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Lower bound of the lifespan of solutions to nonlinear elastic wave equation

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1. Introduction

In this paper we consider the Cauchy problem for homogeneous, isotropic, hyperelastic wave equations:

$$(\partial_t^2 - L)u(t,x) = F(\nabla u, \nabla^2 u), \quad (t,x) \in (0,T) \times \mathbf{R}^3, \tag{1.1}$$

$$u(0,x) = \varepsilon f(x),\ (\partial_t u)(0,x) = \varepsilon g(x), \quad x \in \mathbf{R}^3, \tag{1.2}$$

where $u(t,x) = {}^t(u_1(t,x), u_2(t,x), u_3(t,x))$ is the displacement vector from the configuration, $\nabla u = (\partial_1 u, \partial_2 u, \partial_3 u)$, $\partial_j = \partial/\partial x_j$ $(j = 1,2,3)$, and

$$L = c_2^2 \Delta + (c_1^2 - c_2^2)\,\mathrm{grad}\,\mathrm{div}, \quad \Delta = \mathrm{div}\,\mathrm{grad}$$

with material constants c_1, c_2 satisfying $0 < c_2 < c_1$. Here grad and div stand for the spatial gradient and divergence, respectively. Besides, f, g are smooth functions with compact support and ε is a positive parameter. In addition, the nonlinearity is expressed as

$$\begin{aligned} F(\nabla u, \nabla^2 u) =& A_1\,\mathrm{grad}\,(\mathrm{div} u)^2 + A_2\,\mathrm{grad}\,|\mathrm{rot} u|^2 \\ &+ A_3\,\mathrm{rot}\,((\mathrm{div} u)(\mathrm{rot} u)) + N(u). \end{aligned} \tag{1.3}$$

Here, A_1, A_2 and A_3 are real constants and each components of $N(u)$ is a linear combination of the so-called null-forms. (for the detail, see Appendix below; also [1]).

We denote the lifespan of the problem (1.1)-(1.2) by T_ε which is the supremum of all $T > 0$ such that the problem admits a unique smooth solution in $[0,T) \times \mathbf{R}^3$. In John [10] the lower bound for the lifespan $T_\varepsilon \geq e^{C/\varepsilon}$ with a positive number C was obtained for sufficiently small ε (see also [13]). Moreover, if $A_1 = 0$, then the global solvability of the problem for sufficiently small initial data was proved by Agemi [1] and Sideris [14], independently.

On the other hand, concerning the Cauchy problem for scalar wave equations:

$$(\partial_t^2 - \Delta)v(t,x) = \sum_{j,k,l=0}^{3} g_{jkl}(\partial_j v)(\partial_k \partial_l v), \quad (t,x) \in (0,T) \times \mathbf{R}^3, \tag{1.4}$$

$$v(0,x) = \varepsilon\phi(x),\ (\partial_t v)(0,x) = \varepsilon\psi(x), \quad x \in \mathbf{R}^3, \tag{1.5}$$

not only the estimate of the lifespan $\widetilde{T}_\varepsilon$ of this problem from below but also much precise information of $\widetilde{T}_\varepsilon$ are known (here, g_{jkl} are real constants and ϕ, $\psi \in C_0^\infty(\mathbf{R}^3)$). More explicitly, it was independently shown by Hörmander [5] and John [9] that

(1.6)
$$\liminf_{\varepsilon\to+0} \varepsilon \log \widetilde{T}_\varepsilon \geq \Big(\max\{-2^{-1}G(\theta)\,\partial_s^2\widetilde{\mathcal{R}}[\phi,\psi](s,\theta)\,;\ s\in\mathbf{R},\ \theta\in S^2\}\Big)^{-1},$$

provided the right-hand side is a finite number. Here, the functions G and $\widetilde{\mathcal{R}}[\phi,\psi]$ are defined by

$$G(\theta)=\sum_{j,k,l=0}^{3} g_{jkl}\,\theta_j\theta_k\theta_l \quad \text{with } \theta_0=-1,\ (\theta_1,\theta_2,\theta_3)\in S^2,$$

$$\widetilde{\mathcal{R}}[\phi,\psi](s,\theta)=\frac{1}{4\pi}(\mathcal{R}[\psi](s,\theta)-\partial_s\mathcal{R}[\phi](s,\theta)),\quad (s,\theta)\in\mathbf{R}\times S^2,$$

where $\mathcal{R}[\phi]$ is the Radon transform of ϕ, that is,

$$\mathcal{R}[\phi](s,\theta)=\int_{\theta\cdot y=s}\phi(y)\,dS_y,\quad (s,\theta)\in\mathbf{R}\times S^2. \tag{1.7}$$

The counter part of the estimate (1.6) has been studied by Alinhac [2]. We remark that $G\equiv 0$ on S^2 is equivalent to the null condition introduced by Klainerman [12], and the condition implies $\widetilde{T}_\varepsilon=+\infty$ (see also [3]). While, $\widetilde{\mathcal{R}}[\phi,\psi]\equiv 0$ on $\mathbf{R}\times S^2$ is equivalent to $\phi\equiv\psi\equiv 0$ on $\mathbf{R}^3$.

Therefore, a natural question is if it is possible to derive an analogous estimate to (1.6) for the lifespan T_ε of the problem (1.1)-(1.2) or not. The difficulty for dealing with the elastic wave equation (1.1) comes from the fact that the equation has two distinct propagation speeds. For this, the hyperbolic boosts $x_j\partial_t+t\partial_j$ do not work well, and construction of a nonlinear approximate solution is not straightforward as in the case of the wave equation. Nevertheless, by using a higher order approximation (see (5.36) below) together with careful treatments of the decay factor $(1+|c_it-|x|\,|)^{-1}$, we are able to overcome the difficulty.

In order to state our result, we define

$$\widetilde{\mathcal{R}}_i[f,g](s,\theta)=\frac{1}{4\pi}\left(c_i^{-1}\mathcal{R}[g](s,\theta)-\partial_s\mathcal{R}[f](s,\theta)\right)\quad (i=1,2) \tag{1.8}$$

for $(s,\theta)\in\mathbf{R}\times S^2$, where the Radon transform $\mathcal{R}[f]$ of $f={}^t(f_1,f_2,f_3)\in(C_0^\infty(\mathbf{R}^3))^3$ is given by $\mathcal{R}[f]={}^t(\mathcal{R}[f_1],\mathcal{R}[f_2],\mathcal{R}[f_3])$. We note that $\widetilde{\mathcal{R}}_i[f,g]$ is bounded on $\mathbf{R}\times S^2$ and compactly supported in s for f,

$g \in (C_0^\infty(\mathbf{R}^3))^3$. In particular, if

$$p_0(s,\theta) := \theta \cdot \widetilde{\mathcal{R}}_1[f,g](s,\theta) \tag{1.9}$$

is not identically zero on $\mathbf{R} \times S^2$, then $\partial_s^2 p_0(s,\theta)$ takes both positive and negative values. Therefore, one can define a positive number

$$\tau_* = \left(\max\{-c_1^{-2} A_1\, \partial_s^2 p_0(s,\theta)\,;\; s \in \mathbf{R},\ \theta \in S^2\}\right)^{-1}, \tag{1.10}$$

provided $A_1 \neq 0$ and $p_0 \not\equiv 0$ on $\mathbf{R} \times S^2$.

Then, our main result is the following:

Theorem 1.1. *Let f, $g \in (C_0^\infty(\mathbf{R}^3))^3$. If $A_1 \neq 0$ and $p_0 \not\equiv 0$ on $\mathbf{R} \times S^2$, then we have*

$$\liminf_{\varepsilon \to +0} \varepsilon \log T_\varepsilon \geq \tau_*. \tag{1.11}$$

Remark 1.2. *(i) Unfortunately, we do not have the estimate in the opposite direction to* (1.11)*, that is to say*

$$\limsup_{\varepsilon \to +0} \varepsilon \log T_\varepsilon \leq \tau_* \tag{1.12}$$

in general. But, when the initial data take the following form:

$$f(x) = \phi(r)x, \quad g(x) = \psi(r)x, \quad x \in \mathbf{R}^3,$$

(1.12) *was shown by John* [8], *provided $A_1 \neq 0$ and the corresponding p_0 does not identically vanish on $\mathbf{R} \times S^2$. Hence, the lower bound* (1.11) *seems to be optimal.*

(ii) The number τ_* is related to the lifespan of the following Cauchy problem for $p = p(s,\theta,\tau)$:

$$2c_1^2\, \partial_\tau p + A_1(\partial_s p)^2 = 0 \quad \text{in } \mathbf{R} \times S^2 \times [0,\tau_*), \tag{1.13}$$

$$p(s,\theta,0) = p_0(s,\theta) \quad \text{for } (s,\theta) \in \mathbf{R} \times S^2. \tag{1.14}$$

Indeed, it is known that the solution to the above problem uniquely exists in $\mathbf{R} \times S^2 \times [0,\tau_*)$ (for the proof, see Lemma 6.5.4 with $G(\omega) \equiv 2A_1/c_1^2$ in [6]*).*

This paper is organized as follows. In the next section we gather notation. In Section 3 we give some preliminaries. Baisc results on the linear elastic wave equation are introdued in Section 4. An approximate solution is constructed in Section 5, and useful estimates for the approximation are established in Proposition 5.5. Outline of the proof of Theorem 1.1 is given in Section 6. In Appendix a way to deduce (1.1) is discussed.

2. NOTATION

In this section, we introduce notation which will be used throughout this paper. We denote $r = |x|$ and $\omega = x/r$. We set $\partial_r = \sum_{j=1}^{3}(x_j/r)\partial_j$ and $O = {}^t(O_1, O_2, O_3) = x \wedge {}^t(\partial_1, \partial_2, \partial_3)$, where $\wedge$ stands for the outer product in $\mathbf{R}^3$. Then we have

$$
{}^t(\partial_1, \partial_2, \partial_3) = \omega\partial_r - r^{-1}\omega \wedge O. \tag{2.1}
$$

We denote $Z = \{Z_0, Z_1, \ldots, Z_6\} = \{\partial_t, \partial_1, \partial_2, \partial_3, O_1, O_2, O_3\}$. We write Z^α for $Z_0^{\alpha_0} \cdots Z_6^{\alpha_6}$ with a multi-index $\alpha = (\alpha_0, \ldots, \alpha_6)$. Note that we have $[Z_a, \partial_t^2 - \Delta] = 0$ $(a = 0, \ldots, 6)$, where we have set $[A, B] = AB - BA$.

We also use $\widetilde{Z} = \{\widetilde{Z}_0, \widetilde{Z}_1, \ldots, \widetilde{Z}_6\} = \{\partial_t I, \partial_1 I, \partial_2 I, \partial_3 I, \widetilde{O}_1, \widetilde{O}_2, \widetilde{O}_3\}$ for $\mathbf{R}^3$-valued functions, where I is the 3×3 identity matrix and

$$
\widetilde{O}_j = O_j I + U_j \quad (j = 1, 2, 3) \tag{2.2}
$$

with

$$
U_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad U_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad U_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
$$

The vector fields $\widetilde{O}_j$ is closely related to the fact that if $u(t, x)$ solves (1.1), then so does $A^{-1}u(t, Ax)$ for any orthogonal matix A. This observation leads to the good algebraic relations $[\widetilde{Z}_j, L] = 0$ for $a = 0, \ldots, 6$. We write $\widetilde{Z}^\alpha$ for $\widetilde{Z}_0^{\alpha_0} \cdots \widetilde{Z}_6^{\alpha_6}$ with a multi-index $\alpha = (\alpha_0, \ldots, \alpha_6)$.

