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Some problems concerning mad families

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Abstract

We explain some of the well-known, and also some less-known, open problems about maximal almost disjoint families.

Introduction

Almost disjoint families of sets of natural numbers play an important role in set theory and its applications. For example, they are used in forcing theory for almost disjoint coding or in set-theoretic topology for the construction of the Isbell-Mrówka space.

Let κ be a cardinal. Recall that a family $\mathcal{A} \subseteq [\kappa]^{\kappa}$ is an *almost disjoint family* (*a.d. family*, for short) if $|A \cap B| < \kappa$ for any two distinct members A and B of $\mathcal{A}$. It is a *maximal almost disjoint family* (*mad family*) if it is a.d. and maximal with this property, i.e., for every $C \in [\kappa]^{\kappa}$ there is an $A \in \mathcal{A}$ with $|C \cap A| = \kappa$. Any partition of κ into less than $cf(\kappa)$ many pieces trivially is mad. Accordingly define the *almost disjointness number* $\mathfrak{a}_{\kappa}$ as the least cardinality of a mad family of size $\geq cf(\kappa)$. A simple diagonal argument shows that $\mathfrak{a}_{\kappa} > cf(\kappa)$. If $\kappa = \omega$, we omit the subscript and simply write $\mathfrak{a}$.

While there has been substantial research on a.d. families and mad families in the last decades – and part of this research even lead to the development of deep novel techniques, like Shelah's iteration along templates [Sh3] (see below, Section 1) – there are still many astonishingly simple (in their formulation) open problems concerning such families. The purpose of this short note is to point to some well-known and some less-known problems in the area, with

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particular emphasis on purely set-theoretic problems, many of them connected with forcing theory. Some of these problems were mentioned during my talk on the RIMS workshop. We do not strive towards completeness, even within the narrow area we set ourselves. For example, we shall not mention Shelah's important recent work on completely separable mad families [Sh4] and questions resulting from this work. Nor shall we mention problems connected with forcing (in)destructibility of mad families (see, e.g., [BY]). Thus this note has a strongly personal flavor. For a much more encompassing recent survey on mad families see [Hr2].

1 Classical cardinal invariants

As usual, we let $\mathfrak{b}$ denote the *(un)bounding number*, $\mathfrak{d}$ the *dominating number*, $\mathfrak{s}$ the *splitting number*, $\mathfrak{r}$ the *(un)reaping number*, $\mathfrak{u}$ the *ultrafilter number*, and $\mathfrak{i}$ the *independence number.* See e.g. [Bl] for the definitions as well as for known ZFC-inequalities between, and consistency results about the order relationship of, these cardinals.

We are interested in the question of how these cardinals compare with $\mathfrak{a}$ and, in particular, whether some of them are lower or upper bounds for $\mathfrak{a}$. An old and easy result says that $\mathfrak{b} \leq \mathfrak{a}$ (see [vD] or [Bl, Proposition 8.4]), and it has been known for a while that this was the best lower bound in terms of classical cardinal invariants of the continuum. In particular, all of the other cardinals mentioned above are consistently larger than $\mathfrak{a}$: for example the Cohen model satisfies $\mathfrak{a} = \aleph_1$ [Ku, Theorem VIII.2.3] and $\mathfrak{d} = \mathfrak{r} = \mathfrak{u} = \mathfrak{i} = \mathfrak{c}$.[1] The consistency of $\mathfrak{a} < \mathfrak{s}$ is more difficult and was first obtained by Shelah [Sh1] with a countable support iteration of proper forcing. Hence his model satisfies $\mathfrak{a} = \aleph_1$ and $\mathfrak{s} = \mathfrak{c} = \aleph_2$. A model with $\mathfrak{a} = \kappa$ and $\mathfrak{s} = \mathfrak{c} = \lambda$ for arbitrary regular uncountable $\kappa < \lambda$ was obtained by Fischer and the author in [BF], using the method of matrix iterations originally developed by Blass and Shelah [BS].

