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On the ABP maximum principle and applications

Shigeaki Koike (Saitama University)
小池 茂昭 (埼玉大学)

1 Introduction

We are concerned with fully nonlinear second order uniformly elliptic PDEs:

$$F(x, Du, D^2u) = f(x) \quad \text{in } \Omega. \tag{E}$$

Here, $\Omega \subset \mathbf{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$, and $f \in L^p(\Omega)$ for $p > \hat{p}$, where $\hat{p} \in [\frac{n}{2}, n)$ is called Escauriaza's constant depending only on known quantities. For simplicity, we suppose that F does not contain u-variable.

We suppose that the following hypotheses: the first one is the uniform ellipticity;

$$\mathcal{P}^-(X - Y) \leq F(x, \xi, X) - F(x, \xi, Y) \leq \mathcal{P}^+(X - Y) \tag{$A1$}$$

for $x \in \Omega$, $\xi \in \mathbf{R}^n$, $X, Y \in S^n$. Here S^n denotes the set of $n \times n$ symmetric matrices, and for fixed $0 < \lambda \leq \Lambda$, the Pucci operators $\mathcal{P}^{\pm}(X) := \mathcal{P}^{\pm}_{\lambda,\Lambda}(X)$ are given by

$$\mathcal{P}^+(X) := \max\{-\text{trace}(AX) \mid A \in S^n_{\lambda,\Lambda}\} \quad \text{and} \quad \mathcal{P}^-(X) := -\mathcal{P}^+(-X),$$

where $S^n_{\lambda,\Lambda} := \{Z \in S^n \mid \lambda I \leq Z \leq \Lambda I\}$, There exists nonnegative $\mu \in L^q(\Omega)$ with $q \geq n$ such that

$$|F(x, \xi, X) - F(x, \eta, X)| \leq \mu(x)|\xi - \eta| \tag{$A2$}$$

for $x \in \Omega$, $\xi, \eta \in \mathbf{R}^n$ and $X \in S^n$.

Without loss of generality, we may suppose

$$F(x, 0, O) = 0 \quad \text{for } x \in \Omega \tag{$A3$}$$

by considering $F(x, \xi, X) - F(x, 0, O)$ in place of F if necessary. We note that $(A3)$ together with $(A2)$ yields

$$|F(x, \xi, O)| \leq \mu(x)|\xi| \quad \text{for } x \in \Omega \text{ and } \xi \in \mathbf{R}^n.$$

A typical example for F is the following linear operator:

$$F(x, Du, D^2u) = -\mathrm{trace}(A(x)D^2u) + \langle b(x), Du \rangle,$$

where $A(x) \in S^n_{\lambda,\Lambda}$ for $x \in \Omega$, and $b = (b_1, \ldots, b_n)$ with $b_i \in L^q(\Omega)$ for $i = 1, 2, \ldots, n$. Note that we will not assume continuity on A and b. Therefore, we cannot use the notion of weak solutions in the distribution sense. It is known that viscosity solutions are correct weak solutions when fully nonlinear PDEs are (degenerate) elliptic. However, if we only suppose that the inhomogeneous term f in (E) belong to $L^p(\Omega)$ (not necessarily $C(\Omega)$), the notion of viscosity solutions introduced by Crandall and Lions is not appropriate. Thus, our weak solutions will be L^p-viscosity solutions introduced by Caffarelli-Crandall-Kocan-Święch [3] in 1996, which was motivated by a pioneering work by Caffarelli [1] in 1989. We also refer to [2] as a nice survey book.

Throughout this article, we only state the assertions for subsolutions but it is easy to obtain the results associated with supersolutions. We also suppose

$$\mathrm{diam}(\Omega) = 1$$

for simplicity. It is easy to obtain the dependence on $\mathrm{diam}(\Omega)$ in the results below by scaling argument.

The definition of L^p-viscosity solutions of (E) is as follows:

Definition We call $u \in C(\Omega)$ an L^p-viscosity subsolution (resp., supersolution) of (E) if for $\phi \in W^{2,p}_{\mathrm{loc}}(\Omega)$,

$$ess \liminf_{y \to x} \left\{ F(y, D\phi(y), D^2\phi(y)) - f(y) \right\} \leq 0$$

$$\left(\text{resp.,} \quad ess \limsup_{y \to x} \left\{ F(y, D\phi(y), D^2\phi(y)) - f(y) \right\} \geq 0 \right)$$

provided $u - \phi$ attains its local maximum (resp., minimum) at $x \in \Omega$.

