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A Consideration on Functions Preserving Set Inclusion Relation

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Abstract– *This paper discusses functions over the set of non-empty subsets of* $\{0, 1, \dots, r-1\}$ *that are monotonic in the set inclusion relation. Min, Max and Literal operations play an important role in multiple-valued logic design/circuits because they can realize any function over* $\{0, 1, \dots, r-1\}$. *Operations over the set of non-empty subsets of* $\{0, 1, \dots, r-1\}$ *that preserve the set inclusion relation are introduced from Min, Max and Literal operations over* $\{0, 1, \dots, r-1\}$. *Then, this paper proves some of mathematical properties of functions over the set of non-empty subsets of* $\{0, 1, \dots, r-1\}$ *that are composed of the operations introduced.*

Keywords: Multiple-Valued Logic Design/Circuits, Set Inclusion Relation, Clone Theory

1 Introduction

S. C. Kleene [1] first introduced regularity into ternary operations over the set of truth values $\{0, 1, \mathrm{u}\}$ in the following way.

> A truth table for a ternary operation is *regular* if it satisfies the condition that "*A given column (row) contains* 1 *in the* u *row (column), only if the column (row) consists entirely of* 1*'s; and likewise for* 0".

Kleene's regularity is one of the ways how binary operations can be expanded into ternary operations. Table 1 is the truth tables of regular ternary operations, which are given from the traditional binary operations AND, OR and NOT.

It is worth to notice that M. Goto [2] independently introduced ternary operations that are identical with the Kleene's ternary operations in Table 1. He showed that the ternary operations can be a model for analyzing undetermined behavior existing in binary systems, such as hazards in binary logic circuits. After Goto's work, M. Mukaidono studied mathematical properties of functions over $\{0, 1, \mathrm{u}\}$ that can be expressed by a formula composed of the three ternary operations (He called the ternary functions regular ternary logic functions). One of Mukaidono's main results[3] is that a function f over $\{0, 1, \mathrm{u}\}$ is a regular ternary logic function if and only if the function f is monotonic in the partial ordered relation, defined by Figure 1. I. G. Rosenberg [8] indicated that the set of regular ternary logic functions is this clone generated by the Kleene's ternary logic, i.e., the clone is identical with the clone over the 3-element universe $\{\{0\}, \{1\}, \{0, 1\}\}$ that preserves the set inclusion relation $\subseteq$.

This paper discusses functions over the set of non-empty subsets of $\{0, 1, \dots, r-1\}$ when r is more than 2. In the following, E_r and P_r denote the r-valued set $\{0, 1, \dots, r-1\}$ and the set of non-empty subsets of E_r, respectively.

Table 1: Truth Tables of Regular Ternary Operations NOT, AND and OR

	NOT	AND			OR		
		0	1	u	0	1	u
0	1	0	0	0	0	1	u
1	0	0	1	u	1	1	1
u	u	0	u	u	u	1	u

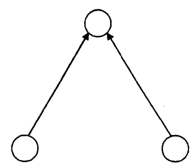

Figure 1: Partial Ordered Relation on $\{0, 1, \mathrm{u}\}$

First, this paper shows a definition for expanding operations over E_r into operations over P_r. This definition is identical with the Kleene's regularity when r is equal to 2, and it has already been shown by M. Mukaidono [4] and I. G. Rosenberg [8]. Min, Max, and Literal operations play an important role in multiple-valued logic design/circuits, because they can realize any multiple-valued logic function over E_r. Therefore, Min, Max, and Literal operations are focused on in this paper. This paper then clarifies mathematical properties of functions over P_r, which are expressed by formulas composed of the operations given from Min, Max, and Literal operations over E_r.

This paper is organized below. Section 2 is for preliminaries. This section shows the definition for expanding operations over E_r into operations over P_r, and then gives some of their mathematical properties. Section 3 focuses on Min, Max, and Delta Literal operations over E_r. They are expanded into operations over P_r, and then this section proves a necessary and sufficient condition for a function over P_r to be expressed by a formula composed of these operations. Section 4 shows examples for the results obtained in Section 3. Section 5 discusses mathematical properties of functions over P_r when we selected Min, Max, and Universal Literal operations over E_r. Then, Section 6 gives examples for the results appeared in Section 5 Section 7 concludes the paper.

2 Preliminaries

Let E_r be the r-valued set $\{0, \ldots, r-1\}$, and let P_r be the set of all non-empty subsets of E_r, i.e., $P_r = 2^{E_r} - \{\emptyset\}$, where 2^{E_r} is the power set of E_r. If a subset of E_r consists of only one element, then it is called a singleton. The set of all singletons of E_r is denoted by S_r, i.e., $S_r = \{\{0\}, \ldots, \{r-1\}\}$. It is evident that the set P_r is a partial ordered set in the set inclusion $\subseteq$. In this paper, elements of the set E_r are denoted by small letters such as a, b, c, x, y, etc., while elements of the set P_r (i.e., non-empyt subsets of E_r) are denoted by capital letters such as A, B, C, X, Y etc.

Definition 1 Let o be an n-ary operation on E_r. Then, an n-ary operation $\hat{o}$ on P_r with respect to o is defined by setting

$$\hat{o}(A_1, \ldots, A_n) = \{o(a_1, \ldots, a_n) \mid a_1 \in A_1, \ldots, a_n \in A_n\}$$

for any element $(A_1, \ldots, A_n) \in P_r^n$. (End of Definition)

The following three operations play an important role in multiple-valued logic design because r-valued functions consisting of these operations and the constants $0, \ldots, r-1$ are

Table 2: Truth Table of $\wedge$

$X\backslash Y$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{01}$	$\underline{02}$	$\underline{12}$	$\underline{012}$
$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$
$\underline{1}$	$\underline{0}$	$\underline{1}$	$\underline{1}$	$\underline{01}$	$\underline{01}$	$\underline{1}$	$\underline{01}$
$\underline{2}$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{01}$	$\underline{02}$	$\underline{12}$	$\underline{012}$
$\underline{01}$	$\underline{0}$	$\underline{01}$	$\underline{01}$	$\underline{01}$	$\underline{01}$	$\underline{01}$	$\underline{01}$
$\underline{02}$	$\underline{0}$	$\underline{01}$	$\underline{02}$	$\underline{01}$	$\underline{02}$	$\underline{012}$	$\underline{012}$
$\underline{12}$	$\underline{0}$	$\underline{1}$	$\underline{12}$	$\underline{01}$	$\underline{012}$	$\underline{12}$	$\underline{012}$
$\underline{012}$	$\underline{0}$	$\underline{01}$	$\underline{012}$	$\underline{01}$	$\underline{012}$	$\underline{012}$	$\underline{012}$

Table 3: Truth Table of $\sqcup$

$X\backslash Y$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{01}$	$\underline{02}$	$\underline{12}$	$\underline{012}$
$\underline{0}$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{01}$	$\underline{02}$	$\underline{12}$	$\underline{012}$
$\underline{1}$	$\underline{1}$	$\underline{1}$	$\underline{2}$	$\underline{1}$	$\underline{12}$	$\underline{12}$	$\underline{12}$
$\underline{2}$	$\underline{2}$	$\underline{2}$	$\underline{2}$	$\underline{2}$	$\underline{2}$	$\underline{2}$	$\underline{2}$
$\underline{01}$	$\underline{01}$	$\underline{1}$	$\underline{2}$	$\underline{01}$	$\underline{012}$	$\underline{12}$	$\underline{012}$
$\underline{02}$	$\underline{02}$	$\underline{12}$	$\underline{2}$	$\underline{012}$	$\underline{02}$	$\underline{12}$	$\underline{012}$
$\underline{12}$	$\underline{12}$	$\underline{12}$	$\underline{2}$	$\underline{12}$	$\underline{12}$	$\underline{12}$	$\underline{12}$
$\underline{012}$	$\underline{012}$	$\underline{12}$	$\underline{2}$	$\underline{012}$	$\underline{012}$	$\underline{12}$	$\underline{012}$

Table 4: Truth Table of X^S

X	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{01}$	$\underline{02}$	$\underline{12}$	$\underline{012}$
$X^{\underline{0}}$	$\underline{2}$	$\underline{0}$	$\underline{0}$	$\underline{02}$	$\underline{02}$	$\underline{0}$	$\underline{02}$
$X^{\underline{1}}$	$\underline{0}$	$\underline{2}$	$\underline{0}$	$\underline{02}$	$\underline{0}$	$\underline{02}$	$\underline{02}$
$X^{\underline{2}}$	$\underline{0}$	$\underline{0}$	$\underline{2}$	$\underline{0}$	$\underline{02}$	$\underline{02}$	$\underline{02}$
$X^{\underline{01}}$	$\underline{2}$	$\underline{2}$	$\underline{0}$	$\underline{2}$	$\underline{02}$	$\underline{02}$	$\underline{02}$
$X^{\underline{02}}$	$\underline{2}$	$\underline{0}$	$\underline{2}$	$\underline{02}$	$\underline{2}$	$\underline{02}$	$\underline{02}$
$X^{\underline{12}}$	$\underline{0}$	$\underline{2}$	$\underline{2}$	$\underline{02}$	$\underline{02}$	$\underline{2}$	$\underline{02}$
$X^{\underline{012}}$	$\underline{2}$	$\underline{2}$	$\underline{2}$	$\underline{2}$	$\underline{2}$	$\underline{2}$	$\underline{2}$

functionally complete on the set E_r [5].

$$\begin{aligned} a \cdot b &= \min(a, b), \\ a + b &= \max(a, b), \\ x^S &= \begin{cases} r-1 & \text{if } x \in S \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

where $a, b \in E_r$ and $S \subseteq E_r$. The unary operations x^S are often called the universal literals. However, when S is a singleton, x^S is sometimes called a delta literal.