For functions of $(s, \theta, \tau) \in \mathbf{R} \times S^2 \times [0, \infty)$, we denote the differentiation with respect to s, θ and τ by

$$
\Lambda_0 = \partial_s, \quad \Lambda_1 = o_1, \quad \Lambda_2 = o_2, \quad \Lambda_3 = o_3, \quad \Lambda_4 = \partial_\tau, \tag{2.3}
$$

where differential operators o_i on S^2 are (formally) defined by ${}^t(o_1, o_2, o_3) = \theta \wedge {}^t(\partial_{\theta_1}, \partial_{\theta_2}, \partial_{\theta_3})$. We write Λ^β for $\Lambda_0^{\beta_0} \cdots \Lambda_4^{\beta_4}$ and $\Lambda_*^\gamma = \Lambda_0^{\gamma_0} \cdots \Lambda_3^{\gamma_3}$ with multi-indeceis $\beta = (\beta_0, \ldots, \beta_4)$ and $\gamma = (\gamma_0, \ldots, \gamma_3)$.

For a non-negative integer k, and a real-valued smooth function $\varphi(t, x)$, we define

$$
|\varphi(t, x)|_k = \sum_{|\alpha| \le k} |(Z^\alpha \varphi)(t, x)|, \quad |\partial\varphi(t, x)|_k = \sum_{|\alpha| \le k} \sum_{a=0}^{3} |(Z^\alpha \partial_a \varphi)(t, x)|
$$

For a $\mathbf{R}^3$-valued function $u(t, x)$, we use the same notation $|u(t, x)|_k$ and $|\partial u(t, x)|_k$ with Z replaced by $\widetilde{Z}$.

For $\nu \geq 0$, a non-negative integer k, and $\phi \in \mathcal{S}(\mathbf{R}^3)$, we define

$$\|\phi\|_{k,\nu} = \left(\sup_{x \in \mathbf{R}^3} \sum_{|\alpha| \leq k} (1+|x|^2)^{\nu} \, |\partial_x^{\alpha} \phi(x)|^2 \right)^{1/2} .$$

Here, $\mathcal{S}(\mathbf{R}^3)$ is the Schwartz class, the set of rapidly decreasing real-valued functions. Besides, for $f,\ g \in \left(\mathcal{S}(\mathbf{R}^3)\right)^3$, we set

$$\mathcal{A}_{k,\nu}[f,g] = \sum_{j=1}^{3} (\|f_j\|_{k+1,\nu} + \|g_j\|_{k,\nu}) \,. \tag{2.4}$$

As usual, various positive constants which may change line by line are denoted just by the same letter C throughout this paper.

3. Preliminaries

First we recall basic properties of the Radon transform discussed in the section 4 of [11] for the case of $n = 3$ and $\chi \equiv 1$ (note that when $\chi \equiv 1$, $\mathcal{S}_{\chi}(\mathbf{R}^3)$ and $\|\varphi\|_{\chi,k,\nu}$ in [11] become to $\mathcal{S}(\mathbf{R}^3)$ and $\|\varphi\|_{k,\nu}$, respectively). It holds that

$$\partial_s \mathcal{R}[\varphi](s,\theta) = \mathcal{R}[(\theta \cdot \mathrm{grad})\varphi](s,\theta), \tag{3.1}$$

$$o_i \mathcal{R}[\varphi](s,\theta) = \mathcal{R}[O_i \varphi](s,\theta), \quad i = 1,2,3, \tag{3.2}$$

$$\mathcal{R}[\partial_i \varphi](s,\theta) = \theta_i \, \partial_s \mathcal{R}[\varphi](s,\theta), \quad i = 1,2,3 \tag{3.3}$$

for a real-valued function $\varphi \in \mathcal{S}(\mathbf{R}^3)$. Moreover, for $\nu \geq 0$, a nonnegative integer k, and a multi-indix α, we have

$$\left|\partial_s^k o^{\alpha} \mathcal{R}[\varphi](s,\theta)\right| \leq C \|\varphi\|_{k+|\alpha|,\nu+3+|\alpha|} (1+s^2)^{-\frac{\nu}{2}} \tag{3.4}$$

for $(s,\theta) \in \mathbf{R} \times S^2$. Here $C = C(k,\nu,\alpha)$ is a positive constant.

Next we define

$$Q_{\gamma}[\varphi](t,x) = \frac{1}{4\pi} \int_{\theta \in S^2} \theta^{\gamma} \varphi(x + t\theta) dS'_{\theta}, \quad (t,x) \in (0,\infty) \times \mathbf{R}^3 \tag{3.5}$$

for a multi-index $\gamma = (\gamma_1, \gamma_2, \gamma_3)$, a real-valued function $\varphi \in \mathcal{S}(\mathbf{R}^3)$. Here, dS'_{θ} is the area element on S^2. Note that $Q_0[\varphi]$ is the *spherical mean* of φ. We shall derive decay property of $Q_{\gamma}[\varphi]$.

Proposition 3.1. *Let k be a nonnegative integer, $\nu > 0$, and γ be a muti-index. Then there exists a positive constant C such that we have*

$$\left|\partial_t^k Q_{\gamma}[\varphi](t,x)\right| \leq C \|\varphi\|_{k,\nu+2} (1+t+r)^{-2} (1+|r-t|)^{-\nu} \tag{3.6}$$

for $(t,x) \in (0,\infty) \times \mathbf{R}^3$ with $r = |x|$, provided that $\varphi \in \mathcal{S}(\mathbf{R}^3)$.

Proof. It follows that

$$\partial_t^k Q_\gamma[\varphi](t,x) = \sum_{|\alpha|=k} \frac{1}{4\pi} \int_{\theta\in S^2} c_\alpha\, \theta^{\gamma+\alpha} (\partial_x^\alpha \varphi)(x+t\theta) dS'_\theta \tag{3.7}$$

with some approriate constants c_α. Therefore, we get

$$\begin{aligned} \left|\partial_t^k Q_\gamma[\varphi](t,x)\right| &\le C\|\varphi\|_{k,\nu+2} \int_{\theta\in S^2} (1+|x+t\theta|)^{-\nu-2} dS'_\theta \\ &= C\|\varphi\|_{k,\nu+2} \times \frac{2\pi}{tr} \int_{|t-r|}^{t+r} \lambda(1+\lambda)^{-\nu-2} d\lambda \end{aligned}$$

Hence, the desired estimate follows from

$$\frac{1}{tr} \int_{|t-r|}^{t+r} \lambda(1+\lambda)^{-\nu-2} d\lambda \le C(1+t+r)^{-2}(1+|r-t|)^{-\nu} \tag{3.8}$$

for $t, r > 0$. By symmetry, it suffices to show (3.8) for $0 < r \le t$.

First suppose $0 < r \le t < 1$. Then the desire estimate follows from

$$\frac{1}{tr} \int_{|t-r|}^{t+r} \lambda(1+\lambda)^{-\nu-2} d\lambda \le \frac{1}{tr} \int_{|t-r|}^{t+r} \lambda d\lambda = 2.$$

Next suppose $t \ge 1$ and $0 < r \le t$. Since $t \ge (t+r+1)/3$, we get

$$\frac{1}{tr} \int_{|t-r|}^{t+r} \lambda(1+\lambda)^{-\nu-2} d\lambda \le \frac{3}{(1+t+r)r} \int_{|t-r|}^{t+r} (1+\lambda)^{-\nu-1} d\lambda.$$

Observing that $t - r \ge (t+r)/3$ for $t \ge 2r$ and that $r \ge (t+r)/3$ for $t \le 2r$, we obtain (3.8). This completes the proof. □

The following proposition shows that the leading term of $Q_\gamma[\varphi]$ is described by the Radon transform. Since the proof of the proposition is similar to that of Lemma 4.3 in [11], we omit it.

Proposition 3.2. *Let k be a nonnegative integer, $\nu \ge 0$, γ be a muti-index, and $c_* \ge 1$. Then there exist a positive constant C and an integer $N_0\,(\ge \nu+4)$ such that we have*

$$\begin{aligned} &\left|t\partial_t^k Q_\gamma[\varphi](t,x) - (4\pi r)^{-1}(-\omega)^\gamma \left((-\partial_s)^k \mathcal{R}[\varphi]\right)(r-t,\omega)\right| \\ &\qquad \le C\|\varphi\|_{k+1,N_0}(1+t+r)^{-2}(1+|r-t|)^{-\nu} \end{aligned} \tag{3.9}$$

for $(t,x) \in (0,\infty)\times \mathbf{R}^3$ satisfying $r \ge t/(2c_*) \ge 1$ with $r = |x|$ and $\omega = x|x|^{-1}$, provided that $\varphi \in \mathcal{S}(\mathbf{R}^3)$.

Next we derive a couple of estimates of the following integral operator for the latter sake:

$$T_\gamma[\varphi](t,x) = \int_{c_2 t}^{c_1 t} \tau^{-1} Q_\gamma[\varphi](\tau,x) d\tau, \quad (t,x) \in (0,\infty)\times \mathbf{R}^3. \tag{3.10}$$

Proposition 3.3. *Let k be a nonnegative integer, $\nu > 0$, γ be a muti-index, and $\varphi \in \mathcal{S}(\mathbf{R}^3)$. When $(t,x) \in (0,\infty) \times \mathbf{R}^3$ satisfies one of $r > 2c_1 t$, $r < c_2 t/2$ or $0 < t + r \le 1$, we have*

$$|T_\gamma[\varphi](t,x)| \le C\|\varphi\|_{0,\nu+2}\,(1+t+r)^{-2-\nu}. \tag{3.11}$$

While, when $(t,x) \in (0,\infty)\times\mathbf{R}^3$ satisfies $c_2t/2 < r < 2c_1t$ and $t+r \ge 1$, we have

$$|T_\gamma[\varphi](t,x)| \le C\|\varphi\|_{0,\nu+2}\,(1+t+r)^{-3}, \tag{3.12}$$

provided $\nu > 1$. Moreover, if $k \ge 1$, then we have

$$\begin{aligned} &\left|\partial_t^k T_\gamma[\varphi](t,x)\right| \\ &\quad \le C\|\varphi\|_{k,\nu+2}\,(1+t)^{-1}(1+t+r)^{-2}\max_{i=1,2}\{(1+|r-c_it|)^{-\nu}\} \end{aligned} \tag{3.13}$$

for $(t,x) \in (0,\infty) \times \mathbf{R}^3$. Furthermore, we have

$$|T_\gamma[\partial_j\varphi](t,x)| \le C\|\varphi\|_{2,N_0}\,(1+t+r)^{-3}\max_{i=1,2}\{(1+|r-c_it|)^{-1}\}, \tag{3.14}$$

where N_0 is the number from Lemma 3.2.