We also recall the definition of L^p-strong solutions:

Definition We call $u \in C(\Omega)$ an L^p-strong subsolution (resp., supersolution, solution) of (E) if $u \in W^{2,p}_{\mathrm{loc}}(\Omega)$, and

$$F(x, Du(x), D^2u(x)) \leq f(x) \quad (\text{resp.,} \geq f(x), = f(x)) \quad a.e. \text{ in } \Omega.$$

We recall a classical version of the Aleksandrov-Bakelman-Pucci (ABP for short) maximum principle for L^n-strong solutions.

ABP maximum principle There exists $C_0 > 0$ such that if $f, \mu \in L^n(\Omega)$, and $u \in C(\overline{\Omega})$ is an L^n-strong subsolution of (E), then it follows that

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u + C_0 e^{C_0\|\mu\|^n_{L^n(\Omega)}} \|f^+\|_{L^n(\Gamma[u;\Omega])}.$$

Here, for $g : \Omega \to \mathbf{R}$, the upper contact set for g in Ω is defined by

$$\Gamma[g;\Omega] = \{x \in \Omega \mid \exists \xi \in \mathbf{R}^n \text{ such that } g(y) \leq g(x) + \langle \xi, y - x\rangle \text{ for all } y \in \Omega\}.$$

The upper contact set plays an important role when we study the L^p-regularity theory although we will not treat it here.

We note that in the above, we only need u to be an L^p-viscosity subsolution of

$$\mathcal{P}^-(D^2u) - \mu(x)|Du| = f^+(x) \quad \text{in } \Omega, \tag{E'}$$

which L^p-viscosity subsolutions of (E) satisfy in the L^p-viscosity sense under $(A1), (A2), (A3)$.

This article is based on a series of joint works with A. Święch (Georgia Institute of Technology).

2 Motivation

The ABP maximum principle for viscosity solutions was first obtained by Caffarelli [1]. However, we had to assume that the inhomogeneous term $f \in L^n(\Omega)$ is continuous. Afterwards, introducing the notion of L^p-viscosity solutions, we can remove this restriction in [3].

Fok [5] first studied viscosity solutions when (E) may have unbounded coefficients to the first derivatives; μ in $(A2)$.

We shall explain why we impose $(A2)$ while in the literature, we have investigated the case of $\mu \equiv \gamma$ for a constant $\gamma \geq 0$.

Consider the follwoing quasilinear equation

$$-\text{trace}(A(x, Du)D^2u) + b(x, Du) = f(x), \tag{1}$$

where the symmetic matrix $A(x,\xi)$ is uniformly elliptic, and Lipschitz continuous in the second variable; there exists $\gamma \geq 0$ such that for $x \in \Omega$, $\xi,\eta \in \mathbf{R}^n$,

$$\lambda I \leq A(x,\xi) \leq \Lambda I, \quad \text{and} \quad \|A(x,\xi) - A(x,\eta)\| \leq \gamma|\xi - \eta|. \qquad (a0)$$

When we study the equation (1), it is difficult to use the standard perturbation technique introduced by Caffarelli in [1] and further developed in [3] since we only consider Pucci extremal equations

$$\mathcal{P}^{\pm}(D^2u) \pm \gamma|Du| = f(x).$$

Hence, we cannot prove basic properties of solutions of (1). However, using the L^p-strong solutions of

$$\mathcal{P}^{\pm}(D^2u) \pm \mu(x)|Du| = f(x),$$

where $\mu \in L^q(\Omega)$, we can proceed our theory.

Note that if A satisfies $(a0)$, then it follows that for $x \in \Omega$, $\xi,\eta \in \mathbf{R}^n$, $X, Y \in S^n$,

$$\begin{array}{rl} & \mathcal{P}^-(X-Y) - \gamma\min\{\|X\|,\|Y\|\}|\xi - \eta| \\ \leq & -\mathrm{trace}\{A(x,\xi)X - A(x,\eta)Y\} \\ \leq & \mathcal{P}^+(X-Y) + \gamma\min\{\|X\|,\|Y\|\}|\xi - \eta|. \end{array} \qquad (a1)$$

We suppose that B satisfies that for $x \in \Omega$, $\xi,\eta \in \mathbf{R}^n$,

$$|b(x,\xi) - b(x,\eta)| \leq \gamma|\xi - \eta|. \qquad (a2)$$

Example of A: We give an example of A which satisfies $(a0)$ with $\gamma = \lambda + \Lambda$ when $n = 2$.

$$A(x,\xi) := \frac{1}{|\xi|+1}\begin{pmatrix} \lambda|\xi| + \Lambda & 0 \\ 0 & \Lambda|\xi| + \lambda \end{pmatrix}.$$

We denote by $B_r(x)$ for $x \in \mathbf{R}^n$ and $r > 0$, the open ball with the center $x \in \mathbf{R}^n$ and the radius $r > 0$. We will also write B_r for $B_r(0)$.