For simplicity, in writing elements of P_r, we will remove brackets and put an underline if no confusion arises. That is, for example, $\underline{0}$, $\underline{02}$ and $\underline{012}$ stand for $\{0\}$, $\{0,2\}$ and $\{0,1,2\}$, respectively. Tables 2, 3 and 4 are truth tables of operations on P_3 with respect to $\cdot$, $+$ and x^S, respectively. Because this paper focuses on the operations on P_r with respect to $\cdot$, $+$ and x^S, they are denoted by $\wedge$, $\sqcup$ and X^S, respectively. [1]

This paper does not allow any kinds of compositions of the operations $\wedge$, $\sqcup$ and X^S on P_r. Compositions are restricted by the form of the formulas defined below.

Definition 2 Formulas are defined inductively in the following way.

(1) Constants $\{0\}, \ldots, \{r-1\}$ and literals X_i^S ($i = 1, \ldots, n$ and $S \in P_r$) are formulas.

(2) If G and H are formulas, then $(G \wedge H)$ and $(G \sqcup H)$ are also formulas.

(3) It is a formula if and only if we get it from (1) and (2) in a finite number of steps.

(End of Definition)

[1] The operations $\wedge$ and $\sqcup$ do not satisfy the absorption laws and the distributive laws. Thus, the algebraic system $(P_r, \wedge, \sqcup)$ do not form a lattice.

In writing formulas, we sometimes omit the operation $\wedge$ for simplicity.

It is evident that every formula expresses a function on P_r when each variable X_i takes an element of P_r. Furthermore, it is easy to verify that the formulas can not express all of the functions on P_r, i.e., the functions on P_r expressed by the formulas are not functionally complete on P_r. Thus, one of the main subjects of the paper is to clear what functions on P_r can be expressed by the formulas.

In the following, for any elements $(A_1, \dots, A_n)$ and $(B_1, \dots, B_n)$ of P_r^n, $(A_1, \dots, A_n) \subseteq (B_1, \dots, B_n)$ stands for $A_i \subseteq B_i$ for all i's. Moreover, $(A_1, \dots, A_n) \cap (B_1, \dots, B_n) = \emptyset$ stands for $A_i \cap B_i = \emptyset$ for some i.

Theorem 1 [2] Suppose a function f on P_r can be expressed by a formula. Then, $f(A_1, \dots, A_n) \in S_r$ holds for any element $(A_1, \dots, A_n) \in S_r^n$.

Theorem 2 Suppose a function f on P_r can be expressed by a formula. Then, $f(A_1, \dots, A_n) \subseteq f(B_1, \dots, B_n)$ holds for any elements $(A_1, \dots, A_n)$ and $(B_1, \dots, B_n)$ of P_r^n such that $(A_1, \dots, A_n) \subseteq (B_1, \dots, B_n)$.

3 Functions Expressed by Formulas Composed of $\wedge$, $\sqcup$, and Delta Literals

This section shows a necessary and sufficient condition for functions on P_r that can be expressed by formulas with the operations $\wedge$, $\sqcup$ and delta literals.

Theorem 3 Let $A_1, \dots, A_{i-1}, A_{i+1}, \dots, A_n$ be elements of P_r. If a function f on P_r is expressed by a formula, then the least element of $f(A_1, \dots, A_{i-1}, A, A_{i+1}, \dots, A_n)$ (which is a subset of E_r) is equal to the least element of $f(A_1, \dots, A_{i-1}, B, A_{i+1}, \dots, A_n)$ for any elements A and B of $P_r - S_r$, i.e.,

$$\min f(a_1, \dots, A_{i-1}, A, A_{i+1}, \dots, A_n) = \min f(A_1, \dots, A_{i-1}, B, A_{i+1}, \dots, A_n)$$

holds for any elements A and B of $P_r - S_r$.

From Theorems 1, 2 and 3, any function f on P_r expressed by a formula satisfies the following Condition A.

Condition A: Let f be a function on P_r.

(1) If $(A_1, \dots, A_n) \in S_r^n$, then $f(A_1, \dots, A_n) \in S_r$.

(2) For any elements $(A_1, \dots, A_n)$ and $(B_1, \dots, B_n)$ of P_r^n, $(A_1, \dots, A_n) \subseteq (B_1, \dots, B_n)$ implies $f(A_1, \dots, A_n) \subseteq f(B_1, \dots, B_n)$.

(3) Let $A_1, \dots, A_{i-1}, A_{i+1}, \dots, A_n$ be elements of P_r. Then, the least element of $f(A_1, \dots, A_{i-1}, A, A_{i+1}, \dots, A_n)$ is equal to the least element of $f(A_1, \dots, A_{i-1}, B, A_{i+1}, \dots, A_n)$ for any elements A and B of $P_r - S_r$, i.e.,

$$\min f(A_1, \dots, A_{i-1}, A, A_{i+1}, \dots, A_n) = \min f(A_1, \dots, A_{i-1}, B, A_{i+1}, \dots, A_n)$$

holds for any elements A and B of $P_r - S_r$.

[2] All of the proofs in this paper are omitted because of the limitation of the space.

In the remainder of this section, it is proven that Condition A is a necessary and sufficient condition for a function on P_r to be expressed by a formula with the operations $\wedge$, $\sqcup$, and delta literals.

Definition 3 Let f be a function on P_r, and let $A = (A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n)$ be an element of S_r^{n-1}. Then, we define one-variable functions $\check{f}_A^i(X)$ and $\hat{f}_A^i(X)$ $(i = 1, \ldots, n)$ expressed by the following formulas.

$$\check{f}_A^i(X) = \bigsqcup_{s \in E_r} \left\{ \{s\} \wedge \bigsqcup_{B \in \check{P}_A^i(s)} X^B \right\}, \tag{1}$$

where $\check{P}_A^i(s)$ is the set of all maximal elements of the set

$$P_A^i(s) = \{B \in P_r \mid \min f(A_1, \ldots, A_{i-1}, B, A_{i+1}, \ldots, A_n) = s\}, \tag{2}$$

and

$$\hat{f}_A^i(X) = \bigsqcup_{S \in P_r - S_r} \left(\bigsqcup_{t \in S} \left[\{t\} \wedge \left\{ \bigsqcup_{B \in \hat{Q}_A^i(S)} \left(\bigwedge_{e \in B} X^{\{e\}} \right) \right\} \right] \right), \tag{3}$$

where $\hat{Q}_A^i(S)$ is the set of all minimal elements of

$$Q_A^i(S) = \{B \in P_r - S_r \mid f(A_1, \ldots, A_{i-1}, B, A_{i+1}, \ldots, A_n) = S\}. \tag{4}$$

In the formulas (1) and (3), if $\check{P}_A^i(s)$ and $\hat{Q}_A^i(S)$ are the empty set, then

$$\bigsqcup_{B \in \check{P}_A^i(s)} X^B \text{ and } \bigsqcup_{B \in \hat{Q}_A^i(S)} \left(\bigwedge_{k \in B} X^{\{k\}} \right)$$

are defined as $\{0\}$, respectively. Moreover, in the formula (1), when $B = E_r$, then X^B is the constant $\{r-1\}$. (End of Definition)

In the formula (1), if f is a function satisfying Condition A, then any subset $\check{P}_A^i(s)$ is a subset of S_r, or it is equal to $\{E_r\}$. Now, let us show this property. Suppose an element B of $P_r - S_r$ is a member of $P_A^i(s)$. Then, by Condition A(3), it follows that E_r is also a member of $P_A^i(s)$. Therefore, when an element of $P_r - S_r$ is a member of $P_A^i(s)$, then E_r is also a member of $P_A^i(s)$. This fact implies that $\check{P}_A^i(s)$ is a subset of S_r, or it is equal to $\{E_r\}$. So, the formula (1) is well-defined when f is a function satisfying Condition A.

Lemma 1 Let $A = (A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n)$ be an element of S_r^{n-1}. Then, for a function f satisfying Condition A,

$$\check{f}_A^i(B) = \begin{cases} f(A_1, \ldots, A_{i-1}, B, A_{i+1}, \ldots, A_n) & \text{if } f(A_1, \ldots, A_{i-1}, B, A_{i+1}, \ldots, A_n) \in S_r \\ K & \text{otherwise} \end{cases}$$

holds for any element B of P_r, where K is an element of P_r such that

$$\{f_0\} \subseteq K \subseteq f(A_1, \ldots, A_{i-1}, B, A_{i+1}, \ldots, A_n)$$

and f_0 is the least element of $f(A_1, \ldots, A_{i-1}, B, A_{i+1}, \ldots, A_n)$. (End of Lemma)

Lemma 2 Let $A = (A_1, \dots, A_{i-1}, A_{i+1}, \dots, A_n)$ be an element of S_r^{n-1}. Then, for a function f satisfying Condition A,

$$\hat{f}_A^i(B) = \begin{cases} \{0\} & \text{if } f(A_1, \dots, A_{i-1}, B, A_{i+1}, \dots, A_n) \in S_r \\ \{0\} \cup f(A_1, \dots, A_{i-1}, B, A_{i+1}, \dots, A_n) & \text{otherwise} \end{cases}$$

holds for any element $B \in P_r$. (End of Lemma)

Lemma 3 Let $A = (A_1, \dots, A_{i-1}, A_{i+1}, \dots, A_n)$ be an element of S_r^{n-1}. Then, for a function f satisfying Condition A,

$$\check{f}_A^i(B) \sqcup \hat{f}_A^i(B) = f(A_1, \dots, A_{i-1}, B, A_{i+1}, \dots, A_n)$$

holds for any element $B \in P_r$.

In the following, this section proves that any function satisfying Condition A can be expressed by a formula, and also shows a method how a formula can be formulated by a function satisfying Condition A.