Proof. First we prove (3.11). By (3.6) we have

$$\begin{aligned} |T_\gamma[\varphi](t,x)| &\le C\|\varphi\|_{0,\nu+2}\int_{c_2t}^{c_1t} \tau^{-1}(1+\tau+r)^{-2}(1+|r-\tau|)^{-\nu}d\tau \\ &\le C\|\varphi\|_{0,\nu+2}\,(1+c_2t+r)^{-2}\int_{c_2t}^{c_1t} \tau^{-1}(1+|r-\tau|)^{-\nu}d\tau. \end{aligned} \tag{3.15}$$

Observe that if $r \le c_2t/2$ and $\tau \ge c_2t$ then $|\tau - r| \ge (c_2t+r)/3$, and that if $r \ge 2c_1t$ and $\tau \le c_1t$, then $|r-\tau| \ge (c_1t+r)/3$. Thus we get (3.11) for $r \le c_2t/2$ or $r \ge 2c_1t$. On the one hand, from (3.15) we have

$$|T_\gamma[\varphi](t,x)| \le C\|\varphi\|_{0,\nu+2}\int_{c_2t}^{c_1t}\tau^{-1}d\tau \le C\|\varphi\|_{0,\nu+2},$$

which yields (3.11) for $0 < t+r \le 1$.

Next we prove (3.12). Since $\tau > C(1+t+r)$ for $\tau > c_2t$, $c_2t/2 < r < 2c_1t$, and $t+r \ge 1$, we get (3.12) from (3.15) by $\nu > 1$.

Next we prove (3.13). It follows from (3.10) that

$$\partial_t T_\gamma[\varphi](t,x) = t^{-1}\left(Q_\gamma[\varphi](c_1t,x) - Q_\gamma[\varphi](c_2t,x)\right). \tag{3.16}$$

When $t \ge 1$, we easily have (3.13) by (3.6). While, when $0 < t < 1$, we rewrite the right-hand side of (3.16) as

$$(c_1-c_2)\int_0^1 \left(\partial_t Q_\gamma[\varphi]\right)(c_1t\sigma + c_2t(1-\sigma),x)d\sigma.$$

Since $0 \le c_1 t\sigma + c_2 t(1-\sigma) \le C$ for $0 < \sigma, t < 1$, we get from (3.6)

$$\left|\partial_t^k T_\gamma[\varphi](t,x)\right| \le C\|\varphi\|_{k,\nu+2}\,(1+r)^{-2-\nu},$$

which yields (3.13) for $0 < t \le 1$.

Finally, we prove (3.14). When one of $r > 2c_1 t$, $r < c_2 t/2$ or $0 < t + r \le 1$ holds, (3.11) with $\nu = 2$ yields (3.14). Therefore, we have only to consider the case where $c_2 t/2 \le r \le 2c_1 t$ and $t + r \ge 1$. We rewrite

$$T_\gamma[\partial_j\varphi](t,x) = (4\pi r)^{-1}\int_{c_2 t}^{c_1 t} \tau^{-2}(-\omega)^\gamma R[\partial_j\varphi](r-\tau,\omega)d\tau$$
$$+\int_{c_2 t}^{c_1 t} \tau^{-2}\left(\tau Q_\gamma[\partial_j\varphi](\tau,x) - (4\pi r)^{-1}(-\omega)^\gamma R[\partial_j\varphi](r-\tau,\omega)\right)d\tau.$$

Let $\nu > 1$ in the following. Then, by (3.9) with $k = 0$ the second term on the right-hand side is estimated by

$$C\|\varphi\|_{2,N_0}\int_{c_2 t}^{c_1 t} \tau^{-2}(1+\tau+r)^{-2}(1+|r-\tau|)^{-\nu}d\tau$$
$$\le C\|\varphi\|_{2,N_0}\,(1+t+r)^{-4},$$

because $\tau \ge C(1+t+r)$ in this case. Using (3.3), we can make integration by parts in τ in the first term. Then it is rewritten as

$$(4\pi r)^{-1}\int_{c_2 t}^{c_1 t}(-2\tau^{-3})\omega_j(-\omega)^\gamma R[\varphi](r-\tau,\omega)d\tau$$
$$-(4\pi r)^{-1}\left((c_1 t)^{-2}\omega_j(-\omega)^\gamma R[\varphi](r-c_1 t,\omega)\right.$$
$$\left.-(c_2 t)^{-2}\omega_j(-\omega)^\gamma R[\varphi](r-c_2 t,\omega)\right).$$

By (3.4) we have $|R[\varphi](s,\omega)| \le C\|\varphi\|_{0,\nu+3}\,(1+s)^{-\nu}$. Since $\nu > 1$, we thus find (3.14) in this case. This completes the proof. □

4. Linear elastic wave equations

First of all, we consider the Cauchy problem:

$$(4.1)\qquad (\partial_t^2 - L)u_0(t,x) = 0, \qquad (t,x) \in (0,\infty)\times\mathbf{R}^3,$$

$$(4.2)\qquad u_0(0,x) = f(x),\ (\partial_t u_0)(0,x) = g(x), \qquad x \in \mathbf{R}^3,$$

where $f,\ g \in \left(\mathcal{S}(\mathbf{R}^3)\right)^3$. We recall the explicit representation of the solution u_0. We define

$$(4.3)\qquad E[g](t,x) = E_1[g](t,x) + E_2[g](t,x) + E_3[g](t,x),$$

with

$$E_1[g](t,x) = \frac{t}{4\pi}\int_{\theta\in S^2} \Pi(\theta) g(x+c_1t\theta)\,dS'_\theta, \tag{4.4}$$

$$E_2[g](t,x) = \frac{t}{4\pi}\int_{\theta\in S^2} (I-\Pi(\theta)) g(x+c_2t\theta)\,dS'_\theta, \tag{4.5}$$

$$\begin{aligned} E_3[g](t,x) = &-\frac{t}{4\pi}\int_{c_2t}^{c_1t} \tau^{-1}d\tau \\ &\times \int_{\theta\in S^2} (g(x+\tau\theta) - 3(\theta\cdot g(x+\tau\theta))\theta)\,dS'_\theta. \end{aligned} \tag{4.6}$$

Here, for each fixed $\theta \in S^2$, $\Pi(\theta) : \mathbf{R}^3 \to \mathbf{R}^3$ is the projection defined by $\Pi(\theta)v = (\theta\cdot v)\theta$ for $v \in \mathbf{R}^3$. Then it is known that

$$u_0(t,x) = \partial_t E[f](t,x) + E[g](t,x), \quad (t,x) \in (0,\infty)\times\mathbf{R}^3 \tag{4.7}$$

holds (see, e.g., John [10]). By virtue of Propositions 3.1 and 3.3, we can prove the following estimates which are refinement of those in Theorem 1 in [10] in the sense that we can replace the decaying factor $1+r$ by $1+t+r$ and that the derivatives enjoy better decay property with respect to $1+|r-c_it|$ with $i=1,2$.

Proposition 4.1. *Let k be a nonnegative integer, f, $g \in (\mathcal{S}(\mathbf{R}^3))^3$, $\nu > 1$, and N_0 be the number from Proposition 3.2. Then, for $(t,x) \in (0,\infty)\times\mathbf{R}^3$, we have*

$$|u_0(t,x)|_k \le C\mathcal{A}_{k,\nu+2}[f,g]\,(1+t+r)^{-1}\,W_{-1}(t,r) \tag{4.8}$$

and

$$|\partial u_0(t,x)|_k \le C\mathcal{A}_{k+2,N_0}[f,g]\,(1+t+r)^{-1}\,W_{-2}(t,r), \tag{4.9}$$

where $\mathcal{A}_{k,\nu}[f,g]$ is defined by (2.4), and for $\nu \in \mathbf{R}$ we put

$$W_\nu(t,r) = \max_{i=1,2}\{(1+|r-c_it|)^\nu\}. \tag{4.10}$$

Next we consider the radiation field for the free elastic wave (for the case of the scalar wave equation, see Friedlander [4], and also [11]). Having Proposition 3.2 in mind, we define the radiation field $\mathcal{F}_i[f,g]$ $(i=1,2)$ for u_0 associated with the propagation speed c_i by

$$\mathcal{F}_1[f,g](s,\theta) = \Pi(\theta)\widetilde{\mathcal{R}}_1[f,g](s,\theta), \tag{4.11}$$

$$\mathcal{F}_2[f,g](s,\theta) = (I-\Pi(\theta))\widetilde{\mathcal{R}}_2[f,g](s,\theta) \tag{4.12}$$

for $(s,\theta) \in \mathbf{R}\times S^2$, and f, $g \in (\mathcal{S}(\mathbf{R}^3))^3$. Here, $\widetilde{\mathcal{R}}_i[f,g](s,\theta)$ is defined by (1.8). We remark that (3.4) implies

$$|\partial_s^k o^\alpha \mathcal{F}_i[f,g](s,\theta)| \le C(1+s)^{-\nu}, \quad (s,\theta) \in \mathbf{R}\times S^2 \tag{4.13}$$

for any $\nu > 0$, nonnegative integer k, multi-indix α, and f, $g \in (\mathcal{S}(\mathbf{R}^3))^3$. Then we have the following.

Proposition 4.2. *Let f, $g \in (\mathcal{S}(\mathbf{R}^3))^3$ and let u_0 be the solution to the problem (4.1)-(4.2). Then for any non-negative integer k and any multi-index α with $|\alpha| \geq 1$, there exists a positive constant C such that*

$$\left| u_0(t,x) - \sum_{m=1}^{2} r^{-1}\mathcal{F}_m[f,g](r - c_m t, \omega) \right|_k \leq C(1+t+r)^{-2}, \tag{4.14}$$

and

$$\begin{aligned} &\left| \partial_t u_0(t,x) - \sum_{m=1}^{2} (-c_m) r^{-1}(\partial_s \mathcal{F}_m[f,g])(r - c_m t, \omega) \right|_k \\ &+ \left| \partial_x^\alpha u_0(t,x) - \sum_{m=1}^{2} \omega^\alpha r^{-1}(\partial_s^{|\alpha|} \mathcal{F}_m[f,g])(r - c_m t, \omega) \right|_k \\ \leq & C(1+t+r)^{-2} W_{-1}(t,r) \end{aligned} \tag{4.15}$$

for $(t,x) \in (0,\infty) \times \mathbf{R}^3$ with $r \geq c_2 t/2 \geq 1$. Here, $\omega = (\omega_1, \omega_2, \omega_3) = r^{-1}x$.

Next we consider the inhomogeneous elastic wave equation with zero initial data:

$$\begin{cases} (\partial_t^2 - L)u(t,x) = h(t,x) & \text{for } (t,x) \in (0,T) \times \mathbf{R}^3, \\ u(0,x) = 0, \ (\partial_t u)(0,x) = 0 & \text{for } x \in \mathbf{R}^3. \end{cases} \tag{4.16}$$

The following estimate is an improvement of the corresponding estimate given by [1, Proposition 5.1] in the sense that the exponent of the weight in the right hand side $1+\mu$ is replaced by $1-\mu$. This kind of modification was well studied in the case of the scalar wave equation, and the detail of the proof of (4.17) will appear elsewhere.