Proposition 1. Let $A, A_k : \Omega \times \mathbf{R}^n \to S^n$ satisfy $(a1)$, and $b, b_k : \Omega \times \mathbf{R}^n \to \mathbf{R}$ satisfy $(a2)$, and let $f, f_k \in L^p(\Omega)$ for $k \in \mathbf{N}$ and $p > n$. Let $u_k \in C(\Omega)$ $(k \in \mathbf{N})$ be L^p-viscosity subsolutions of

$$-\mathrm{trace}(A_k(x,Du)D^2u) + b_k(x,Du) = f_k(x) \quad \text{in } \Omega.$$

Assume also that for every $B_{2r}(z) \subset \Omega$, $u_k \to u$ uniformly in $B_r(z)$ as $k \to \infty$, and for $\phi \in W^{2,p}(B_r(z))$

$$\lim_{k\to\infty} \|(G[\phi] - G_k[\phi])^+\|_{L^p(B_r(z))} = 0$$

where

$$G_k[\phi](x) = -\mathrm{trace}(A_k(x, D\phi(x))D^2\phi(x)) + b_k(x, D\phi(x)) - f_k(x),$$

and

$$G[\phi](x) = -\mathrm{trace}(A(x, D\phi(x))D^2\phi(x)) + b(x, D\phi(x)) - f(x).$$

Then, u is an L^p-viscosity subsolution of (1).

Proof. Suppose that u is not an L^p-viscosity subsolution of (1). Then, there exist $\hat{x} \in \Omega, r, \theta > 0, \varphi \in W^{2,p}(B_r(\hat{x}))$ such that $u - \varphi$ has a maximum at $\hat{x}$ over $B_{2r}(\hat{x}) \subset \Omega$ but

$$G[\varphi](x) \geq 2\theta \quad a.e. \text{ in } B_r(\hat{x}).$$

We may assume that $\hat{x} = 0 \in \Omega$ and $(u - \varphi)(0) = 0$. Setting $\psi(x) = \varphi(x) + \eta|x|^2$, where $\eta = \theta/(2n\Lambda)$, we have $u - \psi \leq -\eta r^2$ on ∂B_r.

In view of Theorem 7.1 in [8], by setting $h = 1 + \|D^2\varphi\|$, and $g_k = (G[\varphi] - G_k[\varphi])^+$, there exists an L^p-strong subsolution $w \in C(\overline{B}_r)$ of

$$\mathcal{P}^+(D^2w) + \gamma h(x)|Dw| = -2\eta\gamma h(x)|x| - g_k(x) \quad \text{in } B_r, \tag{2}$$

such that $w = 0$ on ∂B_r, and

$$0 \leq -w \leq Cr^{2-\frac{n}{p}}\left(r + \|g_k\|_{L^p(B_r)}\right) \quad \text{in } B_r.$$

Hence, for small r, there is $K(r) \in \mathbf{N}$ such that if $k \geq K(r)$, then $u_k - (\psi - w)$ has a local maximum at $x_k \in B_r$ such that $x_k \to 0$ as $k \to \infty$. Therefore, we have

$$ess \liminf_{x\to x_k} G_k[\psi - w](x) \leq 0. \tag{3}$$

On the other hand, we have

$$\begin{aligned} G_k[\psi - w](x) &\geq G_k[\varphi](x) - \mathcal{P}^+(D^2w) + \mathcal{P}^-(2\eta I) \\ &\quad -\gamma h(x)(|Dw(x)| + 2\eta|x|) \\ &\geq G_k[\varphi](x) + g_k(x) - 2n\eta\Lambda \\ &\geq G[\varphi](x) - \theta \\ &\geq \theta \qquad a.e. \text{ in } B_r, \end{aligned}$$

which contradicts to (3). *Q.E.D.*

In Proposition 1, the assumption $p > n$ is essential since we do not know if we have L^p-strong solution of (2) when $n = p$.

3 Main results

First, we recall Escauriaza's constant $\hat{p} \in [\frac{n}{2}, n)$ in [4].