Definition 4 Let f be a function on P_r. Then, f_1 is defined as a function on P_r expressed by the following formula.

$$f_1(X_1, \dots, X_n) = \bigsqcup_{i=1}^{n} f^i(X_1, \dots, X_n), \tag{5}$$

where

$$f^i(X_1, \dots, X_n) = \bigsqcup_{A=(A_1,\dots,A_{i-1},A_{i+1},\dots,A_n)\in S_r^{n-1}} \left(\bigwedge_{j=1(j\neq i)}^{n} X_j^{A_j} \wedge \left(\check{f}_A^i(X_i) \sqcup \hat{f}_A^i(X_i) \right) \right).$$

(End of Definition)

Here, let us introduce a subset of P_r^n, which will be denoted by $I(r,n)$, below.

$$I(r,n) = \bigcup_{i=1}^{n} \{(A_1, \dots, A_n) \in P_r^n \mid A_i \in P_r - S_r \text{ and } A_1, \dots, A_{i-1}, A_{i+1}, \dots, A_n \in S_r\}$$

That is, each element $(A_1, \dots, A_n)$ of $I(r,n)$ consists of elements of S_r, but except for one.

Lemma 4 Let F be a function satisfying Condition A. Then,

$$f_1(A_1, \dots, A_n) = \begin{cases} f(A_1, \dots, A_n) & \text{if } f(A_1, \dots, A_n) \in S_r \cup I(r,n) \\ K & \text{otherwise} \end{cases}$$

holds for any element $(A_1, \dots, A_n) \in P_r^n$, where K is an element of P_r such that $\{0\} \subseteq K \subseteq \{0\} \cup f(A_1, \dots, A_n)$. (End of Lemma)

Definition 5 Let f be a function on P_r, let S be an element of $P_r - S_r$, and let $\hat{T}(f,S)$ is the set of all minimal elements of the following subset of P_r^n.

$$T(f,S) = \{(A_1,\dots,A_n) \in P_r^n \mid f(A_1,\dots,A_n) = S \text{ and } (A_1,\dots,A_n) \notin S_r^n \cup I(r,n)\} \quad (6)$$

Then, f_2 is defined as a function on P_r expressed by the following formula.

$$f_2(X_1,\dots,X_n) = \{s_0\} \sqcup \left[\bigsqcup_{S\in P_r - S_r} \left\{ \bigsqcup_{t\in S} \{t\} \wedge f_S(X_1,\dots,X_n) \right\} \right], \quad (7)$$

where

$$f_S(X_1,\dots,X_n) = \begin{cases} \displaystyle\bigsqcup_{(A_1,\dots,A_n)\in \hat{T}(f,S)} \left\{ \bigwedge_{b\in A_1} X_1^{\{b\}} \wedge \cdots \wedge \bigwedge_{b\in A_n} X_n^{\{b\}} \right\} & \text{if } \hat{T}(f,S) \neq \emptyset \\ \{0\} & \text{otherwise} \end{cases} \quad (8)$$

and s_0 is the least element of $\displaystyle\bigcup_{(A_1,\dots,A_n)\in P_r^n} f(A_1,\dots,A_n)$. (End of Definition)

Lemma 5 Let f be a function on P_r satisfying Condition A. Then,

$$f_2(A_1,\dots,A_n) = \begin{cases} \{s_0\} & \text{if } (A_1,\dots,A_n) \in S_r^n \cup I(r,n) \\ f(A_1,\dots,A_n) & \text{otherwise,} \end{cases}$$

holds for any element $(A_1,\dots,A_n) \in P_r^n$, where s_0 is the least element of the union $\displaystyle\bigcup_{(A_1,\dots,A_n)\in P_r^n} f(A_1,\dots,A_n)$. (End of Lemma)

Theorem 4 Let f be a function on P_r satisfying Condition A. Then,

$$f(A_1,\dots,A_n) = f_1(A_1,\dots,A_n) \sqcup f_2(A_1,\dots,A_n)$$

holds for any element $(A_1,\dots,A_n) \in P_r^n$, where f_1 and f_2 are the formulas (5) and (7), respectively. (End of Theorem)

4 Examples of Functions Satisfying Condition A

Consider the function f on P_3 whose truth table is given in Table 5. It is not difficult to verify that f satisfies Condition A. Then, this section illustrates how we can form the formula that expresses the function f.

Example 1 Let us first consider the formulas (1) and (3). It follows by Eq. (2) that we have the following three subsets of P_3.

$$\begin{aligned} P_{\underline{0}}^1(0) &= \{B \in P_3 | \min f(B,\underline{0}) = 0\} = \{\underline{0}, \underline{2}, \underline{01}, \underline{02}, \underline{12}, \underline{012}\} \\ P_{\underline{0}}^1(1) &= \{B \in P_3 | \min f(B,\underline{0}) = 1\} = \{\underline{1}\} \\ P_{\underline{0}}^1(2) &= \{B \in P_3 | \min f(B,\underline{0}) = 2\} = \emptyset \end{aligned}$$

Table 5: Example of Function f Satisfying Condition A

$X \backslash Y$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{01}$	$\underline{02}$	$\underline{12}$	$\underline{012}$
$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{02}$	$\underline{02}$
$\underline{1}$	$\underline{1}$	$\underline{1}$	$\underline{2}$	$\underline{1}$	$\underline{12}$	$\underline{12}$	$\underline{12}$
$\underline{2}$	$\underline{0}$	$\underline{2}$	$\underline{0}$	$\underline{02}$	$\underline{0}$	$\underline{02}$	$\underline{02}$
$\underline{01}$	$\underline{01}$	$\underline{01}$	$\underline{02}$	$\underline{012}$	$\underline{012}$	$\underline{012}$	$\underline{012}$
$\underline{02}$	$\underline{0}$	$\underline{02}$	$\underline{0}$	$\underline{02}$	$\underline{0}$	$\underline{02}$	$\underline{02}$
$\underline{12}$	$\underline{012}$	$\underline{012}$	$\underline{02}$	$\underline{012}$	$\underline{012}$	$\underline{012}$	$\underline{012}$
$\underline{012}$	$\underline{012}$	$\underline{012}$	$\underline{02}$	$\underline{012}$	$\underline{012}$	$\underline{012}$	$\underline{012}$

Thus, since $\check{P}^1_{\underline{0}}(0) = \{\underline{012}\}$, $\check{P}^1_{\underline{0}}(1) = \{\underline{1}\}$, and $\check{P}^1_{\underline{0}}(2) = \emptyset$, we have the formula $\check{f}^1_{\underline{0}}(X)$ by Eq. (1).

$$\begin{aligned} \check{f}^1_{\underline{0}}(X) &= (\underline{0} \wedge \underline{012}) \sqcup (\underline{1} \wedge X^{\underline{1}}) \sqcup (\underline{2} \wedge \underline{0}) \\ &= \underline{1} \wedge X^{\underline{1}} \end{aligned} \tag{9}$$

In a similar way, we have the formulas $\check{f}^1_{\underline{1}}(X)$, $\check{f}^1_{\underline{2}}(X)$, $\check{f}^2_{\underline{0}}(Y)$, $\check{f}^2_{\underline{1}}(Y)$, and $\check{f}^2_{\underline{2}}(Y)$, below.

$$\left.\begin{aligned} &\check{f}^1_{\underline{1}}(X) = \underline{1}X^{\underline{1}} \sqcup X^{\underline{2}} \\ &\check{f}^1_{\underline{2}}(X) = X^{\underline{1}} \\ &\check{f}^2_{\underline{0}}(Y) = \underline{0} \\ &\check{f}^2_{\underline{1}}(Y) = \underline{1} \sqcup Y^{\underline{2}} \\ &\check{f}^2_{\underline{2}}(Y) = Y^{\underline{1}} \end{aligned}\right\} \tag{10}$$

Moreover, it follows by Eq. (4) that we have

$$\begin{aligned} Q^1_{\underline{0}}(\underline{01}) &= \{B \in P_3 - S_3 \mid f(B, \underline{0}) = \underline{01}\} = \{\underline{01}\}, \\ Q^1_{\underline{0}}(\underline{02}) &= \{B \in P_3 - S_3 \mid f(B, \underline{0}) = \underline{02}\} = \emptyset, \\ Q^1_{\underline{0}}(\underline{12}) &= \{B \in P_3 - S_3 \mid f(B, \underline{0}) = \underline{12}\} = \emptyset, \text{ and} \\ Q^1_{\underline{0}}(\underline{012}) &= \{B \in P_1 - S_3 \mid f(B, \underline{0}) = \underline{012}\} = \{\underline{12}, \underline{012}\}. \end{aligned}$$

Thus, since $\hat{Q}^1_{\underline{0}}(\underline{01}) = \{\underline{01}\}$, $\hat{Q}^1_{\underline{0}}(\underline{02}) = \hat{Q}^1_{\underline{0}}(\underline{12}) = \emptyset$ and $\hat{Q}^1_{\underline{0}}(\underline{012}) = \{\underline{12}\}$, we have the formula $\hat{f}^1_{\underline{0}}(X)$ by Eq. (3).

$$\begin{aligned} \hat{f}^1_{\underline{0}}(X) &= (\underline{0}X^{\underline{0}}X^{\underline{1}} \sqcup \underline{1}X^{\underline{0}}X^{\underline{1}}) \sqcup \underline{0} \sqcup \underline{0} \sqcup (\underline{0}X^{\underline{2}}X^{\underline{2}} \sqcup \underline{1}X^{\underline{2}}X^{\underline{2}} \sqcup \underline{2}X^{\underline{2}}X^{\underline{2}}) \\ &= \underline{1}X^{\underline{0}}X^{\underline{1}} \sqcup \underline{1}X^{\underline{1}}X^{\underline{2}} \sqcup X^{\underline{1}}X^{\underline{2}}. \end{aligned} \tag{11}$$