The question of whether any of the classical cardinal invariants could possibly be an upper bound of $\mathfrak{a}$ proved to be much deeper and more complicated. The simplest result in this direction is the consistency of $\mathfrak{s} < \mathfrak{b}$ and, a fortiori, $\mathfrak{s} < \mathfrak{a}$, due to Baumgartner and Dordal [BD]. In fact they proved, using a rank argument for Hechler forcing modifications of which have been widely used since for all kinds of forcing notions adjoining dominating reals, that $\mathfrak{s} = \aleph_1 < \mathfrak{b} = \mathfrak{c}$ in the Hechler model (the model obtained by a finite support iteration of length a regular cardinal larger than $\aleph_1$ of Hechler forcing). Modifying his result mentioned above, Shelah [Sh1] also proved the consistency of $\mathfrak{b} < \mathfrak{a} = \mathfrak{s}$, again with a countable support iteration of proper forcing.[2] The author obtained a model for $\mathfrak{b} = \kappa$ and $\mathfrak{s} = \mathfrak{a} = \mathfrak{c} = \kappa^+$ for arbitrary regular uncountable κ using a

[1] Similarly one has $\mathfrak{a} = \mathfrak{d} = \aleph_1$ [Bl, Subsection 11.4] and $\mathfrak{r} = \mathfrak{u} = \mathfrak{i} = \mathfrak{c}$ in the random model.

[2] The reason why the consistency of $\aleph_1 = \mathfrak{b} < \mathfrak{a}$ is more difficult than the one of, say, $\aleph_1 = \mathfrak{s} < \mathfrak{b} = \mathfrak{a}$ is that the former also requires certain combinatorial principles on ω_1 to fail. For example, $\diamondsuit(\mathfrak{b})$, a strengthening of $\mathfrak{b} = \aleph_1$ implies $\mathfrak{a} = \aleph_1$ [MHD], and $\mathfrak{b} = \aleph_1$ together with the assumption that a cardinal invariant of 2^{ω_1} is larger than $\mathfrak{c}$ also implies $\mathfrak{a} = \aleph_1$ [Hy].

finite support iteration of ccc forcing [Br1]. Notice, however, that in all known models where the smaller of $\mathfrak{s}$ and $\mathfrak{b}$ is $\aleph_1$ and $\mathfrak{a}$ is $\aleph_2$ (or larger), the larger of the former two cardinals is (at least) $\aleph_2$. Accordingly we ask

Problem 1 (Brendle and Raghavan). *Is $\mathfrak{b} = \mathfrak{s} = \aleph_1 < \mathfrak{a} = \mathfrak{c} = \aleph_2$ consistent?*

Using Shelah's technique of taking ultrapowers of p.o.'s [Sh3] (see also [Br6]), Fischer and the author [BF] proved the consistency of $\mathfrak{b} = \kappa$ and $\mathfrak{a} = \mathfrak{s} = \mathfrak{c} = \lambda$ for arbitrary regular uncountable $\kappa < \lambda$ larger than a measurable cardinal.

Comparing $\mathfrak{d}$, $\mathfrak{r}$, $\mathfrak{u}$, and $\mathfrak{i}$ with $\mathfrak{a}$ turned out to be an even more interesting problem. A breakthrough was made by Shelah in 1999 [Sh3]: he first observed that *taking the ultrapower* of a partial order $\mathbb{P}$ via a κ-complete ultrafilter on a measurable cardinal κ and forcing with it destroys the maximality of any a.d. family of size $\geq \kappa$ in the intermediate extension via $\mathbb{P}$, while it preserves any scale of size $\neq \kappa$. This means that taking as $\mathbb{P}$ the partial order adding μ Hechler reals for some regular $\mu > \kappa$, and then iteratively taking the ultrapower for λ many steps for some regular $\lambda > \mu$, one obtains the consistency of $\mathfrak{b} = \mathfrak{d} = \mu$ and $\mathfrak{a} = \mathfrak{c} = \lambda$. Some care has to be taken in limit stages. See [Sh3] or [Br6] for details. Replacing Hechler forcing by Laver forcing with an ultrafilter, one even gets $\mathfrak{b} = \mathfrak{d} = \mathfrak{r} = \mathfrak{u} = \mu$ and $\mathfrak{a} = \mathfrak{c} = \lambda$ with the same method. In the same paper, Shelah also developed a new iteration technique (for ccc forcing), *iteration along templates*, which he used to prove the consistency of $\mathfrak{b} = \mathfrak{d} = \mu$ and $\mathfrak{a} = \mathfrak{c} = \lambda$ for any regular $\lambda > \mu \geq \aleph_2$, on the basis of *ZFC* alone.[3] Very roughly speaking, for destroying madness, the ultrapower argument is replaced by an *isomorphism-of-names argument.* See [Sh3] or [Br3] (or even [Br5]) for details. In the latter paper, the consistency of $\mathfrak{d} < \mathfrak{a}$ is strengthened to $\mathsf{cof}(\mathcal{N}) < \mathfrak{a}$, still on the basis of *ZFC*. Since the fragments of the iteration are built up in a more complicated way with this second method, it so far works only for easily definable (Suslin) ccc forcing. In particular, the following is still open:

Problem 2. (Shelah [Sh2, Question 10.1(2)]) *Is $\mathfrak{u} < \mathfrak{a}$ consistent on the basis of ZFC?*

The simplest instance of the problem one encounters when one tries to build up fragments of the iteration using Laver forcing with an ultrafilter as iterands is

Problem 3. [Br6, Question 2] *Assume $\mathbb{P}_{0\wedge 1} \lessdot \mathbb{P}_i \lessdot \mathbb{P}_{0\vee 1}$, $i \in \{0,1\}$, are forcing notions with correct projections and $\dot{\mathcal{U}}_i$ are $\mathbb{P}_i$-names for ultrafilters, $i \in \{0\wedge 1, 0, 1\}$, such that $\Vdash_{\mathbb{P}_i} \dot{\mathcal{U}}_{0\wedge 1} \subseteq \dot{\mathcal{U}}_i$, $i \in \{0,1\}$. Is there a $\mathbb{P}_{0\vee 1}$-name $\dot{\mathcal{U}}_{0\vee 1}$ for an ultrafilter such that $\Vdash_{\mathbb{P}_{0\wedge 1}} \dot{\mathcal{U}}_0, \dot{\mathcal{U}}_1 \subseteq \dot{\mathcal{U}}_{0\vee 1}$?*

This may be false in general. The real question, then, would be whether it is true for a sufficiently large class of forcing notions so that the iteration can be built up.

[3]The proof using the measurable and ultrapowers can also be formulated in the template framework, but the argument without templates is simpler in this case. However, for preservation results the template framework can be more convenient, as observed by Mejía [Me] who used it to prove the consistency of $\mathfrak{s} = \theta < \kappa < \mathfrak{b} = \mu < \mathfrak{a} = \lambda$ for arbitrary regular θ, μ, λ with κ being measurable in the ground model.

Both the ultrapower and the isomorphism-of-names arguments mentioned in the previous paragraph also destroy maximality of independent families, by the very same reason they destroy madness. Therefore, neither of the methods can be used to solve the following old problem:

Problem 4. (Vaughan [Va, Problem 1.1(c)]) *Is* $\mathfrak{i} < \mathfrak{a}$ *consistent?*

All the models (mentioned above) for $\mathfrak{d} < \mathfrak{a}$ have $\mathfrak{d} \geq \aleph_2$ and, in fact, the following famous problem from the seventies is still open.

Problem 5. (Roitman [Mi2, Problem 4.1]) *Does* $\mathfrak{d} = \aleph_1$ *imply* $\mathfrak{a} = \aleph_1$*?*

The ultrapower method for destroying madness works only for a.d. families of size at least the measurable (of the ground model), and in the template models, the isomorphism-of-names argument applies only to families of at least $\aleph_2$ names (this is so because names for reals in ccc forcing are countable objects and, under CH, one needs at least $\aleph_2$ of them so that "many" are isomorphic) and thus can be used only to show that there are no mads of size κ were $\aleph_2 \leq \kappa < \mathfrak{c}$. In either case, another argument is needed to get rid of small mads: in the former one makes $\mathfrak{b}$ larger than the measurable and in the latter one makes $\mathfrak{b} \geq \aleph_2$ for this purpose. In particular, $\mathfrak{d} \geq \aleph_2$. There is even a deeper reason why Roitman's Problem cannot be solved by the template technique: namely, if the template framework is set up to force $\mathfrak{d} = \aleph_1$ it will automatically force $\diamondsuit_{\mathfrak{d}}$ [Br3], and the latter is known to imply $\mathfrak{a} = \aleph_1$ [Hr1].