Fix $p > \hat{p}$ (Notice that we may choose $p < n$). Then, there exist $\hat{C} > 0$ such that for any $g \in L^p(\Omega)$ and $h \in C(\partial\Omega)$, there exists an L^p-strong subsolution $u \in C(\overline{\Omega})$ of

$$\mathcal{P}^+(D^2u) = g(x) \quad \text{in } \Omega$$

such that $u = h$ on $\partial\Omega$,

$$-\hat{C}\|g^-\|_{L^p(\Omega)} - \min_{\partial\Omega} h \le u \le \max_{\partial\Omega} h + \hat{C}\|g^+\|_{L^p(\Omega)} \quad \text{in } \Omega,$$

and

$$\|u\|_{W^{2,p}(\Omega)} \le \hat{C}\left(\max_{\partial\Omega}|h| + \|g\|_{L^p(\Omega)}\right).$$

The last estimate was local one in [1, 2, 4]. However, recently, Winter in [9] established the global estimate.

Our ABP maximum principle for L^p-viscosity subsolutions of (E') is as follows:

Theorem 2. Assume $f \in L^p(\Omega)$ and $\mu \in L^q(\Omega)$. Let $u \in C(\overline{\Omega})$ be an L^p-viscosity subsolution of (E').
(i) If $q \ge p \ge n$ and $q > n$, then it follows that

$$\sup_\Omega u \le \sup_{\partial\Omega} u + C_0 e^{C_0\|\mu\|^n_{L^n(\Omega)}}\|f^+\|_{L^n(\Omega)},$$

where $C_0 > 0$ is from Proposition 1.
(ii) If $q > n > p > \hat{p}$, then it follows that

$$\sup_\Omega u \le \sup_{\partial\Omega} u + C_1\left\{e^{C_0\|\mu\|^n_{L^n(\Omega)}}\|\mu\|^N_{L^q(\Omega)} + \sum_{k=0}^{N-1}\|\mu\|^k_{L^q(\Omega)}\right\}\|f^+\|_{L^p(\Omega)}, \qquad (4)$$

where $C_1 = C_1(n, \lambda, \Lambda, p, q) > 0$ and $N = N(n, p, q) \in \mathbf{N}$ are universal constants.
(iii) If $q = n > p > \hat{p}$, and $\|\mu\|_{L^n(\Omega)} \leq \varepsilon_0$, then it follows that

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u + C_2 \|f^+\|_{L^p(\Omega)},$$

where $\varepsilon_0 = \varepsilon_0(n, \lambda, \Lambda, p) > 0$ and $C_2 = C_2(n, \lambda, \Lambda, p) > 0$ are universal constants.

Remark. It is possible to change $L^n(\Omega)$ by $L^n(\Gamma[u; \Omega])$ in (i).

We shall simply write $\|\cdot\|_p$ for $\|\cdot\|_{L^p(\Omega)}$. We will use $C > 0$ for various constants depending on known quantities.

Idea of proof. (i) For each $\delta > 0$, it is enough to find an L^p-strong subsolution $v := v_\delta \in C(\overline{\Omega})$ of

$$\mathcal{P}^+(D^2 v) + \mu(x)|Dv| = -\delta - f^+(x) \quad \text{in } \Omega \tag{5}$$

such that $v = 0$ on $\partial\Omega$, and $0 \leq -v \leq Ce^{C\|\mu\|_n^n}(\delta + \|f^+\|_n)$ in Ω. In fact, since $\mathcal{P}^-(X+Y) \leq \mathcal{P}^-(X) + \mathcal{P}^+(Y)$ holds for $X, Y \in S^n$, it is easy to verify that $w := u + v$ is an L^p-viscosity subsolution of

$$\mathcal{P}^-(D^2 w) - \mu(x)|Dw| = -\delta \quad \text{in } \Omega, \tag{6}$$

which yields

$$\sup_{\Omega} w = \sup_{\partial\Omega} w.$$

Because, if w attains its maximum at $x \in \Omega$, then the definition immediately gives a contradiction to (6).

Therefore, we have

$$\sup_{\Omega} u \leq \sup_{\Omega} w + \sup_{\Omega}(-v) \leq \sup_{\partial\Omega} u + C(\delta + \|f^+\|_n),$$

which shows our assertion by sending $\delta \to 0$.