In a similar way, we have the formulas $\hat{f}^1_{\underline{1}}(X)$, $\hat{f}^1_{\underline{2}}(X)$, $\hat{f}^2_{\underline{0}}(Y)$, $\hat{f}^2_{\underline{1}}(Y)$, and $\hat{f}^2_{\underline{2}}(Y)$, below.

$$\left.\begin{aligned} &\hat{f}^1_{\underline{1}}(X) = \underline{1}X^{\underline{0}}X^{\underline{1}} \sqcup X^{\underline{0}}Y^{\underline{2}} \sqcup \underline{1}X^{\underline{1}}X^{\underline{2}} \sqcup X^{\underline{1}}X^{\underline{2}}, \\ &\hat{f}^1_{\underline{2}}(X) = X^{\underline{0}}X^{\underline{1}} \sqcup X^{\underline{1}}X^{\underline{2}}, \\ &\hat{f}^2_{\underline{0}}(Y) = Y^{\underline{1}}Y^{\underline{2}}, \\ &\hat{f}^2_{\underline{1}}(Y) = \underline{1}Y^{\underline{0}}Y^{\underline{2}} \sqcup X^{\underline{0}}Y^{\underline{2}} \sqcup \underline{1}Y^{\underline{1}}Y^{\underline{2}} \sqcup Y^{\underline{1}}Y^{\underline{2}}, \\ &\hat{f}^2_{\underline{2}}(Y) = Y^{\underline{0}}Y^{\underline{1}} \sqcup Y^{\underline{1}}Y^{\underline{2}}. \end{aligned}\right\} \tag{12}$$

Table 6: Truth Tables of $\check{f}_A^1(X)$ and $\hat{f}_A^1(X)$

X	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{01}$	$\underline{02}$	$\underline{12}$	$\underline{012}$
$\check{f}_{\underline{0}}^1(X)$	$\underline{0}$	$\underline{1}$	$\underline{0}$	$\underline{01}$	$\underline{0}$	$\underline{01}$	$\underline{01}$
$\check{f}_{\underline{1}}^1(X)$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{01}$	$\underline{02}$	$\underline{012}$	$\underline{012}$
$\check{f}_{\underline{2}}^1(X)$	$\underline{0}$	$\underline{2}$	$\underline{0}$	$\underline{02}$	$\underline{0}$	$\underline{02}$	$\underline{02}$
$\hat{f}_{\underline{0}}^1(X)$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{01}$	$\underline{0}$	$\underline{012}$	$\underline{012}$
$\hat{f}_{\underline{1}}^1(X)$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{01}$	$\underline{02}$	$\underline{012}$	$\underline{012}$
$\hat{f}_{\underline{2}}^1(X)$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{02}$	$\underline{0}$	$\underline{02}$	$\underline{02}$

Table 7: Truth Tables of $\check{f}_A^2(Y)$ and $\hat{f}_A^2(Y)$

Y	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{01}$	$\underline{02}$	$\underline{12}$	$\underline{012}$
$\check{f}_{\underline{0}}^2(Y)$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$
$\check{f}_{\underline{1}}^2(Y)$	$\underline{1}$	$\underline{1}$	$\underline{2}$	$\underline{1}$	$\underline{12}$	$\underline{12}$	$\underline{12}$
$\check{f}_{\underline{2}}^2(Y)$	$\underline{0}$	$\underline{2}$	$\underline{0}$	$\underline{02}$	$\underline{0}$	$\underline{02}$	$\underline{02}$
$\hat{f}_{\underline{0}}^2(Y)$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{02}$	$\underline{02}$
$\hat{f}_{\underline{1}}^2(Y)$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{012}$	$\underline{012}$	$\underline{012}$
$\hat{f}_{\underline{2}}^2(Y)$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{02}$	$\underline{0}$	$\underline{02}$	$\underline{02}$

Table 8: Truth Table of $f_A^i = \check{f}_A^i \sqcup \hat{f}_A^i$

X or Y	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{01}$	$\underline{02}$	$\underline{12}$	$\underline{012}$
$f_{\underline{0}}^1(X)$	$\underline{0}$	$\underline{1}$	$\underline{0}$	$\underline{01}$	$\underline{0}$	$\underline{012}$	$\underline{012}$
$f_{\underline{1}}^1(X)$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{01}$	$\underline{02}$	$\underline{012}$	$\underline{012}$
$f_{\underline{2}}^1(X)$	$\underline{0}$	$\underline{2}$	$\underline{0}$	$\underline{02}$	$\underline{0}$	$\underline{02}$	$\underline{02}$
$f_{\underline{0}}^2(Y)$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{02}$	$\underline{02}$
$f_{\underline{1}}^2(Y)$	$\underline{1}$	$\underline{1}$	$\underline{2}$	$\underline{1}$	$\underline{12}$	$\underline{12}$	$\underline{12}$
$f_{\underline{2}}^2(Y)$	$\underline{0}$	$\underline{2}$	$\underline{0}$	$\underline{02}$	$\underline{0}$	$\underline{02}$	$\underline{02}$

Tables 6 and 7 show the truth tables of $\check{f}_A^i$ and $\hat{f}_A^i$ for which $i = 1, 2$ and $A \in \{\underline{0}, \underline{1}, \underline{2}\}$.

(End of Example)

It follows by Lemma 3 that

$$f(X, B) = \check{f}_A^1(X) \sqcup \hat{f}_A^1(X) \text{ and } f(B, Y) = \check{f}_A^2(Y) \sqcup \hat{f}_A^2(Y)$$

hold for every $A \in \{\underline{0}, \underline{1}, \underline{2}\}$ and every $B \in P_3$, where $\check{f}_A^1(X)$, $\hat{f}_A^1(X)$, $\check{f}_A^2(Y)$ and $\hat{f}_A^2(Y)$ have been obtained in Eqs. (9), (10), (11) and (12). Table 8 shows the truth tables of $\check{f}_A^i \sqcup \hat{f}_A^i$, where $i = 1, 2$ and $A \in \{\underline{0}, \underline{1}, \underline{2}\}$.

Example 2 Let us next consider the formula (5) in Definition 4. It follows by Eqs. (9) and (11) that we have the formula $\check{f}_{\underline{0}}^1(X) \sqcup \hat{f}_{\underline{0}}^1(X)$ below.

$$\begin{aligned}\check{f}_{\underline{0}}^1(X) \sqcup \hat{f}_{\underline{0}}^1(X) &= \underline{1}X^{\underline{1}} \sqcup \underline{1}X^{\underline{0}}X^{\underline{1}} \sqcup \underline{1}X^{\underline{1}}X^{\underline{2}} \sqcup X^{\underline{1}}X^{\underline{2}} \\ &= \underline{1}X^{\underline{1}} \sqcup X^{\underline{1}}X^{\underline{2}}\end{aligned}$$

In a similar way, by Eqs. (10), (11) and (12), we have the formulas

$$\begin{aligned}\check{f}_{\underline{1}}^1(X) \sqcup \hat{f}_{\underline{1}}^1(X) &= \underline{1}X^{\underline{1}} \sqcup X^{\underline{2}}, \\ \check{f}_{\underline{2}}^1(X) \sqcup \hat{f}_{\underline{2}}^1(X) &= X^{\underline{1}}, \\ \check{f}_{\underline{0}}^2(Y) \sqcup \hat{f}_{\underline{0}}^2(Y) &= Y^{\underline{1}}Y^{\underline{2}}, \\ \check{f}_{\underline{1}}^2(Y) \sqcup \hat{f}_{\underline{1}}^2(Y) &= \underline{1} \sqcup Y^{\underline{2}}, \\ \check{f}_{\underline{2}}^2(Y) \sqcup \hat{f}_{\underline{2}}^2(Y) &= Y^{\underline{1}}.\end{aligned}$$

Therefore, the formula $f_1(X, Y)$ of (5) in Definition 4 is given as

$$f_1(X, Y) = f^1(X, Y) \sqcup f^2(X, Y), \tag{13}$$

Table 9: Truth Table of $f_1(X,Y)$

$X\backslash Y$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{01}$	$\underline{02}$	$\underline{12}$	$\underline{012}$
$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{02}$	$\underline{02}$
$\underline{1}$	$\underline{1}$	$\underline{1}$	$\underline{2}$	$\underline{1}$	$\underline{12}$	$\underline{12}$	$\underline{12}$
$\underline{2}$	$\underline{0}$	$\underline{2}$	$\underline{0}$	$\underline{02}$	$\underline{0}$	$\underline{02}$	$\underline{02}$
$\underline{01}$	$\underline{01}$	$\underline{01}$	$\underline{02}$	$\underline{01}$	$\underline{012}$	$\underline{012}$	$\underline{012}$
$\underline{02}$	$\underline{0}$	$\underline{02}$	$\underline{0}$	$\underline{02}$	$\underline{0}$	$\underline{02}$	$\underline{02}$
$\underline{12}$	$\underline{012}$	$\underline{012}$	$\underline{02}$	$\underline{012}$	$\underline{012}$	$\underline{012}$	$\underline{012}$
$\underline{012}$	$\underline{012}$	$\underline{012}$	$\underline{02}$	$\underline{012}$	$\underline{012}$	$\underline{012}$	$\underline{012}$