Since $\max\{\mathfrak{b}, \mathfrak{s}\} \leq \mathfrak{d}$, Roitman's Problem is closely related to Problem 1. We shall come back to (ramifications of) Problem 5 in Sections 3 and 4.

2 Variations on mad families

Let f and g be (partial) functions (with countable domain) from ω to ω. f and g are *eventually different* if $f(n) \neq g(n)$ for all but finitely many $n \in \text{dom}(f) \cap \text{dom}(g)$. Since we identify functions with their graphs, this is the same as saying that f and g are almost disjoint as subsets of $\omega \times \omega$. Say a family of functions $\mathcal{A}$ is an *eventually different family* if its members are pairwise eventually different. $\mathcal{A}$ is *maximal eventually different* if it is maximal with this property for the functions in the considered class. Let $\mathfrak{a}_\mathfrak{s}$ denote the least size of a maximal eventually different family of partial functions with countable domain. Similarly define $\mathfrak{a}_\mathfrak{e}$ to be the least size of a maximal eventually different family of total functions.

Also consider $\text{Sym}(\omega)$, the group of all permutations of the countable set ω. Let $\mathfrak{a}_\mathfrak{p}$ be the least size of a maximal eventually different family of permutations. Say a permutation $g \in \text{Sym}(\omega)$ is *cofinitary* if g has only finitely many fixed points. A subgroup $G \leq \text{Sym}(\omega)$ is *cofinitary* if every non-identity element of G is cofinitary. It is easy to see that G is cofinitary iff G is an eventually different family and a group. Finally, $\mathfrak{a}_\mathfrak{g}$ denotes the least size of a *maximal cofinitary group* (i.e., the least size of a maximal eventually different family of permutations which is also a group).

Using Bartoszyński's characterization of $\mathsf{non}(\mathcal{M})$ [BJ, Theorem 2.4.7], it is easy to see that $\mathsf{non}(\mathcal{M}) \leq \mathfrak{a}_\mathfrak{s}$ and $\mathsf{non}(\mathcal{M}) \leq \mathfrak{a}_\mathfrak{e}$. In joint work with Spinas and Zhang, the author proved $\mathsf{non}(\mathcal{M}) \leq \mathfrak{a}_\mathfrak{p}$ and $\mathsf{non}(\mathcal{M}) \leq \mathfrak{a}_\mathfrak{g}$ as well [BSZ]. In particular, all of these cardinals are larger than $\max\{\mathfrak{b}, \mathfrak{s}\}$. Also, they are consistently larger than $\mathfrak{a}$, for in the random model one obtains $\mathfrak{a} = \aleph_1$ and $\mathsf{non}(\mathcal{M}) = \mathfrak{a}_\mathfrak{s} = \mathfrak{a}_\mathfrak{e} = \mathfrak{a}_\mathfrak{p} = \mathfrak{a}_\mathfrak{g} = \mathfrak{c}$ (see footnote 1). Using either the ultrapower or the template technique mentioned in the previous section one obtains the consistency of $\mathfrak{a} = \mathfrak{a}_\mathfrak{s} = \mathfrak{a}_\mathfrak{e} = \mathfrak{a}_\mathfrak{p} = \mathfrak{a}_\mathfrak{g} > \mathsf{non}(\mathcal{M})$ and, by replacing Hechler forcing by eventually different reals forcing in either framework, even the consistency of $\mathfrak{a}_\mathfrak{s} = \mathfrak{a}_\mathfrak{e} = \mathfrak{a}_\mathfrak{p} = \mathfrak{a}_\mathfrak{g} > \max\{\mathsf{non}(\mathcal{M}), \mathfrak{a}\}$ [Br3]. However, in all these models $\mathsf{non}(\mathcal{M})$ is at least $\aleph_2$. Accordingly we ask

Problem 6. (Hyttinen [Hy, Introduction], also Fleissner [Mi2, Problem 4.7]) *Is $\mathsf{non}(\mathcal{M}) = \aleph_1 < \mathfrak{a}_\mathfrak{e}$ consistent? Same question for $\mathfrak{a}$ instead of $\mathfrak{a}_\mathfrak{e}$ and for any of the other relatives discussed above.*