In order to find the L^p-strong subsolution of (5), we approximate μ and f^+ by smooth functions μ_k and f_k, and let u_k be classical solutions of

$$\mathcal{P}^+(D^2 u_k) + \mu_k(x)|Du_k| = -\delta - f_k(x) \quad \text{in } \Omega \tag{7}$$

such that $u_k = 0$ on $\partial\Omega$. Notice that in view of Evans-Krylov estimates, we can find $u_k \in C(\overline{\Omega}) \cap C^2(\Omega)$ since $(\xi, X) \in \mathbf{R}^n \times S^n \to \mathcal{P}^+(X) + \mu_k(x)|\xi|$ are convex for any $x \in \Omega$.

We get L^∞ estimate by the classical ABP maximum principle with $\|\mu_k\|_n$ and $\|f_k\|_n$ while we can proceed L^p-estimates on D^2u_k by the argument in [3] with $\|\mu\|_q$ for $q > n$.

It is worth mentioning that our hypothesis $q > n$ is crucial to estimate $\mu|Du|$ terms.

(ii) In this case, we do not know the existence of L^p-strong subsolutions of (5) at this stage. Because we do not have L^∞ estimate for the associated approximate PDE (7). Instead, according to [4], we know the existence of L^p-strong subsolutions of

$$\mathcal{P}^+(D^2v) = -f^+(x) \quad \text{in } \Omega$$

such that $v = 0$ on $\partial\Omega$, $0 \le -v(x) \le C\|f^+\|_p$ in Ω, and

$$\|v\|_{W^{2,p}(\Omega)} \le C\|f^+\|_p.$$

Setting $w := u + v$, we verify that it is an L^p-viscosity subsolution of

$$\mathcal{P}^-(D^2w) - \mu(x)|Dw| = f_1(x) := \mu(x)|Dv(x)| \quad \text{in } \Omega.$$

Though we have non-zero inhomogeneous term f_1, it is easy to show

$$f_1 \in L^{p_1}(\Omega) \quad \text{for some } p_1 > p.$$

Since $Dv \in W^{1,p}(\Omega) \subset L^{p^*}(\Omega)$, where $p^* = np/(n-p)$, and $\mu \in L^q(\Omega)$ for $q > n$, we can take $p_1 = p^*q/(p^* + q)$.

If $p_1 \ge n$, then we may apply the assertion (i) to this w. If $p_1 < n$, then we can proceed this argument; find an L^{p_1}-strong subsolution v_1 of

$$\mathcal{P}^+(D^2v_1) = -f_1(x) \quad \text{in } \Omega$$

such that $v_1 = 0$ on $\partial\Omega$, and $0 \le -v_1 \le C\|f_1\|_{p_1} \le C\|\mu\|_q\|f^+\|_p$ in Ω.

By setting $w_1 := u + v + v_1$, it is easy to check that it is an L^p-viscosity subsolution of

$$\mathcal{P}^-(D^2w_1) - \mu(x)|Dw_1| = f_2(x) := \mu(x)|Dv_1(x)| \quad \text{in } \Omega.$$

Again, we can check that $f_2 \in L^{p_2}(\Omega)$ for $p_2 > p_1$. Continuing this procedure until $p_N \geq n$ for some integer N, we apply the assertion (i) to $w_N := u + v + v_1 + \cdots + v_N$.

(iii) Note that we cannot find an L^p-strong subsolution of (5) when $q > n > p > \hat{p}$. In this case, it seems hard to construct an L^p-strong subsolution of (5) in general. However, we can get enough estimates on (classical) solutions of (7) provided $\|\mu\|_n$ is small. More precisely, for nonnegative $\mu_k \in L^n(\Omega) \cap C^\infty(\Omega)$, and $g_k \in L^p(\Omega) \cap C^\infty(\Omega)$ such that $\|\mu - \mu_k\|_n \to 0$ and $\|g - g_k\|_p \to 0$ as $k \to \infty$, let $v_k \in C(\overline{\Omega}) \cap C^2(\Omega)$ be classical solutions of

$$\mathcal{P}^+(D^2 v_k) + \mu_k(x)|Dv_k| = g_k(x) \quad \text{in } \Omega$$

such that $v_k = 0$ on $\partial\Omega$.

Thanks to Caffarelli-Escauriaza's estimates, we have

$$\|D^2 v_k\|_p \leq C\|\mathcal{P}^+(D^2 v_k)\|_p \leq C\{\|f_k\|_p + \|\mu_k\|_n \|Dv_k\|_{p^*}\}.$$

On the other hand, we have

$$\|v_k\|_\infty \leq C\{\|f_k\|_p + \|\mu_k\|_n \|Dv_k\|_{p^*}\}.$$

These estimates imply $W^{2,p}$ estimates on v_k if $\|\mu_k\|_n$ is small enough. *Q.E.D.*

Remark. We obtained (i), (ii) in [7], and (iii) in [8].