Proposition 4.3. *Let u be the solution to (4.16) and let $\mu > 0$, $c_0 = 0$. Then we have*

$$\begin{aligned} |\partial u(t,x)| \leq & C(1+r)^{-1} W_{-1}(t,r) \sup_{(s,x) \in [0,t] \times \mathbf{R}^3} (1+|x|) \\ & \times (1+s+|x|)^{1+\mu} \big(\max_{i=0,1,2} \{1 + |c_i s - |x||\} \big)^{1-\mu} |h(s,x)|_1 \end{aligned} \tag{4.17}$$

for $(t,x) \in [0,T) \times \mathbf{R}^3$.

On the other hand, the following estimate was proved by [10, Theorem 3].

Proposition 4.4. *Let u be the solution to* (4.16). *Then we have*

$$(4.18)\quad |\partial u(t,x)| \le C(1+r)^{-1}W_{-1}(t,r) \times \log(2+t+r) \sup_{s\in[0,t]} \int_{\mathbf{R}^3} \min_{i=1,2}\{1+|c_i s-|x||\}|h(s,x)|_7 dy$$

for $(t,x)\in[0,T)\times\mathbf{R}^3$, *and*

$$(4.19)\quad \int_{\mathbf{R}^3} |\partial u(t,x)| \frac{dx}{|x|} \le C\log(2+t) \times \sup_{s\in[0,t]} \left(\int_{\mathbf{R}^3} ((1+|y|)\min_{i=1,2}\{1+|c_i s-|y||\}|h(s,x)|_1)^2 dy\right)^{1/2}$$

for $t\in[0,T)$.

5. Approximate solutions

This section is the core of the present paper. We shall construct an approximate solution and derive important estimates given in Proposition 5.5 below in proving Theorem 1.1. Throughout this section we assume that f, $g\in (C_0^\infty(\mathbf{R}^3))^3$ satisfy

$$(5.1)\qquad f(x)=g(x)=0 \text{ for } |x|\ge R$$

with some $R>1$, and that $A_1\ne 0$ and $p_0\not\equiv 0$ on $\mathbf{R}\times S^2$, where p_0 is defined by (1.9).

Lemma 5.1. *Let $p(s,\theta,\tau)$ be the solution to* (1.13)-(1.14) *vanishing for $|s|\ge R$. Let $0<\tau_0<\tau_*$ with τ_* being defined by* (1.10). *Then for any $N>0$, and for any multi-indicies $\beta=(\beta_0,\dots,\beta_4)$ and $\gamma=(\gamma_0,\dots,\gamma_3)$, there exists a positive constant $C=C(\tau_0,\beta,\gamma,N)$ such that*

$$(5.2)\qquad |\Lambda^\beta p(s,\theta,\tau)|\le C,$$

$$(5.3)\qquad |\Lambda^\beta \partial_s p(s,\theta,\tau)|\le C(1+s)^{-N},$$

$$(5.4)\qquad |\Lambda_*^\gamma\{p(s,\theta,\tau)-p_0(s,\theta)\}|\le C\tau,$$

$$(5.5)\qquad |\Lambda_*^\gamma\partial_s\{p(s,\theta,\tau)-p_0(s,\theta)\}|\le C\tau(1+s)^{-N}$$

for all $(s,\theta,\tau)\in\mathbf{R}\times S^2\times[0,\tau_0]$.

Proof. First of all, we note that (5.2) and (5.4) follows from (5.3) and (5.5) with $N>1$ respectively, because both $p(s,\theta,\tau)$ and $p_0(s,\theta)$ vanish for $|s|\ge R$.

Next we prove (5.3). If we set $P = \partial_s p$, then it satisfies

(5.6) $$c_1^2\,\partial_\tau P + A_1 P \partial_s P = 0 \quad \text{in } \mathbf{R} \times S^2 \times [0, \tau_*),$$

(5.7) $$P(s,\theta,0) = \partial_s\, p_0(s,\theta) \quad \text{for } (s,\theta) \in \mathbf{R} \times S^2.$$

Observe that for $(s, s_0, \theta, \tau) \in \mathbf{R} \times \mathbf{R} \times S^2 \times [0, \tau_0)$, the equation

(5.8) $$F(s, s_0, \theta, \tau) := c_1^2(s_0 - s) + \partial_s p_0(s_0, \theta) A_1 \tau = 0$$

determies the implicit function $s_0 = s_0(s, \theta, \tau)$, because

$$\partial_{s_0} F(s, s_0, \theta, \tau) = c_1^2 + \partial_s^2 p_0(s_0, \theta) A_1 \tau \geq c_1^2(1 - \tau/\tau_*) > 0.$$

Therefore, the solution to (5.6)-(5.7) is given by $P(s,\theta,\tau) = (\partial_s p_0)(s_0(s,\theta,\tau),\theta)$, and hence for $(s,\theta,\tau) \in \mathbf{R} \times S^2 \times [0,\tau_0)$, we have

(5.9) $$\partial_s p(s,\theta,\tau) = (\partial_s p_0)(s_0(s,\theta,\tau),\theta).$$

Since (3.4) implies $|\Lambda^\beta p_0(s,\theta)| \leq C(1+s)^{-N}$ for any $(s,\theta,\tau) \in \mathbf{R} \times S^2$ and $N > 0$, we see that $\Lambda^\beta s_0(s,\theta,\tau)$ is bounded for any $(s,\theta,\tau) \in \mathbf{R} \times S^2 \times [0,\tau_0)$, because we have

$$\partial_s s_0(s,\theta,\tau) = \frac{c_1^2}{c_1^2 + (\partial_s^2 p_0)(s_0(s,\theta,\tau),\theta) A_1 \tau},$$

$$\partial_\tau s_0(s,\theta,\tau) = \frac{-A_1(\partial_s p_0)(s_0(s,\theta,\tau),\theta)}{c_1^2 + (\partial_s^2 p_0)(s_0(s,\theta,\tau),\theta) A_1 \tau},$$

$$o_i s_0(s,\theta,\tau) = \frac{-A_1 \tau (o_i \partial_s p_0)(s_0(s,\theta,\tau),\theta)}{c_1^2 + (\partial_s^2 p_0)(s_0(s,\theta,\tau),\theta) A_1 \tau}.$$

Therefore, we get (5.3) by using (5.9).

Next we prove (5.5). Since $s_0(s,\theta,0) = s$, we get

$$\partial_s p(s,\theta,\tau) - \partial_s p_0(s,\theta) = \tau \int_0^1 (\partial_s^2 p_0)(s_0(s,\theta,\sigma\tau),\theta)\partial_\tau s_0(s,\theta,\sigma\tau)d\sigma.$$

In view of (5.8), we see that $(1+s_0(s,\theta,\tau))^{-N}$ is equivalent to $(1+s)^{-N}$ for $(s,\theta,\tau) \in \mathbf{R} \times S^2 \times [0,\tau_0)$, because $|\Lambda_*^\gamma(\partial_s p_0(s_0(s,\theta,\tau),\theta)A_1\tau)|$ is bounded. Thus we find (5.5) holds. This completes the proof. □

For a real-valued function $\varphi = \varphi(s,\theta,\tau)$, we shall write

$$\widetilde{\varphi}(t,x) := \varphi(r - c_1 t, \omega, \varepsilon \log(\varepsilon t))$$

with $r = |x|$ and $\omega = r^{-1}x$. Then we have

(5.10) $$\partial_t \widetilde{\varphi} = -c_1 \widetilde{\partial_s \varphi} + \varepsilon t^{-1} \widetilde{\partial_\tau \varphi}, \quad O_i \widetilde{\varphi} = \widetilde{o_i \varphi} \quad (i = 1,2,3),$$

(5.11) $$\operatorname{grad} \widetilde{\varphi} = \omega \widetilde{\partial_s \varphi} - r^{-1}\, \omega \wedge \widetilde{o \varphi},$$

where we have used (2.1) to get (5.11).

Let $p(s,\theta,\tau)$ be the solution to (1.13)-(1.14) vanishing for $|s| \ge R$. Using the above notation, we define

$$(5.12) \qquad w_1(t,x) = \varepsilon r^{-1}\left(\widetilde{p}(t,x)\omega + \mathcal{F}_2[f,g](r-c_2t,\omega)\right)$$

for $(t,x) \in \left[1/\varepsilon, \exp(\tau_*/\varepsilon)\right) \times (\mathbf{R}^3 \setminus \{0\})$. Note that

$$(5.13) \qquad w_1(t,x) = 0 \quad \text{for } |x| \ge c_1t + R.$$

The following eastimates, which shows that w_1 is a good approximation of u_0 near the characteristic cones $r = c_it$ $(i = 1,2)$, are reduced from Lemma 5.1.

Corollary 5.2. *Let $0 < \tau_0 < \tau_*$ and let $0 < \varepsilon \le 1$. Then for any nonnegative integer k, there exists a positive constant $C = C(\tau_0, k)$ such that*

$$(5.14) \qquad |w_1(t,x)|_k \le C\varepsilon(1+t+r)^{-1},$$

$$(5.15) \qquad |\partial w_1(t,x)|_k \le C\varepsilon(1+t+r)^{-1}W_{-1}(t,r),$$

$$(5.16) \qquad |dvi\, w_1(t,x)|_k \le C\varepsilon(1+t+r)^{-1}(1+|r-c_1t|)^{-1},$$

$$(5.17) \qquad |rot\, w_1(t,x)|_k \le C\varepsilon(1+t+r)^{-1}(1+|r-c_2t|)^{-1}$$

for $c_2t/2 \le |x| \le c_1t + R$ and $1 \le t \le \exp(\tau_0/\varepsilon)$. Moreover, we have

$$(5.18) \qquad |w_1(t,x) - \varepsilon u_0(t,x)|_k \le C\varepsilon(1+t+r)^{-2},$$

$$(5.19) \qquad |\partial_t\{w_1(t,x) - \varepsilon u_0(t,x)\}|_k \le C\varepsilon(1+t+r)^{-2}W_{-1}(t,r),$$

for $c_2t/2 \le |x| \le c_1t + R$ and $1/\varepsilon \le t \le 2/\varepsilon$. Here, u_0 is the solution of the Cauchy problem (4.1)-(4.2).