Fleissner's original question asked whether the existence of a Luzin set, an assumption stronger than $\mathsf{non}(\mathcal{M}) = \aleph_1$, implies $\mathfrak{a} = \aleph_1$. This is also still open. Since $\max\{\mathfrak{b}, \mathfrak{s}\} \leq \mathsf{non}(\mathcal{M})$ in ZFC, the $\mathfrak{a}$-version of Problem 6 is related to Problem 1. Hyttinen proved that if $\mathsf{non}(\mathcal{M}) = \aleph_1$ and a cardinal invariant of 2^{ω_1} is larger than $\mathfrak{c}$, then $\mathfrak{a}_\mathfrak{e} = \aleph_1$ [Hy] (see also footnote 2).

Since adding the vertical sections in $\omega \times \omega$ to a maximal eventually different family of partial functions results in a mad family on $\omega \times \omega$, $\mathfrak{a} \leq \mathfrak{a}_\mathfrak{s}$ is immediate. However, nothing else is known about the relationship between this cardinals.

Problem 7. (Zhang [Zh1, Question 4.2], [Zh2, Question 4.1]) *Is $\mathfrak{a}_\mathfrak{s} = \mathfrak{a}_\mathfrak{e} = \mathfrak{a}_\mathfrak{p} = \mathfrak{a}_\mathfrak{g}$ or are any two of these cardinals consistently distinct?*

Problem 8. (Zhang [Zh1, Question 4.1], [Zh3, Question 4.1]) *Is $\mathfrak{a}_\mathfrak{e} \geq \mathfrak{a}$ or is $\mathfrak{a}_\mathfrak{e} < \mathfrak{a}$ consistent? Similarly for $\mathfrak{a}_\mathfrak{p}$ and $\mathfrak{a}_\mathfrak{g}$ instead of $\mathfrak{a}_\mathfrak{e}$.*

A breakthrough on the connection between maximal eventually different families of partial and total functions was made by Raghavan [Ra] a couple of years ago: he proved there is a *van Douwen mad family* (in ZFC), that is, a maximal eventually different family of total functions which is also maximal with respect to partial functions (and thus is a mad family on $\omega \times \omega$ when augmented by the vertical sections). His mad family has size $\mathfrak{c}$.

Problem 9. (Raghavan [Ra, Question 2.16]) *Is it consistent that the least size of a van Douwen mad family is strictly larger than $\mathfrak{a}_\mathfrak{e}$?*

Note that if there was a van Douwen mad family of size $\mathfrak{a}_\mathfrak{e}$ (in ZFC) this would immediately imply $\mathfrak{a}_\mathfrak{e} \geq \mathfrak{a}_\mathfrak{s}$ and thus also $\mathfrak{a}_\mathfrak{e} \geq \mathfrak{a}$.

In recent years, the investigation of definability of mad families has gotten a lot of attention. Consider the space $[\omega]^\omega$ of infinite subsets of ω as a subspace of the Cantor space $\mathcal{P}(\omega) = 2^\omega$. Clearly $[\omega]^\omega$ is homeomorphic to ω^ω. Thus one may talk about open, closed, Borel, analytic, coanalytic, or projective subsets of $[\omega]^\omega$. Note an a.d. family never can be open, and the standard example of an a.d. family of size $\mathfrak{c}$ is closed (even perfect). Also notice that (because of maximality) a mad family is $\mathbf{\Sigma}^1_n$ iff it is $\mathbf{\Delta}^1_n$. Törnquist [Tö] recently observed that the existence of a Σ^1_2 mad family implies the existence of a Π^1_1 mad family (and similarly for boldface). A classical result of Mathias [Ma] says that no mad family in $[\omega]^\omega$ can be analytic. Miller [Mil] proved that $V = L$ implies the existence of a Π^1_1 mad family (this also follows from the – rather easy – construction of a Σ^1_2 mad using the Σ^1_2 well order of L and Törnquist's observation). We shall come back to definability of mad families in $[\omega]^\omega$ in the next section.

By Mathias' Theorem there can be no analytic maximal eventually different family of partial functions. In particular, there can be no analytic van Douwen mad family. However, for maximal eventually different families of total functions, it is still unknown whether the analogue of Mathias' Theorem holds.