4 Applications

The most important applications of the ABP maximum principle are the weak Harnack inequality, and L^p-regularity. However, we shall focus on the other applications here.

In this article, in what follows, we shall consider the case of

$$q > n > p > \hat{p}$$

since it is easier to obtain the corresponding results in the case when $q \geq p \geq n$ and $q > n$. We will discuss the remaining case (i.e. $q = n > p > \hat{p}$) in the future.

There are several expected properties whose proofs are not trivial when $q > n > p > \hat{p}$:

(P1) The existence of L^p-strong solutions of Pucci extremal equations (see (8) below) with unbounded coefficients to the first derivative.

(P2) L^p-strong solutions satisfy the ABP maximum principle.

(P3) L^p-strong solutions are L^p-viscosity solutions.

(P4) If L^p-viscosity solutions are $W^{2,p}_{\mathrm{loc}}(\Omega)$, then they are L^p-strong solutions.

In the proof of Theorem 2 (ii), given $g \in L^p(\Omega)$, we did not use the existence of L^p-strong subsolutions of

$$\mathcal{P}^+(D^2u) + \mu(x)|Du| = g(x) \quad \text{in } \Omega \tag{8}$$

such that $u = 0$ on $\partial\Omega$ because we could not prove it. However, since we have established the ABP maximum principle in case when $q > n > p > \hat{p}$, we can obtain enough estimates on the associated approximate solutions as in (i).

In the next proposition, we only show the existence of L^p-strong subsolutions by an approximation procedure. However, we will see that they are L^p-strong solutions by use of the ABP maximum principle for L^p-strong solutions, which is not trivial.

Proposition 3. For $q > n > p > \hat{p}$, let $g \in L^p(\Omega)$ and $\mu \in L^q(\Omega)$. Then, there exists an L^p-strong subsolution $u \in C(\overline{\Omega})$ of (8) such that $u = 0$ on $\partial\Omega$,

$$\|u\|_\infty \leq C_1 \left\{ e^{C_0\|\mu\|_n^n} \|\mu\|_q^N + \sum_{k=0}^{N-1} \|\mu\|_q^k \right\} \|g\|_p, \tag{9}$$

and

$$\|u\|_{W^{2,p}(\Omega)} \leq C_3 \|g\|_p, \tag{10}$$

where $C_3 = C_3(n, \lambda, \Lambda, p, q, \|\mu\|_q) > 0$ is a universal constant.

Proof. Let g_k and μ_k be smooth functions such that

$$\lim_{k\to\infty} \|g_k - g\|_p = \lim_{k\to\infty} \|\mu_k - \mu\|_q = 0.$$

Let $u_k \in C(\overline{\Omega}) \cap C^2(\Omega)$ be classical solutions of

$$\mathcal{P}^+(D^2u_k) + \mu_k(x)|Du_k| = g_k(x) \quad \text{in } \Omega \tag{11}$$

such that $u_k = 0$ on $\partial\Omega$.

We remark that here is not the place to apply the classical ABP maximum principle with $\|g_k\|_p$ (not $\|g_k\|_n$!). Instead, we use the same argument as in the proof of (ii) in Theorem 2 to obtain

$$\sup_\Omega u_k \le C_1 \left\{ e^{C_0\|\mu_k\|_n^n} \|\mu_k\|_q^N + \sum_{j=1}^{N-1} \|\mu_k\|_q^k \right\} \|g_k^+\|_p. \tag{12}$$

By noting that $w := -u_k$ is a classical subsolution of

$$\mathcal{P}^-(D^2 w) - \mu_k(x)|Dw| = g_k^-(x) \quad \text{in } \Omega,$$

(12) holds for $-u_k$ with g_k^- in place of g_k^+. Hence, we have L^∞ estimate on u_k.

Once we have L^∞ estimates, it is rather standard to show the $W^{2,p}$ estimates on u_k as in [7]. To this end, we have to suppose $\|\mu_k\|_q$ is small enough. However, this restriction can be removed by considering this procedure in small subdomains (both inside and near the boundary of Ω).