Proof. We suppose that $c_2t/2 \le r \le c_1t + R$ and $1 \le t \le \exp(\tau_0/\varepsilon)$ in what follows. Then we have

$$(5.20) \qquad |t^{-1}|_k + |r^{-1}|_k + |(1+t+r)^{-1}|_k \le C(1+t+r)^{-1}.$$

First we prove (5.14) and (5.15). It follows from (4.13), (2.1), and (5.20) that

$$|\mathcal{F}_2[f,g](r-c_2t,\omega)|_k \le C(1+|r-c_2t|)^{-1}.$$

While, from (5.2), (5.3) with $N = 1$, (5.10), (5.11), and (5.20), we get

$$(5.21) \qquad |\widetilde{p}(t,x)|_k \le C\sum_{|\beta|\le k}\left|\widetilde{\Lambda^\beta p}(t,x)\right| \le C,$$

(5.22)

$$\begin{aligned}|\partial\widetilde{p}(t,x)|_k \le& C\sum_{|\beta|\le k}\left|\widetilde{\Lambda^\beta\partial_s p}(t,x)\right| + C(1+t+r)^{-1}\sum_{|\beta|\le k+1}\left|\widetilde{\Lambda^\beta p}(t,x)\right|\\ \le& C(1+|r-c_1t|)^{-1}.\end{aligned}$$

Thus we obtain (5.14) and (5.15) from (5.12).

Next we prove (5.16). A direct computation shows that

$$\operatorname{dvi}(r^{-1}\widetilde{p}(t,x)\omega) = r^{-1}\widetilde{\partial_s p}(t,x) + r^{-2}\widetilde{p}(t,x), \tag{5.23}$$

$$\begin{aligned}&\operatorname{dvi}(r^{-1}\mathcal{F}_2[f,g](r-c_2t,\omega)) = \\ &\quad -r^{-2}(2\omega\cdot\widetilde{\mathcal{R}}_2[f,g](r-c_2t,\omega) + \Omega\cdot\widetilde{\mathcal{R}}_2[f,g](r-c_2t,\omega)),\end{aligned} \tag{5.24}$$

where we put $\Omega\cdot f(x) = \sum_{j=1}^{3}\Omega_j f_j(x)$ with $\Omega = \omega\wedge O$ (recall also (4.12)). Therefore, by (5.2), (5.3) with $N=1$, and (3.4), we get (5.16).

Next we prove (5.17). A direct computation shows that

$$\operatorname{rot}(r^{-1}\widetilde{p}(t,x)\omega) = -r^{-2}\Omega\wedge(\widetilde{p}(t,x)\omega), \tag{5.25}$$

$$\begin{aligned}&\operatorname{rot}(r^{-1}\mathcal{F}_2[f,g](r-c_2t,\omega)) = r^{-1}\operatorname{rot}\widetilde{\mathcal{R}}_2[f,g](r-c_2t,\omega)\\ &\quad -r^{-2}(\omega\wedge\widetilde{\mathcal{R}}_2[f,g](r-c_2t,\omega) - \Omega\wedge\Pi(\omega)\widetilde{\mathcal{R}}_2[f,g](r-c_2t,\omega)).\end{aligned} \tag{5.26}$$

Thus (5.2) and (3.4) yields (5.17).

Next we prove (5.19). Suppose that we also have $1/\varepsilon \le t \le 2/\varepsilon$ from now on. In view of (4.15), it suffices to show

$$|\partial_t\{w(t,x) - \sum_{m=1}^{2}\varepsilon r^{-1}\mathcal{F}_m[f,g](r-c_mt,\omega)\}|_k \le C\varepsilon(1+t+r)^{-2}W_{-1}(t,r),$$

or

$$|\partial_t\{\widetilde{p}(t,x)\omega - \mathcal{F}_1[f,g](r-c_1t,\omega)\}|_k \le C(1+t+r)^{-1}W_{-1}(t,r),$$

because of (5.12) and (5.20). We see from (4.11) and (1.9) that the above estimate follows from

$$|\partial_t\{\widetilde{p}(t,x) - p_0(r-c_1t,\omega)\}|_k \le C(1+t+r)^{-1}W_{-1}(t,r). \tag{5.27}$$

It follows from (5.10), (5.11), (5.2), and (5.5) with $N=1$ that the left hand side of (5.27) is bounded by

$$\begin{aligned}&C\sum_{|\gamma|\le k}|\widetilde{\Lambda_*^\gamma\partial_s p}(t,x) - (\Lambda_*^\gamma\partial_s p_0)(r-c_1t,\omega)|\\ &\quad + C\varepsilon(1+t+r)^{-1}\sum_{|\beta|\le k}|\widetilde{\Lambda^\beta\partial_\tau p}(t,x)|\\ &\le C\varepsilon\left((\log(\varepsilon t))(1+|r-c_1|)^{-1} + (1+t+r)^{-1}\right),\end{aligned}$$

which yields (5.27), because $t\le 2/\varepsilon$ implies $\varepsilon \le C(1+t+r)^{-1}$.

Similarly, one can show (5.18) by using (4.14), (5.4) instead of (4.15), (5.5), respectively. This completes the proof. □

Next we examine how well $w_1(t,x)$ satisfies the original equation (1.1) near the characteristic cones $r = c_i t$ $(i = 1,2)$. We set

$$E[u](t,x) = (\partial_t^2 - L)u(t,x) - F\big(\nabla u(t,x), \nabla^2 u(t,x)\big). \tag{5.28}$$

Lemma 5.3. *Let $0 < \tau_0 < \tau_*$ and let $0 < \varepsilon \le 1$. Then for any nonnegative integer k, there exists a positive constant $C = C(\tau_0, k)$ such that*

(5.29)

$$|E[w_1](t,x) - (c_1^2 - c_2^2)\varepsilon r^{-2}\{\omega \wedge \widetilde{o\,\partial_s p}(t,x) + (\partial_s Y)(r - c_2 t, \omega)\omega\} + A_2\, grad\, |rot\, w_1(t,x)|^2|_k \le C\varepsilon(1+t+r)^{-3},$$

for $c_2 t/2 \le |x| \le c_1 t + R$ and $1 \le t \le \exp(\tau_0/\varepsilon)$. Here we have set

$$Y(s,\omega) = 2\omega \cdot \widetilde{\mathcal{R}}_2[f,g](s,\omega) + \Omega \cdot \widetilde{\mathcal{R}}_2[f,g](s,\omega).$$

Proof. Let $c_2 t/2 \le |x| \le c_1 t + R$ and $1 \le t \le \exp(\tau_0/\varepsilon)$. Then, t and r are equivalent to $1 + t + r$.

It holds that

$$\begin{aligned}
&\partial_t^2 \widetilde{p}(t,x) = c_1^2 \widetilde{\partial_s^2 p}(t,x) - 2c_1\varepsilon t^{-1} \widetilde{\partial_\tau \partial_s p} + t^{-2}(\varepsilon^2 \widetilde{\partial_\tau^2 p} - \varepsilon \widetilde{\partial_\tau p}),\\
&\Delta(r^{-1}\widetilde{p}(t,x)\omega) = r^{-1}\widetilde{\partial_s^2 p}(t,x)\omega + r^{-3}\Delta_\omega(\widetilde{p}(t,x)\omega),\\
&\mathrm{grad\,div}\,(r^{-1}\widetilde{p}(t,x)\omega) = r^{-1}\widetilde{\partial_s^2 p}(t,x)\omega - r^{-2}\omega \wedge \widetilde{o\,\partial_s p}(t,x)\\
&\qquad\qquad - 2r^{-3}\widetilde{p}(t,x)\omega - r^{-3}\omega \wedge \widetilde{o p}(t,x),
\end{aligned}$$

where $\Delta_\omega = \sum_{j=1}^3 O_j^2$. Therefore, we have

$$\Big|(\partial_t^2 - L)(\varepsilon r^{-1}\widetilde{p}(t,x)\omega) + 2c_1\varepsilon^2 (tr)^{-1}\widetilde{\partial_\tau \partial_s p}(t,x)\omega - (c_1^2 - c_2^2)\varepsilon r^{-2}\omega \wedge \widetilde{o\,\partial_s p}(t,x)\Big|_k \le C\varepsilon(1+r+t)^{-3}. \tag{5.30}$$

While, we have

$$(\partial_t^2 - c_2^2\Delta)(r^{-1}\mathcal{F}_2[f,g](r - c_2 t,\omega)) = -c_2^2 r^{-3}\Delta_\omega \mathcal{F}_2[f,g](r - c_2 t, \omega),$$

Hence, recalling (5.24), (5.10), and (5.11), we obtain

$$\big|(\partial_t^2 - L)(\varepsilon r^{-1}\mathcal{F}_2[f,g](r_2,\omega)) - (c_1^2 - c_2^2)\,\varepsilon r^{-2}(\partial_s Y)(r - c_2 t, \omega)\omega\big|_k \le C\varepsilon(1+r+t)^{-3}. \tag{5.31}$$

Next we consider the nonlinear term. It follows from (5.16), (5.17) that

$$|\mathrm{rot}\,((\mathrm{div}\, w_1(t,x))(\mathrm{rot}\, w_1(t,x)))|_k \le C\varepsilon^2(1+t+r)^{-3}.$$

By using (2.1), we get from (5.14)

$$|N(w_1(t,x))|_k \le C\varepsilon^2(1+t+r)^{-3}.$$

We see from (5.23), (5.24) that

$$|\mathrm{grad}\,(\mathrm{div}\,w_1)^2 - \mathrm{grad}\,(\varepsilon r^{-1}\widetilde{\partial_s p}(t,x))^2|_k \le C\varepsilon^2(1+t+r)^{-3},$$

and hence

$$|\mathrm{grad}\,(\mathrm{div}\,w_1)^2 - 2\varepsilon^2 r^{-2}\widetilde{\partial_s p}(t,x)\widetilde{\partial_s^2 p}(t,x)\,\omega|_k \le C\varepsilon^2(1+t+r)^{-3}.$$

Thus we obtain

(5.32)

$$\Big|F(\nabla w_1, \nabla^2 w_1) - A_2\,\mathrm{grad}\,|\mathrm{rot}\,w_1|^2 \\ -2A_1\varepsilon^2 r^{-2}\widetilde{\partial_s p}(t,x)\widetilde{\partial_s^2 p}(t,x)\,\omega\Big|_k \le C\varepsilon^2(1+r+t)^{-3}.$$

Observe that (5.2) and (5.3) (with $N=1/2$) yield

$$|\widetilde{\partial_s p}(t,x)|_k \le C(1+|c_1 t - r|)^{-1/2}. \tag{5.33}$$

By (5.6) with $P = \partial_s p$, and (5.33), we obtain

$$\begin{aligned} &\Big|2c_1\,\varepsilon^2(tr)^{-1}\widetilde{\partial_\tau\partial_s p} + 2A_1\,\varepsilon^2 r^{-2}\widetilde{\partial_s p}\,\widetilde{\partial_s^2 p}\Big|_k \\ =&\Big|2A_1(r-c_1t)\varepsilon^2(c_1t)^{-1}r^{-2}\widetilde{\partial_s p}\,\widetilde{\partial_s^2 p}\Big|_k \\ \le& C\varepsilon^2(1+t+r)^{-3}. \end{aligned} \tag{5.34}$$

Now (5.30), (5.31), (5.32), and (5.34) imply (5.29). This completes the proof. □

In order to eliminate $r^{-2}\{\omega\wedge \widetilde{o\,\partial_s p}(t,x) + (\partial_s Y)(r-c_2t,\omega)\omega\}$ in the estimate (5.29), we need to construct a more precise approximation. For this reason, we set

(5.35)

$$q_1(s,\theta,\tau) = \int_s^\infty \theta\wedge(op)(s',\theta,\tau)ds', \quad q_2(s,\theta) = \int_s^\infty Y(s',\theta)\,\theta\,ds',$$

and define

$$w(t,x) = w_1(t,x) + \varepsilon r^{-2}\left(\widetilde{q_1}(t,x) + q_2(r-c_2t,\omega)\right) \tag{5.36}$$

for $(t,x)\in\big[1/\varepsilon, \exp(\tau_*/\varepsilon)\big)\times(\mathbf{R}^3\setminus\{0\})$. Then, w enjoys the same estimates as in Corollary 5.2 togeter with a suitable estimates for $E[w]$ as follows.