Problem 10. (Kastermans, Steprāns, and Zhang [KSZ, Question 1.1]) *Is there an analytic (or even closed) maximal eventually different family of total functions?*

A partial result was proved by Steprāns [KSZ]: say that an eventually different family of total functions $\mathcal{A}$ is *strongly maximal* if given any countable $\mathcal{B} \subseteq \omega^\omega$ such that no member of $\mathcal{B}$ is eventually covered by finitely many members of $\mathcal{A}$, there is $f \in \mathcal{A}$ such that $f \cap g$ is infinite for all $g \in \mathcal{B}$. Steprāns showed that there are no analytic strongly maximal families. Raghavan [Ra] observed that any strongly maximal family is van Douwen mad so that Steprāns' Theorem in fact is a consequence of Mathias' Theorem. However, unlike for van Douwen mad families, it is open whether strongly maximal families exist without additional assumptions.

Problem 11. (Kastermans, Steprāns, and Zhang [KSZ, Question 4.2]) *Do strongly maximal families exist on the basis of ZFC?*

The question analogous to Problem 10 is open for maximal eventually different families of permutations and for maximal cofinitary groups.

Problem 12. (Gao and Zhang [GZ, Conjecture 1.7]) *Is there an analytic maximal cofinitary group?*

In fact the conjecture is that both this problem and Problem 10 have negative answers (see also [Ra, Conjecture 3.27]). On the coanalytic level, one obtains positive results in the constructible universe: under $V = L$, there is a Π^1_1 maximal eventually different family of total functions [KSZ], a Π^1_1 maximal eventually different family of permutations [GZ], and a Π^1_1 maximal cofinitary group [Ka].

3 Mad families built from Borel sets

In this section we look at mad families which involve perfect a.d. families as building blocks (and thus always have size $\mathfrak{c}$). The simplest thing one can do in this direction is to start with a closed a.d. family $\mathcal{A}$ in $[\omega]^\omega$ which is given by branches through a tree isomorphic to $[\omega]^\omega$ and then blow it up to a mad family. Accordingly define the *off-branch number* $\mathfrak{o}$ as the least size of a family $\mathcal{B} \subseteq [\omega]^\omega$ such that $\mathcal{A} \cup \mathcal{B}$ is mad. Also, let $\bar{\mathfrak{o}}$ be the least size of a $\mathcal{B} \subseteq [\omega]^\omega$ which consists of *antichains* in the tree underlying $\mathcal{A}$ such that $\mathcal{A} \cup \mathcal{B}$ is mad. Then (clearly) $\mathfrak{o} \leq \bar{\mathfrak{o}}$ and furthermore $\mathfrak{o} \geq \mathfrak{a}$, $\bar{\mathfrak{o}} \geq \mathfrak{a}_\mathfrak{s}$, $\mathfrak{o} \geq \mathsf{cov}(\mathcal{M})$ (all in [Le]), and $\mathfrak{o} \geq \mathsf{non}(\mathcal{M})$ [Br2]. In particular $\mathfrak{a} = \mathfrak{a}_\mathfrak{s} = \mathfrak{a}_\mathfrak{e} = \mathfrak{a}_\mathfrak{p} = \mathfrak{a}_\mathfrak{g} = \aleph_1$ and $\mathfrak{o} = \bar{\mathfrak{o}} = \mathfrak{c}$ holds in the Cohen model. We do not know:

Problem 13. (Leathrum [Le, Question 8.2]) *Is* $\mathfrak{o} = \bar{\mathfrak{o}}$*? Or is* $\mathfrak{o} < \bar{\mathfrak{o}}$ *consistent?* [Br1, Question 1.13] *Is even* $\mathfrak{o} < \mathfrak{a}_\mathfrak{s}$ *consistent?*

One may also consider mad families that are built up from perfect a.d. families or even more complicated Borel a.d. families of size $\mathfrak{c}$: let $\mathfrak{a}_{\text{closed}}$, the *closed almost disjointness number*, be the least size of an infinite family of closed sets in $[\omega]^\omega$ whose union is mad. Similarly, $\mathfrak{a}_{\text{Borel}}$, the *Borel almost disjointness number*, is the least size of an infinite family of Borel sets whose union is mad. Obviously $\mathfrak{a}_{\text{Borel}} \leq \mathfrak{a}_{\text{closed}} \leq \mathfrak{a}$. Mathias' Theorem quoted above says exactly that $\mathfrak{a}_{\text{Borel}} \geq \aleph_1$ and this can be improved to $\mathfrak{a}_{\text{Borel}} \geq \mathfrak{t}$ (Raghavan, unpublished). A further plausible improvement would be:

Problem 14. (Raghavan, see [BK, Conjecture 4.5] or [BR, Question 51]) *Is* $\mathfrak{h} \leq \mathfrak{a}_{\text{Borel}}$*?*

Both Shelah's [Sh1] and the author's proof [Br1] for the consistency of $\mathfrak{b} < \mathfrak{a}$ (see Section 1) can be modified to yield $\mathfrak{b} < \mathfrak{a}_{\text{closed}}$ [BR], and the ultrapower argument for $\mathfrak{d} = \mathfrak{u} < \mathfrak{a}$ [Sh3] (see Section 1) can be modified to the consistency of $\mathfrak{d} = \mathfrak{u} < \mathfrak{a}_{\text{closed}}$ [RS2].

On the other hand, a strong version of the splitting phenomenon entails $\mathfrak{a}_{\text{closed}} = \aleph_1$ – and this distinguishes $\mathfrak{a}_{\text{closed}}$ from $\mathfrak{a}$: say that $\mathcal{A} = \{A_{\alpha,n} : \alpha < \omega_1 \text{ and } n \in \omega\}$ is a *club-splitting sequence of partitions* if for all $B \in [\omega]^\omega$, $C_B = \{\alpha < \omega_1 : \text{all } A_{\alpha,n} \text{ split } B\}$ contains a club. Clearly the existence of a club-splitting sequence of partitions implies $\mathfrak{s} = \aleph_1$ (and even $\mathfrak{s}_\omega = \aleph_1$). The author proved [BR] that it also implies $\mathfrak{a}_{\text{closed}} = \aleph_1$. Since $\mathfrak{d} = \aleph_1$ implies the existence of a club-splitting sequence of partitions, $\mathfrak{a}_{\text{closed}} = \aleph_1$ follows from $\mathfrak{d} = \aleph_1$ (a result originally proved by Raghavan and Shelah [RS1] using different means). This gives a positive answer to Problem 5 for $\mathfrak{a}_{\text{closed}}$ instead of $\mathfrak{a}$. Also, since there is a club-splitting sequence of partitions in the Hechler model, the consistency of $\mathfrak{a}_{\text{closed}} < \mathfrak{b}$ (and thus also of $\mathfrak{a}_{\text{closed}} < \mathfrak{a}$) follows (a result originally proved by Khomskii and the author [BK]). We do not know whether the existence of a club-splitting sequence of partitions can be replaced by $\mathfrak{s} = \aleph_1$.

Problem 15. (Brendle and Khomskii [BK, Question 4.6]) *Does* $\mathfrak{s} = \aleph_1$ *imply* $\mathfrak{a}_{\text{closed}} = \aleph_1$*?*

Neither do we know whether $\mathfrak{a}_{\text{closed}}$ and $\mathfrak{a}_{\text{Borel}}$ can be distinguished.

Problem 16. (Brendle and Khomskii [BK, Question 4.7]) *Is* $\mathfrak{a}_{\text{closed}} = \mathfrak{a}_{\text{Borel}}$*?*

In fact, we do not know of any construction of mad families which involves Borel a.d. families more complicated than closed ones as building blocks. So, perhaps, the simplest question in this context is:

Problem 17. *Is there a Borel a.d. family that is not contained in an F_σ a.d. family?*

The question of whether the existence of easily definable mad families is consistent with large continuum has gotten quite some attention during the past few years. It had been observed early on that Π^1_1 mad families survive in a number of forcing extensions of L (with the same Π^1_1 definition) – roughly speaking, if $\mathbb{P}$ is one of the classical forcing notions adding reals and there is a $\mathbb{P}$-indestructible mad family, then there is also such a mad family with a Π^1_1 definition; at least, this is known to be true for Cohen, random, Miller, and Sacks forcings. Thus the existence of a Π^1_1 mad family (of size $\aleph_1$) is consistent with a number of cardinal invariants being large, e.g., $\mathsf{cov}(\mathcal{M})$ or $\mathsf{cov}(\mathcal{N})$.