Because of $p > \frac{n}{2}$, we can see that there exists $u \in W^{2,p}(\Omega) \cap C(\overline{\Omega})$ such that $u_k \to u$ uniformly in $\overline{\Omega}$, $Du_k \to Du$ strongly in L^p, and $D^2 u_k \to D^2 u$ weakly in L^p as $k \to \infty$ along a subsequence if necessary. Now it is easy to show that this u has desired properties. *Q.E.D.*

Remark. In [7], we first prove (P1) and then, (P2) because we only knew $W^{2,p}_{\mathrm{loc}}$ estimate on u_k. In fact, we can only prove that $u_k \to u$ uniformly in K for each compact set $K \subset \Omega$ by the local $W^{2,p}$ estimates. Thus, we do not know if $u = 0$ holds on $\partial\Omega$. To recover this difficulty, in [7], we first showed (P2) to derive that u_k is a Cauchy sequence in $L^\infty(\Omega)$. See also [3].

Now, we present a proof of (P2).

Proposition 4. For $q > n > p > \hat{p}$, let $f \in L^p(\Omega)$ and $\mu \in L^q(\Omega)$. If $u \in C(\overline{\Omega})$ is an L^p-strong subsolution of (E'), then (4) holds true.

Proof. We shall recall the proof of Theorem 2.5 in [7] for the reader's convenience since it is not very clearly written in [7].

First, we approximate u by $u_k \in C^2(\Omega) \cap C(\overline{\Omega})$ such that

$$\lim_{k\to\infty} \|u - u_k\|_{L^\infty(\Omega)} = \lim_{k\to\infty} \|u - u_k\|_{W^{2,p}(K)} = 0$$

for each compact set $K \subset \Omega$, and

$$\lim_{k\to\infty}\{\mathcal{P}^+(D^2u_k(x))+\mu(x)|Du_k(x)|\} = \mathcal{P}^+(D^2u(x))+\mu(x)|Du(x)| \quad a.e. \text{ in } \Omega.$$

As in the proof of (ii) of Theorem 2, we have

$$\sup_K u_k \le \sup_{\partial K} u_k + C_1\left\{e^{C_0\|\mu\|_n^n}\|\mu\|_q^N + \sum_{j=1}^{N-1}\|\mu\|_q^j\right\}\|f_k^+\|_p,$$

where $f_k = \mathcal{P}^+(D^2u_k) + \mu(x)|Du_k|$. Note that C_i does not depend on K.

Sending $k \to \infty$ in the above, we can finish the proof because the compact set $K \subset \Omega$ is arbitrary.

We give a proof of (P3).

Proposition 5. For $q > n > p > \hat{p}$, suppose that $(A1)$, $(A2)$ and $(A3)$ hold. If $u \in C(\Omega)$ is an L^p-strong subsolution (resp., supersolution) of (E), then it is an L^p-viscosity subsolution (resp., supersolution) of (E).

Proof. Suppose the contrary; there are $\phi \in W^{2,p}_{\text{loc}}(\Omega)$, $\hat{r} > 0$ and $\theta > 0$ such that $u - \phi$ takes its maximum at $\hat{x} \in B_{2\hat{r}}(\hat{x}) \subset \Omega$, and

$$F(x, D\phi(x), D^2\phi(x)) \ge f(x) + \theta \quad a.e. \text{ in } B_{\hat{r}}(\hat{x}).$$

We may suppose that $\hat{x} = 0 \in \Omega$, and $(u - \phi)(0) = 0$.

Setting $\psi(x) = \phi(x) + \eta|x|^2$, where $\eta = \theta/(2n\Lambda)$, we note that $u - \psi$ takes its maximum at 0. Hence, we have

$$\theta + f(x) \le \mathcal{P}^+(-2\eta I) + 2\eta|x|\mu(x) + F(x, D\psi(x), D^2\psi(x)) \quad a.e. \text{ in } B_{\hat{r}}.$$

Since u is an L^p-strong subsolution of (E), by noting $\mathcal{P}^+(-2\eta I) = \theta$, $(A1)$ and $(A2)$ yield

$$\mathcal{P}^-(D^2(u-\psi)(x)) - \mu(x)|D(u-\psi)(x)| \le 2\eta|x|\mu(x) \quad a.e. \text{ in } B_{\hat{r}}.$$

The scaled version of the ABP maximum principle for L^p-strong subsolutions yields

$$0 = \sup_{B_r}(u-\psi) \le \sup_{\partial B_r}(u-\psi) + Cr^{3-\frac{n}{p}}\|\mu\|_{L^p(B_r)}$$

for $0 < r \le \hat{r}$. Since $\sup_{\partial B_r}(u-\psi) \le -\eta r^2$ and $\|\mu\|_{L^p(B_r)} \le \tilde{C} r^{\frac{n}{p}-\frac{n}{q}} \|\mu\|_{L^q(B_r)}$, the above implies

$$0 \le -\eta r^2 + \tilde{C} C r^{3-\frac{n}{q}},$$

which is a contradiction for small $r > 0$. *Q.E.D.*

In the end, we shall show that the limit u in the proof of Proposition 3 is indeed an L^p-strong solution of (8). For this purpose, we recall a stability result.