Table 10: Truth Table of $f_2(X,Y)$

$X\backslash Y$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{01}$	$\underline{02}$	$\underline{12}$	$\underline{012}$
$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$
$\underline{1}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$
$\underline{2}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$
$\underline{01}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{012}$	$\underline{012}$	$\underline{012}$	$\underline{012}$
$\underline{02}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{02}$	$\underline{0}$	$\underline{02}$	$\underline{02}$
$\underline{12}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{012}$	$\underline{012}$	$\underline{012}$	$\underline{012}$
$\underline{012}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{012}$	$\underline{012}$	$\underline{012}$	$\underline{012}$

where

$$\begin{aligned} f^1(X,Y) &= Y^{\underline{0}}\left(\check{f}^1_{\underline{0}}(X) \sqcup \hat{f}^1_{\underline{0}}(X)\right) \sqcup Y^{\underline{1}}\left(\check{f}^1_{\underline{1}}(X) \sqcup \hat{f}^1_{\underline{1}}(X)\right) \sqcup Y^{\underline{2}}\left(\check{f}^1_{\underline{2}}(X) \sqcup \hat{f}^1_{\underline{2}}(X)\right) \text{ and} \\ f^2(X,Y) &= X^{\underline{0}}\left(\check{f}^2_{\underline{0}}(Y) \sqcup \hat{f}^2_{\underline{0}}(Y)\right) \sqcup X^{\underline{1}}\left(\check{f}^2_{\underline{1}}(Y) \sqcup \hat{f}^2_{\underline{1}}(Y)\right) \sqcup X^{\underline{2}}\left(\check{f}^2_{\underline{2}}(Y) \sqcup \hat{f}^2_{\underline{2}}(Y)\right). \end{aligned}$$

Table 9 is the truth table of $f_1(X,Y)$. (End of Example)

Example 3 In this example, let us consider the formula (7) in Definition 5. It follows by Eq. (6) that we have the following subsets of P_3^2.

$$\begin{aligned} T(f,\underline{01}) &= \emptyset \\ T(f,\underline{02}) &= \{(\underline{02},\underline{01}),(\underline{02},\underline{12}),(\underline{02},\underline{012})\} \\ T(f,\underline{12}) &= \emptyset \\ T(f,\underline{012}) &= \{(A,B) \mid A \in \{\underline{01},\underline{12},\underline{012}\} \text{ and } B \in P_3 - S_3\} \end{aligned}$$

Therefore, since we have

$$\begin{aligned} \hat{T}(f,\underline{01}) &= \emptyset, \\ \hat{T}(f,\underline{02}) &= \{(\underline{02},\underline{01}),(\underline{02},\underline{12})\}, \\ \hat{T}(f,\underline{12}) &= \emptyset, \text{ and} \\ \hat{T}(f,\underline{012}) &= \{(\underline{01},\underline{01}),(\underline{01},\underline{02}),(\underline{01},\underline{12}),(\underline{12},\underline{01}),(\underline{12},\underline{02}),(\underline{12},\underline{12})\}, \end{aligned}$$

it follows by Eq. (8) that we have the following formulas.

$$\begin{aligned} f_{\underline{01}}(X,Y) &= \underline{0} \\ f_{\underline{02}}(X,Y) &= X^{\underline{0}}X^{\underline{2}}Y^{\underline{0}}Y^{\underline{1}} \sqcup X^{\underline{0}}X^{\underline{2}}Y^{\underline{1}}Y^{\underline{2}} \\ f_{\underline{12}}(X,Y) &= \underline{0} \\ f_{\underline{012}}(X,Y) &= X^{\underline{0}}X^{\underline{1}}Y^{\underline{0}}Y^{\underline{1}} \sqcup X^{\underline{0}}X^{\underline{1}}Y^{\underline{0}}Y^{\underline{2}} \sqcup X^{\underline{0}}X^{\underline{1}}Y^{\underline{1}}Y^{\underline{2}} \sqcup \\ &\quad\; X^{\underline{1}}X^{\underline{2}}Y^{\underline{0}}Y^{\underline{1}} \sqcup X^{\underline{1}}X^{\underline{2}}Y^{\underline{0}}Y^{\underline{2}} \sqcup X^{\underline{1}}X^{\underline{2}}Y^{\underline{1}}Y^{\underline{2}} \end{aligned}$$

Thus, the formula $f_2(X,Y)$ of (7) in Definition 5 is obtained as the formula below.

$$\begin{aligned} f_2(X,Y) &= \underline{0} \sqcup \{\underline{0}f_{\underline{02}}(X,Y) \sqcup \underline{2}f_{\underline{02}}(X,Y)\} \sqcup \{\underline{0}f_{\underline{012}}(X,Y) \sqcup \underline{1}f_{\underline{012}}(X,Y) \sqcup \underline{2}f_{\underline{012}}(X,Y)\} \\ &= f_{\underline{02}}(X,Y) \sqcup \underline{1}f_{\underline{012}}(X,Y) \sqcup f_{\underline{012}}(X,Y) \qquad (14) \end{aligned}$$

Table 10 is the truth table of $f_2(X,Y)$. (End of Example)

It follows by Theorem 4 that the function f of Table 5 can be expressed by the formula $f_1(X,Y) \sqcup f_2(X,Y)$, where $f_1(X,Y)$ and $f_2(X,Y)$ are the formulas given in (13) and (14), respectively.

5 Functions Expressed by Formulas Composed of $\wedge$, $\sqcup$ and Universal Literals

This section discusses functions on P_r expressed by formulas, which are composed of the operations $\wedge$, $\sqcup$ and universal literals. Then, a necessary and sufficient condition for a function on P_r to be expressed by a formula when r is equal to 3.

Theorem 5 Let f be a function on P_r. If f can be expressed by a formula, then

$$\bigcap_{A \in P_r - S_r} f(A_1, \dots, A_{i-1}, A, A_{i+1}, \dots, A_n) \neq \emptyset$$

holds for any elements $A_1, \dots, A_{i-1}, A_{i+1}, \dots, A_n$ of P_r. (End of Theorem)

By Theorems 1, 2 and 5, any function f on P_r expressed by a formula satisfies the following Condition B.

Condition B: Let f be a function on P_r.

(1) If $(A_1, \dots, A_n) \in S_r^n$, then $f(A_1, \dots, A_n) \in S_r$.

(2) For any elements $(A_1, \dots, A_n)$ and $(B_1, \dots, B_n)$ of P_r^n, $(A_1, \dots, A_n) \subseteq (B_1, \dots, B_n)$ implies $f(A_1, \dots, A_n) \subseteq f(B_1, \dots, B_n)$.

(3) $\bigcap_{A \in P_r - S_r} f(A_1, \dots, A_{i-1}, A, A_{i+1}, \dots, A_n) \neq \emptyset$ holds for any elements $A_1, \dots, A_{i-1}, A_{i+1}, \dots, A_n$ of P_r.

In the following, this section proves that Condition B is a necessary and sufficient condition for a function on P_3 to be expressed by a formula with the operations $\wedge$, $\sqcup$, and universal literals.

Definition 6 Let $(A_1, \dots, A_n)$ be any element of P_r^n. Then, $\alpha = X_1^{A_1} \wedge \cdots \wedge X_n^{A_n}$ is said to be the type-1 term corresponding to $(A_1, \dots, A_n)$. Next, let $(B_1, \dots, B_n)$ be any element of $P_r^n - S_r^n$. Then, $\beta = \bigwedge_{e \in B_1} X_1^{\{e\}} \wedge \cdots \wedge \bigwedge_{e \in B_n} X_n^{\{e\}}$ is said to be the type-2 term corresponding to $(B_1, \dots, B_n)$. (End of Definition)

Let S be an element of P_r, and let T be an element of $P_r - S_r$. Then, it is easy to verify that the following two equations are valid.

$$X^S = \begin{cases} \{r-1\} & \text{if } X \subseteq S \\ \{0\} & \text{if } X \cap S = \emptyset \\ \{0, r-1\} & \text{otherwise} \end{cases} \tag{15}$$

$$\bigwedge_{e \in T} X^{\{e\}} = \begin{cases} \{0, r-1\} & \text{if } T \subseteq X \\ \{0\} & \text{otherwise} \end{cases} \tag{16}$$

Therefore, for any type-1 term α and any type-2 term β, $\alpha(A_1, \dots, A_n) = \{r-1\}$, $\{0, r-1\}$, or $\{0\}$, and $\beta(A_1, \dots, A_n) = \{0, r-1\}$ or $\{0\}$ hold for any element $(A_1, \dots, A_n) \in P_r^n$.

Lemma 6 For any type-1 term α corresponding to $(A_1, \dots, A_n) \in P_r^n$,

(1) $(B_1, \ldots, B_n) \subseteq (A_1, \ldots, A_n)$ iff $\alpha(B_1, \ldots, B_n) = \{r-1\}$,

(2) $(A_1, \ldots, A_n) \cap (B_1, \ldots, B_n) = \emptyset$ iff $\alpha(B_1, \ldots, B_n) = \{0\}$,

(3) $(B_1, \ldots, B_n) \not\subseteq (A_1, \ldots, A_n)$ and $(A_1, \ldots, A_n) \cap (B_1, \ldots, B_n) = \emptyset$ iff $\alpha(B_1, \ldots, B_n) = \{0, r-1\}$

hold for any $(B_1, \ldots, B_n) \in P_r^n$. (End of Lemma)

Lemma 7 For any type-2 term α corresponding to $(A_1, \ldots, A_n) \in P_r^n - S_r^n$,

(1) $(A_1, \ldots, A_n) \subseteq (B_1, \ldots, B_n)$ iff $\alpha(B_1, \ldots, B_n) = \{0, r-1\}$,

(2) $(A_1, \ldots, A_n) \not\subseteq (B_1, \ldots, B_n)$ iff $\alpha(B_1, \ldots, B_n) = \{0\}$

hold for any $(B_1, \ldots, B_n) \in P_r^n$. (End of Lemma)

Let f be a function satisfying Condition B, and let S be an element of P_r. Then, define two subsets of P_r^n, denoted by $L(f,S)$ and $U(f,S)$, below.

$$\begin{aligned} L(f,S) &= \{(A_1, \ldots, A_n) \in P_r^n \mid f(A_1, \ldots, A_n) \subseteq S\} \text{ and} \\ U(f,S) &= \{(A_1, \ldots, A_n) \in P_r^n \mid f(A_1, \ldots, A_n) \cap S \neq \emptyset\}. \end{aligned}$$

Let $\check{L}(f,S)$ and $U'(f,S)$ be the sets of all maximal elements of $L(f,S)$ and of all minimal elements of $U(f,S)$, respectively. Further, let $\hat{U}(f,S) = U'(f,S) - S_r^n$.