Lemma 5.4. *Let $0<\tau_0<\tau_*$. We assume that $0<\varepsilon\le 1$. Then for any nonnegative integer k, there exists a positive constant $C = C(\tau_0,k)$*

such that

$$|w(t,x)|_k \le C\varepsilon(1+t+r)^{-1}, \tag{5.37}$$

$$|\partial w(t,x)|_k \le C\varepsilon(1+t+r)^{-1}W_{-1}(t,r) \tag{5.38}$$

$$|dvi\, w(t,x)|_k \le C\varepsilon(1+t+r)^{-1}(1+|r-c_1t|)^{-1}, \tag{5.39}$$

$$|rot\, w(t,x)|_k \le C\varepsilon(1+t+r)^{-1}(1+|r-c_2t|)^{-1}, \tag{5.40}$$

(5.41)

$$|E[w](t,x) + A_2\, grad\, |rot\, w_1(t,x)|^2|_k \le C\varepsilon(1+t+r)^{-3}$$

for $c_2t/2 \le |x| \le c_1t + R$ *and* $1 \le t \le \exp(\tau_0/\varepsilon)$. *Moreover, we have*

$$|w(t,x) - \varepsilon u_0(t,x)|_k \le C\varepsilon(1+t+r)^{-2}, \tag{5.42}$$

$$|\partial_t\{w(t,x) - \varepsilon u_0(t,x)\}|_k \le C\varepsilon(1+t+r)^{-2}W_{-1}(t,r), \tag{5.43}$$

for $c_2t/2 \le |x| \le c_1t + R$ *and* $1/\varepsilon \le t \le 2/\varepsilon$. *Here,* u_0 *is the solution of the Cauchy problem* (4.1)-(4.2).

Proof. Since $p(s,\theta,\tau) = 0$ for $|s| \ge R$, we see from (5.2), (5.3), and (3.4) that

$$|\Lambda^\beta q_1(s,\theta,\tau)| \le C, \quad |\Lambda^\beta \partial_s q_1(s,\theta,\tau)| \le C(1+s)^{-1}, \tag{5.44}$$

$$|\Lambda^\beta q_2(s,\theta)| \le C, \quad |\Lambda^\beta \partial_s q_2(s,\theta)| \le C(1+s)^{-1}, \tag{5.45}$$

for multi-indicies β and $(s,\theta,\tau) \in \mathbf{R} \times S^2 \times [0,\tau_0]$. Therefore, if we set

$$w_2(t,x) = \varepsilon r^{-2}\left(\widetilde{q_1}(t,x) + q_2(r-c_2t,\omega)\right),$$

then we get

$$|w_2(t,x)|_k \le C\varepsilon(1+t+r)^{-2}, \tag{5.46}$$

$$|\partial w_2(t,x)|_k \le C\varepsilon(1+t+r)^{-2}W_{-1}(t,r), \tag{5.47}$$

so that the estimates in Lemma 5.4 except for (5.41) immediately follow from Corollary 5.2.

In order to show (5.41), we write

(5.48)

$$\begin{aligned}
&E[w] + A_2\, \mathrm{grad}\, |\mathrm{rot}\, w_1|^2 \\
&= (E[w_1] + (c_1^2 - c_2^2)\varepsilon r^{-2}\{\widetilde{\partial_s^2 q_1}(t,x) + (\partial_s^2 q_2)(r-c_2t,\omega)\} \\
&\qquad\qquad + A_2\, \mathrm{grad}\, |\mathrm{rot}\, w_1|^2) \\
&\quad + \Big((\partial_t^2 - L)w_2 - (c_1^2 - c_2^2)\varepsilon r^{-2}\{\widetilde{\partial_s^2 q_1}(t,x) + (\partial_s^2 q_2)(r-c_2t,\omega)\}\Big) \\
&\quad + \big(F(\nabla w_1, \nabla^2 w_1) - F(\nabla w, \nabla^2 w)\big)
\end{aligned}$$

By (5.15), (5.47) we get

$$|F(\nabla w_1, \nabla^2 w_1) - F(\nabla w, \nabla^2 w)|_k \le C\varepsilon^2(1+t+r)^{-3}. \tag{5.49}$$

Using (5.44), we find

$$|(\partial_t^2 - c_2^2\Delta)(r^{-2}\widetilde{q_1}(t,x)) - (c_1^2 - c_2^2)r^{-2}\widetilde{\partial_s^2 q_1}(t,x)|_k \le C(1+t+r)^{-3}.$$

Since $\theta \cdot q_1(s,\theta) = 0$, we have $\operatorname{dvi}(r^{-2}\widetilde{q_1}(t,x)) = -r^{-3}\Omega \cdot \widetilde{q_1}(t,x)$ by (2.1). Therefore, we get

(5.50)

$$|(\partial_t^2 - L)(r^{-2}\widetilde{q_1}(t,x)) - (c_1^2 - c_2^2)r^{-2}\widetilde{\partial_s^2 q_1}(t,x)|_k \le C(1+t+r)^{-3}.$$

While, we have from (5.45)

$$|(\partial_t^2 - c_2^2\Delta)(r^{-2}q_2(r-c_2t,\omega))|_k \le C(1+t+r)^{-3}.$$

Since $\operatorname{dvi}(r^{-2}q_2(r-c_2t,\omega)) = -r^{-2}Y(r-c_2t,\omega)$, we obtain

(5.51)

$$\begin{aligned}|(\partial_t^2 - L)&(r^{-2}q_2(r-c_2t,\omega)) \\ &- (c_1^2 - c_2^2)r^{-2}(\partial_s^2 q_2)(r-c_2t,\omega)|_k \le C(1+t+r)^{-3}.\end{aligned}$$

Now, in view of (5.48), we see from (5.49), (5.50), (5.51), and (5.29) that (5.41) holds, because $\widetilde{\partial_s^2 q_1}(t,x) + (\partial_s^2 q_2)(r-c_2t,\omega) = \omega \wedge \widetilde{o\,\partial_s p}(t,x) + (\partial_s Y)(r-c_2t,\omega)\omega$. This completes the proof. □

Now we are in a position to construct an approximate solution u_1 for all $(t,x) \in [0, \exp(\tau_*/\varepsilon)) \times (\mathbf{R}^3 \setminus \{0\})$: Let χ and ξ be smooth and nonnegative functions on $[0,\infty)$ such that

$$\chi(s) = \begin{cases} 1, & s \le 1, \\ 0, & s \ge 2, \end{cases} \qquad \xi(s) = \begin{cases} 0, & s \le c_2/2, \\ 1, & s \ge 3c_2/4. \end{cases}$$

Let $0 < \varepsilon \le 1$ in the following. We put $\chi_\varepsilon(t) = \chi(\varepsilon t)$ and $\eta(t,x) = \xi(|x|/t)$. Since

$$\varepsilon \le C(1+t)^{-1} \quad \text{if } 0 \le \varepsilon t \le 2, \tag{5.52}$$

we get

$$\left|\frac{d^m \chi_\varepsilon}{dt^m}(t)\right| = \varepsilon^m \left|\frac{d^m \chi}{dt^m}(\varepsilon t)\right| \le C(1+t)^{-m} \quad \text{for } t \ge 0, \tag{5.53}$$

where m is a nonnegative integer. While, we easily have $O_j\eta(t,x) = 0$ for $1 \le j \le 3$. Since $c_2t/2 \le r \le 3c_2t/4$ for $(t,x) \in \operatorname{supp} \partial\eta$, we have

$$\sum_{|\alpha|=m} |\partial^\alpha \eta(t,x)| \le C(1+t+r)^{-m} \quad \text{for } (t,x) \in [1,\infty) \times \mathbf{R}^3, \tag{5.54}$$

where m is a nonnegative integer, $\partial = (\partial_t, \nabla_x)$, and α is a multi-index. Besides, we get

$$W_{-1}(t,r) \le C(1+t+r)^{-1} \quad \text{if } 0 \le r \le 3c_2t/4. \tag{5.55}$$

Let u_0 be the solution of the Cauchy problem (4.1)–(4.2), and let w be given by (5.36). We define

$$u_1(t,x) = \chi_\varepsilon(t)\varepsilon u_0(t,x) + \big(1-\chi_\varepsilon(t)\big)\eta(t,x)w(t,x) \tag{5.56}$$

for $(t,x) \in \big[0, \exp(\tau_*/\varepsilon)\big) \times \mathbf{R}^3$. By (5.1) and the property of finite propagation, we have $|x| \le c_1t + R$ in $\operatorname{supp} u_0$. Hence, recalling (5.13), we find that

$$u_0(t,x) = w(t,x) = u_1(t,x) = 0 \quad \text{for } |x| \ge c_1t+R. \tag{5.57}$$

Then we have the following:

Proposition 5.5. *Let $0 < \tau_0 < \tau_*$, k be a nonnegative integer, $0 \le \lambda \le 1/2$, $0 < \mu \le 1/4$, and $0 < \varepsilon \le 1$. Then there exists a positive constant $C = C(\tau_0, k, \lambda, \mu)$ such that*

$$|u_1(t,x)|_k \le C\varepsilon(1+t+r)^{-1}, \tag{5.58}$$

$$|\partial u_1(t,x)|_k \le C\varepsilon(1+t+r)^{-1}W_{-1}(t,r), \tag{5.59}$$

$$|E[u_1](t,x)|_k \le C\varepsilon^{1+\lambda}(1+t+r)^{-2+\lambda-\mu}W_{-1+\mu}(t,r) \tag{5.60}$$

for $(t,x) \in [0, \exp(\tau_0/\varepsilon)] \times \mathbf{R}^3$, and

$$\big\| |E[u_1](t,\cdot)|_k \big\|_{L^2} \le C\varepsilon^{1+\lambda}(1+t)^{-(3/2)+\lambda} \tag{5.61}$$

for $t \in [0, \exp(\tau_0/\varepsilon)]$.

Proof. We write $x = r\omega$ with $r = |x|$ and $\omega \in S^2$. First we prove (5.58) and (5.59). It follows from (4.8) that

$$|u_0(t,x)|_k \le C(1+t+r)^{-1}W_{-1}(t,r) \tag{5.62}$$

for $(t,x) \in [0,\infty)\times\mathbf{R}^3$. We see from (5.57) that $(1+t)^{-1} \le C(1+t+r)^{-1}$ for $(t,x) \in \operatorname{supp} w$. Therefore, we get (5.58) and (5.59) from (5.37), (5.38), (5.53), (5.54), and (5.62).