Proposition 6. For $q > n > p > \hat{p}$, let $F, F_k : \Omega \times \mathbf{R}^n \times S^n \to \mathbf{R}$ satisfy $(A1), (A2)$ and $(A3)$ with $\mu, \mu_k \in L^q(\Omega)$, respectively. Let $f, f_k \in L^p(\Omega)$. Suppose that $u_k \in C(\Omega)$ be an L^p-viscosity subsolution (resp., supersolution) of

$$F_k(x, Du_k, D^2 u_k) = f_k(x) \quad \text{in } \Omega$$

such that $u_k \to u$ uniformly in each compact set $K \subset \Omega$ as $k \to \infty$, and

$$\lim_{k\to\infty} \|(G[\phi] - G_k[\phi])^+\|_{L^p(B_r(x))} = 0$$

$$\left(\text{resp., } \lim_{k\to\infty} \|(G[\phi] - G_k[\phi])^-\|_{L^p(B_r(x))} = 0\right)$$

provided that $B_{2r}(x) \subset \Omega$ and $\phi \in W^{2,p}(B_r(x))$, where

$$G[\phi](y) = F(y, D\phi(y), D^2\phi(y)) - f(y)$$

and

$$G_k[\phi](y) = F_k(y, D\phi(y), D^2\phi(y)) - f_k(y).$$

Then, u is an L^p-viscosity subsolution (resp., supersolution) of (E).

Since we may prove this assertion by the argument in the proof of Proposition 1 with minor changes, we leave it to the readers. See [7] for the details.

Proposition 7. For $q > n > p > \hat{p}$, let $g \in L^p(\Omega)$ and $\mu \in L^q(\Omega)$. Then, there exists an L^p-strong solution $u \in C(\overline{\Omega})$ of (8) such that $u = 0$ on $\partial\Omega$, (9) and (10).

Proof. It is sufficient to show that u constructed in the proof of Proposition 3 is indeed an L^p-strong supersolution of (8).

Since u_k is an L^p-strong supersolution of (11), it is also an L^p-viscosity supersolution of (11) due to Proposition 5. Now, in view of Proposition 6, it is easy to verify that the limit u is an L^p-viscosity supersolution of (8). Thus, in view of Proposition 9.1 in [7], it turns out that u is an L^p-strong supersolution of (8). *Q.E.D.*

References

[1] L. A. Caffarelli, Interior a priori estimates for solutions of fully non-linear equations, Ann. Math. (2), **130** (1) (1989), 189-213.

[2] L. A. Caffarelli and X. Cabré, Fully Nonlinear Elliptic Equations, American Mathematical Society Colloquium Publications, **43**, Providence, 1995.

[3] L. A. Caffarelli, M. G. Crandall, M. Kocan and A. Święch, On viscosity solutions of fully nonlinear equations with measurable ingredients, Comm. Pure Appl. Math., **49** (4) (1996), 365-397.

[4] L. Escauriaza, $W^{2,n}$ a priori estimates for solutions to fully non-linear equations, Indiana Univ. Math. J., **42** (2) (1993), 413-423.

[5] P. Fok, Some maximum principles and continuity estimates for fully nonlinear elliptic equations of second order, Ph.D. Thesis, UCSB, 1996.

[6] S. Koike and A. Święch, Maximum principle for fully nonlinear equations via the iterated comparison function methods, Math. Ann., **339** (2) (2007), 461-484.

[7] S. Koike and A. Święch, Weak Harnack inequality for fully nonlinear uniformly elliptic PDE with unbounded ingredients, J. Math. Soc. Japan, **61** (3) (2009), 723-755.

[8] S. Koike and A. Święch, Local maximum principle for L^p-viscosity solutions of fully nonlinear PDEs with unbounded ingredients, to appear in Comm. Pure Appl. Anal.

[9] N. Winter, $W^{2,p}$ and $W^{1,p}$-estimates at the boundary for solutions of fully nonlinear, uniformly elliptic equations, Z. Anal. Anzwend., **28** (2) (2009), 129-164.