Lemma 8 Let f be a function satisfying Condition B, and let S be an element of P_r. Then, $(f)^S$ can be expressed by the following formula.

$$(f)^S = \begin{cases} \bigsqcup_{A \in \check{L}(f,S)} \alpha_A \sqcup \bigsqcup_{A \in \hat{U}(f,S)} \beta_A & \text{if } \check{L}(f,S) \neq \emptyset \text{ or } \hat{U}(f,S) \neq \emptyset \\ \{0\} & \text{otherwise} \end{cases} \tag{17}$$

where α_A and β_A are the type-1 and type-2 terms corresponding to A, respectively. (End of Lemma)

Now, let us consider formulas of one-variable functions satisfying Condition B. Any one-variable function f satisfying Condition B is in at least one of the following three cases [3].

(B-1) $f(A) \neq \underline{01}$ holds for any element $A \in P_3 - S_3$.

(B-2) $f(A) \neq \underline{02}$ holds for any element $A \in P_3 - S_3$.

(B-3) $f(A) \neq \underline{12}$ holds for any element $A \in P_3 - S_3$.

[3] If f is in neither one of the cases (B-1), (B-2), (B-3), then it implies that we have three distinct elements A, B, C in $P_3 - S_3$ such that $f(A) = \underline{01}$, $f(B) = \underline{02}$, and $f(C) = \underline{12}$. However, this contradicts to the fact that f satisfies Condition B(3).

Table 11: Example of (B-4)

$x\backslash y$	0	1	2	01	02	12	012
0	0	1	2	01	02	012	012
1	2	2	2	2	2	2	2
2	0	2	1	02	012	12	012
01	02	12	2	012	012	012	012
02	0	12	12	012	012	012	012
12	02	2	12	02	012	12	012
012	02	12	12	012	012	012	012

Table 12: Example of (B-5)

$x\backslash y$	0	1	2	01	02	12	012
0	0	0	0	0	0	0	0
1	0	1	0	01	0	01	01
2	0	2	1	02	012	12	012
01	0	01	0	01	0	01	01
02	0	02	01	012	012	012	012
12	0	012	01	012	012	012	012
012	0	012	01	012	012	012	012

Table 13: Example of (B-6)

$x\backslash y$	0	1	2	01	02	12	012
0	0	0	0	0	0	0	0
1	0	1	1	01	01	1	01
2	0	2	2	02	02	2	02
01	0	01	012	01	012	012	012
02	0	012	02	012	012	012	012
12	0	12	12	012	012	12	012
012	0	012	012	012	012	012	012

Property 1 Any one-variable function f satisfying Condition B can be expressed by the following formula.

$$f(X) = \begin{cases} f^{\underline{12}}(X) \wedge (\underline{1} \sqcup f^{\underline{02}}(X)) & \text{if } f \text{ is in the case (B-1)} \\ (\underline{1} \wedge f^{\underline{1}}(X)) \sqcup f^{\underline{2}}(X) \sqcup (\underline{1} \wedge f^{\underline{12}}(X)) & \text{if } f \text{ is in the case (B-2)} \\ (\underline{1} \wedge f^{\underline{1}}(X)) \sqcup f^{\underline{2}}(X) & \text{if } f \text{ is in the case (B-3)} \end{cases} \tag{18}$$

(End of Property)

By Property 1, every one-variable function satisfying Condition B can be expressed by a formula.

Next, let us consider the case where functions satisfying Condition B depend more than one variable. Then, any function f satisfying Condition B is in at least one of the three cases below.

(B-4) $f(A_1, \dots, A_n) \neq \underline{01}$ holds for any element $(A_1, \dots A_n) \in (P_3 - S_3)^n$.

(B-5) $f(A_1, \dots, A_n) \neq \underline{12}$ holds for any element $(A_1, \dots A_n) \in (P_3 - S_3)^n$.

(B-6) $\displaystyle\bigcap_{(A_1, \dots, A_n) \in (P_3 - S_3)^n} f(A_1, \dots, A_n) = \underline{1}$.

Tables 11, 12 and 13 are examples of two-variable functions being in the cases (B-4), (B-5), and (B-6), respectively.

Then, we can prove Properties 1 $\sim$ 6, which show a way for constructing formulas of n-variable functions satisfying Condition B.

Let A be an element $(A_1, \dots, A_{i-1}, A_{i+1}, \dots, A_n)$ of P_r^{n-1}. Then, denote the one-variable function $f(A_1, \dots, A_{i-1}, X, A_{i+1}, \dots, A_n)$ by $f_A^i(X)$.

Property 2 Suppose a function f satisfying Condition B is in the case (B-5). Let f' be a function expressed by the formula

$$f' = p^1 \sqcup \cdots \sqcup p^n, \tag{19}$$

where

$$p^i(X_1, \ldots, X_n) = \bigsqcup_{A=(A_1,\ldots,A_{i-1},A_{i+1},\ldots,A_n)\in S_3^{n-1}} (f_A^i(X_i) \wedge X_1^{A_1} \wedge \cdots \wedge X_{i-1}^{A_{i-1}} \wedge X_{i+1}^{A_{i+1}} \wedge \cdots \wedge X_n^{A_n}). \tag{20}$$

Then, for any element $(A_1, \ldots, A_n) \in P_3^n$,

$$f'(A_1, \ldots, A_n) = \begin{cases} f(A_1, \ldots, A_n) & \text{if } (A_1, \ldots, A_n) \notin (P_3 - S_3)^n \\ K & \text{otherwise} \end{cases}$$

where K is an element of P_r such that $\{0\} \subseteq K \subseteq \{0\} \cup F(A_1, \ldots, A_n)$. (End of Property)

Property 3 Suppose a function f satisfying Condition B is in the case (B-5). Let f'' be a function expressed by the formula

$$f''(X_1, \ldots, X_n) = \bigsqcup_{S\in P_3-S_3} \Big\{ \bigsqcup_{t\in S} (\{t\} \wedge f_{\hat{T}(S)}(X_1, \ldots, X_n)) \Big\}, \tag{21}$$

where $\hat{T}(S)$ is the set of all minimal elements of the set

$$T(S) = \{(A_1, \ldots, A_n) \in (P_3 - S_3)^n \mid f(A_1, \ldots, A_n) = S\}$$

and

$$f_{\hat{T}(S)}(X_1, \ldots, X_n) = \bigsqcup_{(A_1,\ldots,A_n)\in\hat{T}(S)} \Big\{ \bigwedge_{e\in A_1} X^{\{e\}} \wedge \cdots \wedge \bigwedge_{e\in A_n} X^{\{e\}} \Big\}. \tag{22}$$

Then, for any element $(A_1, \ldots, A_n) \in P_3^n$,

$$f''(A_1, \ldots, A_n) = \begin{cases} \{0\} & \text{if } (A_1, \ldots, A_n) \notin (P_3 - S_3)^n \\ f(A_1, \ldots, A_n) & \text{otherwise} \end{cases}$$

(End of Property)

Property 4 Any function f satisfying Condition B can be expressed by $f = f' \sqcup f''$, if f is in the case (B-5). (End of Property)

Property 5 Suppose a function f satisfying Condition B is in the case (B-4). Let $A = (A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n)$ be an element of S_3^{n-1}. Then, define g_A^i and h as functions on P_r expressed by the following formulas.

$$g_A^i(X_1, \ldots, X_n) = f_A^i(X_i) \sqcup (X_1^{A_1'} \wedge \cdots \wedge X_{i-1}^{A_{i-1}'} \wedge X_{i+1}^{A_{i+1}'} \wedge \cdots \wedge X_n^{A_n'}), \tag{23}$$

where $A_j' = E_3 - A_j$ $(j = 1, \ldots, i-1, i+1, \ldots, n)$.

$$h(X_1, \ldots, X_n) = \left\{ \left(\bigsqcup_{i=1}^{n} \bigsqcup_{A\in S_3} X_i^A \right) \sqcup f^{\underline{12}}(X_1, \ldots, X_n) \right\} \wedge \left\{ \left(\bigsqcup_{i=1}^{n} \bigsqcup_{A\in S_3} X_i^A \right) \sqcup f^{\underline{02}}(X_1, \ldots, X_n) \sqcup \underline{1} \right\} \tag{24}$$

Then, f can be expressed by the following formula.

$$f(X_1,\dots,X_n) = G(X_1,\dots,X_n) \wedge h(X_1,\dots,X_n), \tag{25}$$

where G is $\wedge$-ing of all the g^i_A's of Eq. (23), i.e.,

$$G(X_1,\dots,X_n) = \bigwedge_{i=1}^{n} \left(\bigwedge_{A\in S_3^{n-1}} g^i_A(X_1,\dots,X_n) \right) \tag{26}$$

(End of Property)

Property 6 Suppose a function f satisfying Condition B is in the case (B-6). Then, f is in either one of the following two cases.

(1) $f(A) \neq \underline{02}$ holds for any element $A \in P_3^n$, or

(2) $f(A) = \underline{02}$ holds for some element $A \in P_3^n$.