Next we consider (5.60) and (5.61). If we set

$$v(t,x) = \eta(t,x)w(t,x) - \varepsilon u_0(t,x), \tag{5.63}$$

then we have $u_1 = \varepsilon u_0 + (1-\chi_\varepsilon)v$ by (5.56). Therefore, it follows that

$$E[u_1] = I_0 + I_1 + I_2 + I_3, \tag{5.64}$$

where we put

$$I_0 = -\chi_\varepsilon(t)F(\nabla u_1, \nabla^2 u_1),$$
$$I_1 = -\chi''_\varepsilon(t)v(t,x),$$
$$I_2 = -2\chi'_\varepsilon(t)\partial_t v(t,x),$$
$$I_3 = \big(1-\chi_\varepsilon(t)\big)\left\{(\partial_t^2 - L)\big(\eta(t,x)w(t,x)\big) - F(\nabla u_1, \nabla^2 u_1)\right\}.$$

We will estimate I_j for $0 \le j \le 3$. Let $0 \le \lambda \le 1/2$ and $0 < \mu \le 1/4$ in the following.

By (5.52) and (5.57), we have

$$\varepsilon \le C(1+t+r)^{-1} \quad \text{for } (t,x) \in \operatorname{supp} I_0 \cup \operatorname{supp} I_1 \cup \operatorname{supp} I_2. \tag{5.65}$$

From (5.59) and (5.65) we get

$$\begin{aligned} |I_0|_k \le& C\varepsilon^2(1+t+r)^{-2}W_{-2}(t,r) \\ \le& C\varepsilon^{1+\lambda}(1+t+r)^{-3+\lambda}W_{-2}(t,r), \end{aligned} \tag{5.66}$$

which yields

$$\big\| |I_0|_k \big\|_{L^2} \le C\varepsilon^{1+\lambda}(1+t)^{-2+\lambda}. \tag{5.67}$$

Next we estimate I_1. We may assume $t \ge 1$, because $\varepsilon t \ge 1$ in $\operatorname{supp} \chi''_\varepsilon$. Therefore, (5.54), (5.55) and (5.62) yield

$$\big|\big(1-\eta(t,x)\big)u_0(t,x)\big|_k \le C(1+t+r)^{-2}. \tag{5.68}$$

Observe that we have $1/\varepsilon \le t \le 2/\varepsilon$ and $c_2 t/2 \le r$ in $\operatorname{supp}(\chi''_\varepsilon \eta)$. Thus, writing $I_1 = -\varepsilon^2\chi''(\varepsilon t)\big(\eta(w-\varepsilon u_0) - \varepsilon(1-\eta)u_0\big)$, by (5.42), (5.68), and (5.65), we get

$$|I_1|_k \le C\varepsilon^3(1+t+r)^{-2} \le C\varepsilon^2(1+t+r)^{-3}. \tag{5.69}$$

In order to evaluate I_2, we use

$$\big|\big(1-\eta(t,x)\big)\partial_t u_0(t,x)\big|_k \le C(1+t+r)^{-3}, \tag{5.70}$$

which follows from (5.54), (5.55) and (4.9). Then, writting

$$\begin{aligned} I_2 =& -2\varepsilon\chi'(\varepsilon t)\big((\partial_t\eta)(w-\varepsilon u_0) + (\partial_t\eta)\varepsilon u_0 \\ & + \eta(\partial_t w - \varepsilon\partial_t u_0) - (1-\eta)\varepsilon\partial_t u_0\big), \end{aligned}$$

by (5.42), (5.43), (5.54), (5.62), and (5.70) that

$$|I_2|_k \le C\varepsilon^2(1+t+r)^{-2}W_{-1}(t,r). \tag{5.71}$$

By (5.69), (5.71), and (5.65) we get

$$|I_1 + I_2|_k \le C\varepsilon^{1+\lambda}(1+t+r)^{-3+\lambda}W_{-1}(t,r), \tag{5.72}$$

$$\big\| |I_1 + I_2|_k \big\|_{L^2} \le C\varepsilon^{1+\lambda}(1+t)^{-2+\lambda}. \tag{5.73}$$

Next we consider I_3 by rewritting it as

$$I_3 = \big(1-\chi_\varepsilon(t)\big)(I_{31} + I_{32} + I_{33}), \tag{5.74}$$

where we have set

$$I_{31} = -F(\nabla u_1, \nabla^2 u_1) + \eta F(\nabla w, \nabla^2 w),$$
$$I_{32} = [\partial_t^2 - L, \eta]w$$
$$I_{33} = \eta\big((\partial_t^2 - L)w - F(\nabla w, \nabla^2 w)\big).$$

In the following, we assume $t \geq 1$, because $\varepsilon t \geq 1$ in $\mathrm{supp}(1-\chi_\varepsilon)$.

We first estimate I_{31}. We may assume $\varepsilon t \leq 2$ or $r \leq 3c_2t/4$, because $I_{31} = 0$ otherwise. If $0 \leq \varepsilon t \leq 2$, then we have (5.65) in $\mathrm{supp}\, u_1 \cup \mathrm{supp}\, w$. Therefore, by (5.38) and (5.59), we get

$$|(1-\chi_\varepsilon)I_{31}|_k \leq C\varepsilon^{1+\lambda}(1+t+r)^{-3+\lambda}W_{-2}(t,r),$$

similarly to (5.66). While, if $r \leq 3c_2t/4$, then (5.38), (5.59), and (5.55) yield

$$|(1-\chi_\varepsilon)I_{31}|_k \leq C\varepsilon^2(1+t+r)^{-4}.$$

Summing up, we have proved

$$|(1-\chi_\varepsilon)I_{31}|_k \leq C\varepsilon^{1+\lambda}(1+t+r)^{-3+\lambda}W_{-2}(t,r) + C\varepsilon^2(1+t+r)^{-4}. \tag{5.75}$$

By (5.37), (5.38), (5.54) with $m = 1, 2$, and (5.55), we get

$$|(1-\chi_\varepsilon)I_{32}|_k \leq C\varepsilon(1+t+r)^{-3}. \tag{5.76}$$

From (5.41), we have

$$|(1-\chi_\varepsilon)I_{33}|_k \leq C\varepsilon(1+t+r)^{-3}. \tag{5.77}$$

Thus, (5.75), (5.76), and (5.77) lead to

$$|I_3|_k \leq C\varepsilon(1+t+r)^{-3} + C\varepsilon^{1+\lambda}(1+t+r)^{-3+\lambda}W_{-2}(t,r). \tag{5.78}$$

Since $\varepsilon t \geq 1$ in $\mathrm{supp}\, I_3$, we have $\varepsilon \geq t^{-1} \geq (1+t+r)^{-1}$. Hence, we get

$$|I_3|_k \leq C\varepsilon^{1+\lambda}(1+t+r)^{-3+\lambda}, \tag{5.79}$$

which yields

$$\big\||I_3|_k\big\|_{L^2} \leq C\varepsilon^{1+\lambda}(1+t)^{-(3/2)+\lambda}. \tag{5.80}$$

Finally (5.60) follows from (5.66), (5.72), and (5.79). We also obtain (5.61) from (5.67), (5.73), and (5.80). This completes the proof. □

6. Outline of the proof of Theorem 1.1

We assume that $0 < \varepsilon \leq 1$ and that (5.1) holds for some $R > 1$. Let $u_1(t,x)$ be the approximation defined by (5.56) for $(t,x) \in [0, \exp(\tau_*/\varepsilon)) \times \mathbf{R}^3$. If we set

$$u_2(t,x) = u(t,x) - u_1(t,x),$$

thent (1.1)-(1.2) is reduced to

$$(\partial_t^2 - L)u_2 = H(u_1,u_2) - E[u_1] \quad \text{in } [0, \exp(\tau_*/\varepsilon)) \times \mathbf{R}^3, \tag{6.1}$$

$$u_2(0,x) = (\partial_t u_2)(0,x) = 0 \quad \text{for } x \in \mathbf{R}^3, \tag{6.2}$$

where $E[u]$ is defined by (5.28), and $H(u_1,u_2)$ is given by

$$H(u_1,u_2) = F(\nabla(u_1+u_2), \nabla^2(u_1+u_2)) - F(\nabla u_1, \nabla^2 u_1).$$

Observe that for any nonnegative integer k, there exists a constant C_k such that

$$\sup_{x\in\mathbf{R}^3} |u_2(0,x)|_k \leq C_k\varepsilon^2, \tag{6.3}$$

because for $0 \leq t \leq \varepsilon^{-1}$ and $x \in \mathbf{R}^3$, we have

$$(\partial_t^2 - L)u_2 = F(\nabla(u_1+u_2), \nabla^2(u_1+u_2)),$$

$u_2(0,x) = \partial_t u_2(0,x) = 0$, and $u_1(t,x) = \varepsilon u_0(t,x)$ by (5.56). Therefore, by the local existence theorem (see [7]), whet we need for proving Theorem 1.1 is to establish a suitable a-priori estimte. More explicitely, for $0 < T < \max\{T_\varepsilon, \exp(\tau_0/\varepsilon)\}$ with $\tau_0 \in (0, \tau_*)$, we wish to evaluate the following quantity:

$$\begin{aligned}\sup_{(t,x)\in[0,T]\times\mathbf{R}^3} \{&(1+r)(W_{-1}(t,r))^{-1}|\partial u_2(t,x)|_K \\ &+ (1+r)(1+|c_1t-r|)|\operatorname{div} u_2(t,x)|_K \\ &+(1+r)(1+|c_2t-r|)|\operatorname{rot} u_2(t,x)|_K\},\end{aligned} \tag{6.4}$$

provided K is an integer large enough and ε is small enough.

In order to carry out this purpose, we employ (4.17) for estimating $E[u_1]$ and (4.18) for evaluating $H(u_1,u_2)$, respectively. Note that (5.60), (5.61) enable us to regard $E[u_1]$ as a harmless term. In addition, when $T < \exp(\tau_0/\varepsilon)$, we see that $0 \leq t \leq T$ implies $\varepsilon \log(2+t) \leq C(1+\tau_0)$. Hence, one can develop the argument as in [1], and find that Theorem 1.1 is valid.

Appendix: Derivation of (1.1)

In this appendix we derive the quadratically perturbed wave equation (1.1) as the Euler-Lagrange equation of the following lagrangian:

$$I(u) = \iint_{\mathbf{R}^{1+3}} \left\{ \frac{1}{2}|\partial_t u|^2 - W(\varepsilon(u)) \right\} dxdt, \tag{A.1}$$

where $u = u(t,x)$ is the displacement vector, $W(\varepsilon(u))$ is the strain energy, and

$$\varepsilon(u) = \frac{1}{2}\left((\nabla \otimes u) + {}^t(\nabla \otimes u)\right) = (\varepsilon_{ij}(u)) \tag{A.2}$$

with $\varepsilon_{ij}(u) = (\partial_i u_j + \partial_j u_i)/2$ for $i,j = 1,2,3$. We underline that one can obtain the same equation as in Agemi [1]. Sideris [14], although our choice of the strain tensor $\varepsilon(u)$ is just the linear approximation of

$$\tilde{\varepsilon}(u) = \{{}^t(I + \nabla u)(I + \nabla u)\}^{1/2} - I,$$

used in [1]. [14].

