-connected triangulations of both the torus and the Klein bottle with an even order which are not distance 3 3-extendable.*

In the above results, we always find a difference between planar case and the toroidal or the Klein bottle case: when we extend Theorem 1 to Theorem 2, the additional condition on representativity is necessary, and in Proposition 7, the number of the edges to extend is deduced compared to the planar case. In contrast to these situations, an extension of Theorem 6 is obtained without adding any extra-condition.

Theorem 8. *Let G be a 5-connected triangulation on the torus or the Klein bottle with an even order. If $m \geq 0$ and G has at least $2m+2$ vertices, then G is distance 5 m-extendable.*

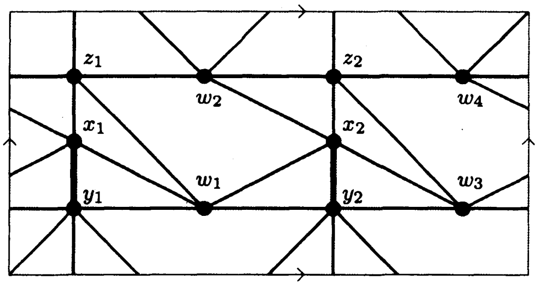

Figure 1: the graph G_0.

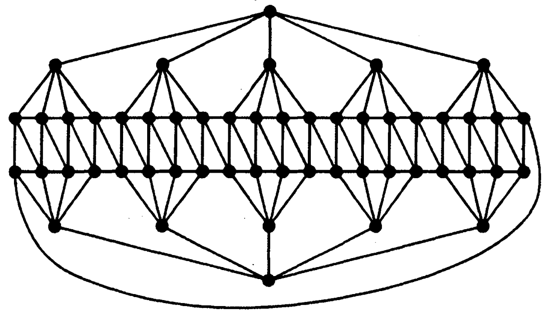

Figure 2: the graph H_0.

It is shown in [5] that there exist 4-connected triangulations of both the torus and the Klein bottle with an even order which are not 1-extendable, and so the condition of the connectivity in Theorem 8 cannot be relaxed. Next we show that the condition "triangulation" cannot be dropped in Theorem 8. Let G_0 and H_0 be the graphs shown in Figures 1 and 2, respectively. Let G be the graph obtained from $G_0 - \{w_1, w_2, w_3, w_4\}$ by adding 4 copies H_1, H_2, H_3, H_4 of $H_0 - \{v\}$ and joining five vertices of degree 4 in H_i and the five neighbors of w_i in G_0 by a perfect matching for $i = 1, 2, 3, 4$ so that the resulting graph is toroidal. Then, since both of G_0 and H_0 are 5-connected, G is also 5-connected. Moreover, since $G - \{x_1, y_1, z_1, x_2, y_2, z_2\}$ has 4 odd components, $\{x_1y_1, x_2y_2\}$ is not extendable in G. Thus G is not distance 5 m-extendable, though G is a 5-connected toroidal graph with an even order. Note that we obtain larger example by recursively exchanging a vertex in $V(G) \setminus \{x_1, y_1, z_1, x_2, y_2, z_2\}$ for the graph $H_0 - \{v\}$ and adding 5 edges as we done in the above. Moreover, notice that the G can be embedded on the Klein bottle (such an embedding is obtained by changing the order of the edges between $\{w_3, w_4\}$ and $\{x_1, y_1, z_1\}$ and changing the direction of the right border in Figure 1).

Though Theorem 8 is best possible in the above senses, it remains to be studied whether we can exchange "distance 5" with "distance 4" in Theorem 8 or not.

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