If f is in the case (1), then f can be expressed by

$$f(X_1,\dots,X_n) = (\underline{1} \wedge f^{\underline{1}}(X_1,\dots,X_n)) \sqcup f^{\underline{2}}(X_1,\dots,X_n) \sqcup (\underline{1} \wedge f^{\underline{12}}(X_1,\dots,X_n)). \tag{27}$$

Let w be a function expressed by the following formula.

$$w(X_1,\dots,X_n) = \bigsqcup_{(A_1,\dots,A_n)\in Q_{\underline{02}}} \xi_1(A_1) \sqcup \dots \sqcup \xi_n(A_n), \tag{28}$$

where $Q_{\underline{02}} = \{(A_1,\dots,A_n) \in P_3^n \mid f(A_1,\dots,A_n) = \underline{02}\}$ [4], and

$$\xi_i(A) = \begin{cases} X_i^A & \text{if } A \in S_3 \\ \underline{0} & \text{otherwise.} \end{cases}$$

Then, f can be expressed by the following formula, if f is in the case (2).

$$f(X_1,\dots,X_n) = G(X_1,\dots,X_n) \wedge w(X_1,\dots,X_n), \tag{29}$$

where $G(X_1,\dots,X_n)$ is given by Eq. (26). (End of Property)

6 Examples of Function Satisfying Condition B

This section shows examples of 2-variable functions satisfying Condition B, and illustrates how they can be expressed by formulas.

Example 4 Consider the function f defined by Table 12, which is in the case (B-5). The formula expressing f is given by Properties 2, 3, and 4. First, consider the formulas $f^1_A(X)$ and $f^2_A(Y)$, which appear in Eq. (20). Table 14 shows the truth tables of the six one-variable functions $f^1_{\underline{0}}(X)$, $f^1_{\underline{1}}(X)$, $f^1_{\underline{2}}(X)$, $f^2_{\underline{0}}(Y)$, $f^2_{\underline{1}}(Y)$ and $f^2_{\underline{2}}(Y)$. Since $f^1_{\underline{2}}(X)$ and $f^2_{\underline{1}}(Y)$ are in (B-2) (or (B-3)), $f^1_{\underline{1}}(X)$ is in (B-3), and $f^2_{\underline{2}}(Y)$ is in (B-1), it follows by Eq. (17) and (18) that these one-variable functions are expressed by the following formulas.

[4] $Q_{\underline{02}} \cap (P_3 - S_3)^n = \emptyset$ holds, since f is in the case (B-6).

Table 14: One-Variable Functions f_A^1 and f_A^2 of Example 4

X or Y	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{01}$	$\underline{02}$	$\underline{12}$	$\underline{012}$
$f_{\underline{0}}^1(X)$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$
$f_{\underline{1}}^1(X)$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{01}$	$\underline{02}$	$\underline{012}$	$\underline{012}$
$f_{\underline{2}}^1(X)$	$\underline{0}$	$\underline{0}$	$\underline{1}$	$\underline{0}$	$\underline{01}$	$\underline{01}$	$\underline{01}$
$f_{\underline{0}}^2(Y)$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$
$f_{\underline{1}}^2(Y)$	$\underline{0}$	$\underline{1}$	$\underline{0}$	$\underline{01}$	$\underline{0}$	$\underline{01}$	$\underline{01}$
$f_{\underline{2}}^2(Y)$	$\underline{0}$	$\underline{2}$	$\underline{1}$	$\underline{02}$	$\underline{012}$	$\underline{12}$	$\underline{012}$

Table 15: Truth Table of f' of Example 4

$X \backslash Y$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{01}$	$\underline{02}$	$\underline{12}$	$\underline{012}$
$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$
$\underline{1}$	$\underline{0}$	$\underline{1}$	$\underline{0}$	$\underline{01}$	$\underline{0}$	$\underline{01}$	$\underline{01}$
$\underline{2}$	$\underline{0}$	$\underline{2}$	$\underline{1}$	$\underline{02}$	$\underline{012}$	$\underline{12}$	$\underline{012}$
$\underline{01}$	$\underline{0}$	$\underline{01}$	$\underline{0}$	$\underline{01}$	$\underline{0}$	$\underline{01}$	$\underline{01}$
$\underline{02}$	$\underline{0}$	$\underline{02}$	$\underline{01}$	$\underline{02}$	$\underline{012}$	$\underline{012}$	$\underline{012}$
$\underline{12}$	$\underline{0}$	$\underline{012}$	$\underline{01}$	$\underline{012}$	$\underline{012}$	$\underline{012}$	$\underline{012}$
$\underline{012}$	$\underline{0}$	$\underline{012}$	$\underline{01}$	$\underline{012}$	$\underline{012}$	$\underline{012}$	$\underline{012}$

Table 16: Truth Table of f'' of Example 4

$X \backslash Y$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{01}$	$\underline{02}$	$\underline{12}$	$\underline{012}$
$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$
$\underline{1}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$
$\underline{2}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{0}$
$\underline{01}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{01}$	$\underline{0}$	$\underline{01}$	$\underline{01}$
$\underline{02}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{012}$	$\underline{012}$	$\underline{012}$	$\underline{012}$
$\underline{12}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{012}$	$\underline{012}$	$\underline{012}$	$\underline{012}$
$\underline{012}$	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\underline{012}$	$\underline{012}$	$\underline{012}$	$\underline{012}$

$$f_{\underline{0}}^1(X) = \underline{0}, \quad f_{\underline{1}}^1(X) = \underline{1}X^{\underline{1}} \sqcup X^{\underline{2}}, \quad f_{\underline{2}}^1(X) = \underline{1}X^{\underline{2}}$$
$$f_{\underline{0}}^2(Y) = \underline{0}, \quad f_{\underline{1}}^2(Y) = \underline{1}Y^{\underline{1}}, \quad f_{\underline{2}}^2(Y) = Y^{\underline{12}}(\underline{1} \sqcup Y^{\underline{01}})$$

Therefore, it follows by Eq. (19) that f' is expressed by the following formula.

$$f' = X^{\underline{0}} f_{\underline{0}}^2(Y) \sqcup X^{\underline{1}} f_{\underline{1}}^2(Y) \sqcup X^{\underline{2}} f_{\underline{2}}^2(Y) \sqcup f_{\underline{0}}^1(X) Y^{\underline{0}} \sqcup f_{\underline{1}}^1(X) Y^{\underline{1}} \sqcup f_{\underline{2}}^1(X) Y^{\underline{2}} \tag{30}$$

Table 15 is the truth table of f'. Next, consider f'' in Eq. (21). Since

$$\begin{aligned}
\hat{T}(\underline{01}) &= \{(\underline{01}, \underline{01}), (\underline{01}, \underline{12})\}, \\
\hat{T}(\underline{02}) &= \hat{T}(\underline{12}) = \emptyset, \text{ and} \\
\hat{T}(\underline{012}) &= \{(\underline{02}, \underline{01}), (\underline{02}, \underline{02}), (\underline{02}, \underline{12}), (\underline{12}, \underline{01}), (\underline{12}, \underline{02}), (\underline{12}, \underline{12})\},
\end{aligned}$$

it follows by Eq. (22) that we have the following formulas.

$$\begin{aligned}
f_{\hat{T}(\underline{01})}(X, Y) &= X^{\underline{0}} X^{\underline{1}} Y^{\underline{0}} Y^{\underline{1}} \sqcup X^{\underline{0}} X^{\underline{1}} Y^{\underline{1}} Y^{\underline{2}} \\
f_{\hat{T}(\underline{012})}(X, Y) &= X^{\underline{0}} X^{\underline{2}} Y^{\underline{0}} Y^{\underline{1}} \sqcup X^{\underline{0}} X^{\underline{2}} Y^{\underline{0}} Y^{\underline{2}} \sqcup X^{\underline{0}} X^{\underline{2}} Y^{\underline{1}} Y^{\underline{2}} \sqcup X^{\underline{1}} X^{\underline{2}} Y^{\underline{0}} Y^{\underline{1}} \sqcup \\
&\quad X^{\underline{1}} X^{\underline{2}} Y^{\underline{0}} Y^{\underline{2}} \sqcup X^{\underline{1}} X^{\underline{2}} Y^{\underline{1}} Y^{\underline{2}}
\end{aligned}$$

We then have $f''(X, Y)$ below by Eq. (21).

$$f''(X, Y) = \underline{1} f_{\hat{T}(\underline{01})}(X, Y) \sqcup \underline{1} f_{\hat{T}(\underline{012})}(X, Y) \sqcup f_{\hat{T}(\underline{012})}(X, Y) \tag{31}$$

Table 16 is the truth table of f''. It follows by Property 4 that $f(X, Y) = f'(X, Y) \sqcup f''(X, Y)$.