Since we assumed that the elastic body is isotropic, the strain energy $W(\varepsilon(u))$ is a function of the principal invariants $\alpha(u)$, $\beta(u)$, and $\gamma(u)$ which are explictely given by

$$\alpha(u) = \varepsilon_{11}(u) + \varepsilon_{22}(u) + \varepsilon_{33}(u) = \operatorname{div} u, \tag{A.3}$$

$$\begin{aligned} \beta(u) &= \varepsilon_{11}(u)\varepsilon_{22}(u) + \varepsilon_{22}(u)\varepsilon_{33}(u) + \varepsilon_{33}(u)\varepsilon_{11}(u) \\ &\quad - ((\varepsilon_{13}(u))^2 + (\varepsilon_{32}(u))^2 + (\varepsilon_{21}(u))^2) \\ &= -\frac{1}{4}|\operatorname{rot} u|^2 + Q_{13}(u_1,u_3) + Q_{32}(u_3,u_2) + Q_{21}(u_2,u_1) \end{aligned} \tag{A.4}$$

$$\gamma(u) = \det \varepsilon(u) \tag{A.5}$$

where for scalar functions ϕ and ψ, we put

$$Q_{ij}(\phi,\psi) = (\partial_i\phi)(\partial_j\psi) - (\partial_j\phi)(\partial_i\psi) \quad (i,j = 1,2,3). \tag{A.6}$$

If we assume that $W(\varepsilon(u))$ is of cubic order with respect to u, then it is expressed as

$$W(\varepsilon(u)) = W_0(\varepsilon(u)) + a(\alpha(u))^3 + b(\alpha(u))^2\beta(u) + c\gamma(u) \tag{A.7}$$

where a, b, and c are constants, while $W_0(\varepsilon(u))$ is the quadratic part of $W(\varepsilon(u))$ definde by

$$W_0(\varepsilon(u)) = \frac{1}{2}(\lambda + 2\mu)(\alpha(u))^2 - 2\mu\beta(u) \tag{A.8}$$

with the Lamé constants λ and μ.

The variational principle tells us that if u describes the phenominum associated with the lagrangian $I(u)$, then it must satisfy

$$\lim_{\eta\to 0}\eta^{-1}\{I(u+\eta\varphi)-I(u)\}=0 \tag{A.9}$$

for any $\varphi = {}^t(\varphi_1,\varphi_2,\varphi_3) \in C_0^\infty(\mathbf{R}^{1+3})$. We shall show that (A.9) implies

$$\begin{aligned}\partial_t^2 u_i &- (\mu\Delta u_i + (\lambda+\mu)\partial_i \operatorname{div} u) - A_1\,\partial_i\,(\operatorname{div} u)^2 \\ &- A_2\,\partial_i\,|\operatorname{rot} u|^2 - A_3\,\{(\operatorname{div} u)(\partial_i \operatorname{div} u - \Delta u_i) \\ &+(\partial_2 \operatorname{div} u)(\partial_2 u_1 - \partial_1 u_2) - (\partial_3 \operatorname{div} u)(\partial_3 u_1 - \partial_1 u_3)\} \\ &+ N_i(u) = 0 \quad (i=1,2,3),\end{aligned} \tag{A.10}$$

where $N_i(u)$ is a linear combination of null-forms Q_{kl} defined by (A.6). Since

$$\operatorname{rot}((\operatorname{div} u)(\operatorname{rot} u)) = (\operatorname{div} u)\operatorname{rot}(\operatorname{rot} u) + (\operatorname{grad}\operatorname{div} u)\wedge \operatorname{rot} u, \tag{A.11}$$

$$\operatorname{rot}(\operatorname{rot} u) = \operatorname{grad}\operatorname{div} u - \Delta u, \tag{A.12}$$

we find (1.1) from (A.10) by setting $c_1^2 = \lambda+2\mu$, $c_2^2=\mu$, $A_1 = 3a$, $A_2 = -b/4$, and $A_3 = b/2$.

For simplicity, we shall write $f \asymp g$ if there exist h_i $(i=1,2,3)$ such that $f(x)-g(x) = \sum_{i=1}^3 \partial_i h_i(x)$. In order to prove (A.10) for $i=1$, we take $\varphi = {}^t(\varphi_1,0,0)$ in the following.

Since $\varepsilon_{ij}(u)$ is linear in u, we see from (A.3) that $\alpha(u)$ is also linear functional, and hence we get

$$\begin{aligned}\lim_{\eta\to 0}\eta^{-1}\{(\alpha(u+\eta\varphi))^2-(\alpha(u))^2\} &= 2\alpha(u)\alpha(\varphi) \\ &= 2(\partial_1\varphi_1)(\operatorname{div} u) \asymp -2\varphi_1\partial_1(\operatorname{div} u).\end{aligned} \tag{A.13}$$

From (A.4) we get

$$\begin{aligned}\beta(u+\eta\varphi) = &-\frac{1}{4}|\operatorname{rot} u+\eta\operatorname{rot}\varphi|^2 + Q_{13}(u_1+\eta\varphi_1,u_3) \\ &+ Q_{32}(u_3,u_2)+Q_{21}(u_2,u_1+\eta\varphi_1).\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}&\lim_{\eta\to 0}\eta^{-1}\{\beta(u+\eta\varphi)-\beta(u)\} \\ &= -\frac{1}{2}\left((\partial_3 u_1)(\partial_3\varphi_1)+(\partial_2 u_1)(\partial_2\varphi_1)\right) \\ &\quad + (\partial_2 u_2+\partial_3 u_3)(\partial_1\varphi_1) - \frac{1}{2}(\partial_1 u_3)(\partial_3\varphi_1) - \frac{1}{2}(\partial_1 u_2)(\partial_2\varphi_1) \\ &\asymp -\frac{1}{2}\varphi_1(\partial_1 \operatorname{div} u - \Delta u_1).\end{aligned} \tag{A.14}$$

Thus we find from (A.13) and (A.14) that

$$\text{(A.15)}\qquad \begin{aligned}&\lim_{\eta\to 0}\eta^{-1}\{W_0(u+\eta\varphi)-W_0(u)\}\\ &\asymp -\varphi_1(\mu\Delta u_1+(\lambda+\mu)\partial_1\mathrm{div}\,u).\end{aligned}$$

In particular, when $a=b=c=0$, we obtain the homogeneous elastic wave eqaurtion (4.1) from (A.9).

Next we consider the higher order terms in $W(\varepsilon(u))$. It is easy to see that

$$\text{(A.16)}\qquad \lim_{\eta\to 0}\eta^{-1}\{(\alpha(u+\eta\varphi))^3-(\alpha(u))^3\}\asymp -3\varphi_1\partial_1(\mathrm{div}\,u)^2.$$

It follows that

$$\begin{aligned}&\lim_{\eta\to 0}\eta^{-1}\{\alpha(u+\eta\varphi)\beta(u+\eta\varphi)-\alpha(u)\beta(u)\}\\ =&\alpha(u)\lim_{\eta\to 0}\eta^{-1}\{\beta(u+\eta\varphi)-\beta(u)\}+\alpha(\varphi)\lim_{\eta\to 0}\beta(u+\eta\varphi).\\ \asymp&-\frac{1}{2}\alpha(u)(\partial_1\mathrm{div}\,u-\Delta u_1)\varphi_1+\frac{1}{2}(\partial_3\alpha(u)(\partial_3u_1)+\partial_2\alpha(u)(\partial_2u_1))\varphi_1\\ &-\partial_1\alpha(u)(\partial_2u_2+\partial_3u_3)\varphi_1+\frac{1}{2}(\partial_3\alpha(u)(\partial_1u_3)+\partial_2\alpha(u)(\partial_1u_2))\varphi_1\\ &-\varphi_1\partial_1\beta(u),\end{aligned}$$

in view of (A.14). Rearranging the terms in the last expression, we get

$$\text{(A.17)}\qquad \begin{aligned}&\lim_{\eta\to 0}\eta^{-1}\{\alpha(u+\eta\varphi)\beta(u+\eta\varphi)-\alpha(u)\beta(u)\}\\ &\asymp-\frac{1}{2}\varphi_1\{\alpha(u)(\partial_1\mathrm{div}\,u-\Delta u_1)\\ &\qquad+(-\partial_3\alpha(u)(\partial_3u_1-\partial_1u_3)+\partial_2\alpha(u)(\partial_2u_1-\partial_1u_2))\}\\ &\quad+\varphi_1\{Q_{12}(u_2,\alpha(u))+Q_{13}(u_3,\alpha(u))\}\\ &\quad+\varphi_1\partial_1(\frac{1}{4}|\mathrm{rot}\,u|^2-Q_{13}(u_1,u_3)-Q_{32}(u_3,u_2)-Q_{21}(u_2,u_1)).\end{aligned}$$

A direct compuation shows that

$$\begin{aligned}\lim_{\eta\to 0}\eta^{-1}\{\gamma(u+\eta\varphi)-\gamma(u)\}=&\begin{vmatrix}\partial_1\varphi_1 & \partial_2\varphi_1 & \partial_3\varphi_1\\ \varepsilon_{12}(u) & \varepsilon_{22}(u) & \varepsilon_{23}(u)\\ \varepsilon_{13}(u) & \varepsilon_{23}(u) & \varepsilon_{33}(u)\end{vmatrix}\\ \asymp-\varphi_1\{&\partial_1(\varepsilon_{22}(u)\varepsilon_{33}(u)-(\varepsilon_{23}(u))^2)\\ &+\partial_2(\varepsilon_{23}(u)\varepsilon_{13}(u)-\varepsilon_{12}(u)\varepsilon_{33}(u))\\ &\quad+\partial_3(\varepsilon_{23}(u)\varepsilon_{12}(u)-\varepsilon_{22}(u)\varepsilon_{13}(u))\},\end{aligned}$$

which implies

$$
\begin{aligned}
\text{(A.18)} \qquad & \lim_{\eta\to 0}\eta^{-1}\{\gamma(u+\eta\varphi)-\gamma(u)\} \\
&= -\varphi_1\{\partial_1 Q_{23}(u_2,u_3)+\frac{1}{4}\partial_2(Q_{23}(u_3,u_1)+Q_{31}(u_2,u_3)) \\
&\qquad +\frac{1}{4}\partial_3(Q_{12}(u_2,u_3)+Q_{23}(u_1,u_2)) \\
&\qquad +\frac{1}{4}(Q_{12}(u_3,\partial_2u_3-\partial_3u_2)+Q_{13}(u_2,\partial_3u_2-\partial_2u_3) \\
&\qquad +Q_{23}(u_2,\partial_1u_3+\partial_3u_1)+Q_{23}(u_3,\partial_2u_1+\partial_1u_2))\}.
\end{aligned}
$$

Finally, one can conclude from (A.15), (A.16), (A.17), and (A.18) that (A.9) yields (A.10), and hence (1.1).

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