(End of Example)

Table 17: Truth Table of G of Example 5

$X\backslash Y$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{01}$	$\underline{02}$	$\underline{12}$	$\underline{012}$
$\underline{0}$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{01}$	$\underline{02}$	$\underline{012}$	$\underline{012}$
$\underline{1}$	$\underline{2}$	$\underline{2}$	$\underline{2}$	$\underline{2}$	$\underline{2}$	$\underline{2}$	$\underline{2}$
$\underline{2}$	$\underline{0}$	$\underline{2}$	$\underline{1}$	$\underline{02}$	$\underline{012}$	$\underline{12}$	$\underline{012}$
$\underline{01}$	$\underline{02}$	$\underline{12}$	$\underline{2}$	$\underline{012}$	$\underline{02}$	$\underline{012}$	$\underline{012}$
$\underline{02}$	$\underline{0}$	$\underline{12}$	$\underline{12}$	$\underline{012}$	$\underline{012}$	$\underline{012}$	$\underline{012}$
$\underline{12}$	$\underline{02}$	$\underline{2}$	$\underline{12}$	$\underline{02}$	$\underline{012}$	$\underline{12}$	$\underline{012}$
$\underline{012}$	$\underline{02}$	$\underline{12}$	$\underline{12}$	$\underline{012}$	$\underline{012}$	$\underline{012}$	$\underline{012}$

Table 18: Truth Table of h of Example 5

$X\backslash Y$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{01}$	$\underline{02}$	$\underline{12}$	$\underline{012}$
$\underline{0}$	$\underline{2}$	$\underline{2}$	$\underline{2}$	$\underline{2}$	$\underline{2}$	$\underline{2}$	$\underline{2}$
$\underline{1}$	$\underline{2}$	$\underline{2}$	$\underline{2}$	$\underline{2}$	$\underline{2}$	$\underline{2}$	$\underline{2}$
$\underline{2}$	$\underline{2}$	$\underline{2}$	$\underline{2}$	$\underline{2}$	$\underline{2}$	$\underline{2}$	$\underline{2}$
$\underline{01}$	$\underline{2}$	$\underline{2}$	$\underline{2}$	$\underline{012}$	$\underline{012}$	$\underline{012}$	$\underline{012}$
$\underline{02}$	$\underline{2}$	$\underline{2}$	$\underline{2}$	$\underline{012}$	$\underline{012}$	$\underline{012}$	$\underline{012}$
$\underline{12}$	$\underline{2}$	$\underline{2}$	$\underline{2}$	$\underline{02}$	$\underline{012}$	$\underline{12}$	$\underline{012}$
$\underline{012}$	$\underline{2}$	$\underline{2}$	$\underline{2}$	$\underline{012}$	$\underline{012}$	$\underline{012}$	$\underline{012}$

Example 5 Consider the function f defined by Table 11, which is in (B-4). The formula expressing f is given by Property 5. First, consider the formula $g_A^1(X)$ and $g_A^2(Y)$ of Eq. (23). It follows by Eq. (17) and (18) that the one-variable functions $f_A^1(X)$ and $f_A^2(Y)$ are obtained below.

$$f_{\underline{0}}^1(X) = X^{\underline{1}}, \qquad f_{\underline{1}}^1(X) = \underline{1} \sqcup X^{\underline{12}}, \quad f_{\underline{2}}^1(X) = \underline{1} \sqcup X^{\underline{01}},$$
$$f_{\underline{0}}^2(Y) = \underline{1}Y^{\underline{1}} \sqcup Y^{\underline{2}}, \quad f_{\underline{1}}^2(Y) = \underline{2}, \qquad f_{\underline{2}}^2(Y) = Y^{\underline{12}}(\underline{1} \sqcup Y^{\underline{01}})$$

Then, by Eq. (23), we have

$$g_{\underline{0}}^1(X,Y) = X^{\underline{1}} \sqcup Y^{\underline{12}}, \qquad g_{\underline{1}}^1(X,Y) = \underline{1} \sqcup X^{\underline{12}} \sqcup Y^{\underline{02}}, \quad g_{\underline{2}}^1(X,Y) = \underline{1} \sqcup X^{\underline{01}} \sqcup Y^{\underline{01}},$$
$$g_{\underline{0}}^2(X,Y) = X^{\underline{12}} \sqcup \underline{1}Y^{\underline{1}} \sqcup Y^{\underline{2}}, \quad g_{\underline{1}}^2(X,Y) = \underline{2}, \qquad g_{\underline{2}}^2(X,Y) = X^{\underline{01}} \sqcup Y^{\underline{12}}(\underline{1} \sqcup Y^{\underline{01}}).$$

By Eq. (26), we have the function $G(X,Y)$ expressed by $\wedge$-ing of all the above $g_A^i(X,Y)$'s. Table 17 is the truth table of $G(X,Y)$. Next, consider h in Eq. (24). It follows by Eq. (17) that the functions $f^{\underline{12}}(X,Y)$ and $f^{\underline{02}}(X,Y)$ are expressed by the following formulas.

$$\begin{aligned} f^{\underline{12}}(X,Y) &= X^{\underline{1}} \sqcup X^{\underline{12}}Y^{\underline{12}} \sqcup Y^{\underline{1}} \sqcup Y^{\underline{2}} \sqcup X^{\underline{0}}X^{\underline{1}}Y^{\underline{0}} \sqcup X^{\underline{1}}X^{\underline{2}}Y^{\underline{0}} \\ f^{\underline{02}}(X,Y) &= X^{\underline{0}}Y^{\underline{02}} \sqcup X^{\underline{1}} \sqcup X^{\underline{01}}Y^{\underline{2}} \sqcup X^{\underline{12}}Y^{\underline{01}} \sqcup Y^{\underline{1}} \end{aligned}$$

Thus, by Eq. (24), we have

$$h(X,Y) = \left(v(X,Y) \sqcup f^{\underline{12}}(X,Y)\right) \wedge \left(v(X,Y) \sqcup f^{\underline{02}}(X,Y) \sqcup \underline{1}\right),$$

where $v(X,Y) = X^{\underline{0}} \sqcup X^{\underline{1}} \sqcup X^{\underline{2}} \sqcup Y^{\underline{0}} \sqcup Y^{\underline{1}} \sqcup Y^{\underline{2}}$. Table 18 is the truth table of $h(X,Y)$. Lastly, by Eq. (25), $f(X,Y)$ are expressed by the following formula.

$$\begin{aligned} f(X,Y) &= G(X,Y) \wedge h(X,Y) \\ &= g_{\underline{0}}^1(X,Y) \wedge g_{\underline{1}}^1(X,Y) \wedge g_{\underline{2}}^1(X,Y) \wedge g_{\underline{0}}^2(X,Y) \wedge g_{\underline{1}}^2(X,Y) \wedge g_{\underline{2}}^2(X,Y) \wedge h(X,Y) \end{aligned}$$

(End of Example)

Example 6 Consider the function f define by Table 13, which is in (B-6). The formula expressing f is given by Property 6. Since f is in the case (2) of Property 6, f is expressed by the formula given in Eq. (29).

First, consider the formula $G(X,Y)$. It follows by Eq. (17) and (18) that one-variable functions $f_A^1(X)$ and $f_A^2(Y)$ are obtained below.

Table 19: Truth Table of G of Example 6

$X\backslash Y$	0	1	2	01	02	12	012
0	0	0	0	0	0	0	0
1	0	1	1	01	01	1	01
2	0	2	2	12	02	2	02
01	0	01	012	012	012	012	012
02	0	012	02	012	02	012	012
12	0	12	12	012	012	12	012
012	0	012	012	012	012	012	012

Table 20: Truth Table of w of Example 6

$X\backslash Y$	0	1	2	01	02	12	012
0	1	1	2	1	12	12	12
1	1	1	2	1	12	12	12
2	2	2	2	2	2	2	2
01	1	1	2	1	12	12	12
02	12	12	2	12	12	12	12
12	12	12	2	12	12	12	12
012	12	12	2	12	12	12	12

$$f^1_{\underline{0}}(X)=\underline{0} \quad f^1_{\underline{1}}(X)=(\underline{1}\wedge X^{\underline{12}})\sqcup X^{\underline{2}} \quad f^1_{\underline{2}}(X)=X^{\underline{12}}\wedge(\underline{1}\sqcup X^{\underline{02}})$$
$$f^2_{\underline{0}}(Y)=\underline{0}, \quad f^2_{\underline{1}}(Y)=\underline{1}\wedge Y^{\underline{12}} \quad f^2_{\underline{2}}(Y)=Y^{\underline{12}}$$

Thus, we obtain $g^1_A(X,Y)$ and $g^2_A(X,Y)$ of Eq. (23) below.

$$g^1_{\underline{0}}(X,Y)=Y^{\underline{12}} \quad g^1_{\underline{1}}(X,Y)=(\underline{1}\wedge X^{\underline{12}})\sqcup X^{\underline{2}}\sqcup Y^{\underline{02}} \quad g^1_{\underline{2}}(X,Y)=X^{\underline{12}}\wedge(\underline{1}\sqcup X^{\underline{02}})\sqcup Y^{\underline{01}}$$
$$g^2_{\underline{0}}(X,Y)=X^{\underline{12}} \quad g^2_{\underline{1}}(X,Y)=X^{\underline{02}}\sqcup(\underline{1}\wedge Y^{\underline{12}}) \quad g^2_{\underline{2}}(X,Y)=X^{\underline{01}}\sqcup Y^{\underline{12}}$$

By Eq. (26), we have the function $G(X,Y)$ expressed by $\wedge$-ing of all the above $g^i_A(X,Y)$'s. Table 19 is the truth table of G.

Next, consider $w(X,Y)$ of Eq. (28). Because

$$Q_{\underline{02}}=\{(\underline{2},\underline{01}),(\underline{2},\underline{02}),(\underline{2},\underline{012}),(\underline{02},\underline{02})\},$$

it follow by Eq. (28) that the formula w is obtained below.

$$w(X,Y)=\underline{1}\sqcup X^{\underline{2}}\sqcup Y^{\underline{2}}$$

Table 20 is the truth table of w. Lastly, it follows by Eq. (29) that the following formula expresses the function f.

$$\begin{aligned} f(X,Y) &= G(X,Y)\wedge w(X,Y) \\ &= g^1_{\underline{0}}(X,Y)\wedge g^1_{\underline{1}}(X,Y)\wedge g^1_{\underline{2}}(X,Y)\wedge g^2_{\underline{0}}(X,Y)\wedge g^2_{\underline{1}}(X,Y)\wedge g^2_{\underline{2}}(X,Y)\wedge w(X,Y). \end{aligned}$$

(End of Example)

7 Conclusions

This paper discussed functions over P_r that preserves the set inclusion relation $\subseteq$. We referred the three kinds of operations Min, Max, and Literals over E_r, because they are functionally complete on the r-valued set E_r. This paper then proved some of the mathematical properties of functions over P_r that can be expressed by formulas. It is one of the open problems that which set of operations $\hat{o}_1, \hat{o}_2, \ldots, \hat{o}_m$ over P_r can realize any function over P_r preserving the set inclusion relation $\subseteq$.

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