Of course, this approach cannot work for obtaining a definable mad family when $\mathfrak{b} > \aleph_1$. Recently, Fischer, Friedman, and Zdomskyy [FZ, FFZ] proved that it is consistent that there is a Π^1_2 mad family and $\mathfrak{b} > \aleph_1$. Subsequently, Khomskii and the author [BK] showed that under CH there is an ω_1-sequence of closed sets whose union is mad and which survives (iterated) Hechler forcing if the closed sets are reinterpreted in the extension, see above. If this sequence is taken in L with a Σ^1_2 definition, one obtains a Σ^1_2 mad family, and thus – by Törnquist's result – also a Π^1_1 mad family, consistent with $\mathfrak{b} > \aleph_1$. In fact, the problem of getting definable mad families of low complexity consistent with large $\mathfrak{b}$ was the original motivation for introducing the cardinals $\mathfrak{a}_{\text{closed}}$ and $\mathfrak{a}_{\text{Borel}}$. Obviously, $\mathfrak{a}_{\text{Borel}} > \aleph_1$ implies there are no $\boldsymbol{\Sigma}^1_2$ mad families (because $\boldsymbol{\Sigma}^1_2$ sets are $\aleph_1$ Borel) and hence so does $\mathfrak{t} > \aleph_1$ (by Raghavan's observation mentioned above). We do not know whether $\mathfrak{h} > \aleph_1$ is sufficient (see Problem 14). Very closely related is:

Problem 18. (Brendle and Khomskii [BK, Question 4.1]) *Does the statement "all $\boldsymbol{\Sigma}^1_2$ sets are Ramsey" imply that there is no $\boldsymbol{\Pi}^1_1$ mad family?*

The connection between the two problems comes from the observation that the Mathias model is the canonical (= "minimal") model for both $\mathfrak{h} > \aleph_1$ and "all $\boldsymbol{\Sigma}^1_2$ sets are Ramsey".[4]

An answer to the following more basic question may help to shed light on this and other problems:

Problem 19. (Brendle and Khomskii [BK, Question 4.3]) *Is there a notion of transcendence over L which is equivalent to the nonexistence of $\boldsymbol{\Pi}^1_1$ mad families?*

Concerning more complicated sets, Mathias [Ma] proved that in the $L(\mathbb{R})$ of the Levy collapse of a Mahlo cardinal, there are no mad families; in particular, in the collapse model, there are no projective mad families. It is not known whether the Mahlo is needed:

[4] This is also related to a more general conjecture of Mathias, see below, Problem 20.

Problem 20. (Mathias [Ma, comment after 5.3]) *What is the consistency strength of $ZF + DC +$ "there are no mad families"?*

He conjectured that $ZF + DC+$ "all set of reals are Ramsey" is enough to get no mad families. This would mean the consistency strength is at most an inaccessible.[5] We even don't know:

Problem 21. *Is it consistent on the basis of ZFC that there are no $\mathbf{\Pi}^1_2$ mad families?*

4 Mad families on larger cardinals

We briefly turn to mad families on cardinals κ larger than ω. We split the discussion into two cases, regular κ and singular κ.

For regular κ, one has $\mathfrak{a}_\kappa \geq \mathfrak{b}_\kappa$ with the same proof as for ω. However, the following basic question is still open:

Problem 22. (Jech and Veličković, see [BHZ, Question 1.13]) *Is $\mathfrak{b}_\kappa < \mathfrak{a}_\kappa$ consistent?*

We do believe this is consistent, at least for large enough κ. In fact, we conjecture that modifying the consistency proof of [Br1] for $\mathfrak{b} < \mathfrak{a}$ using the methods of [DS], the consistency of $\mathfrak{b}_\kappa < \mathfrak{a}_\kappa$ can be shown for supercompact κ.

There is a fundamental obstacle to generalizing the ultrapower method or the template technique to cardinals larger than ω. Both involve having to deal with nonlinear iterations, that is, they canonically add reals $f_n \in \omega^\omega$ such that $f_{n+1} <^* f_n$ for all n. While, by a result of Hjorth, we cannot have that all members of such a sequence are generic over the model containing everything added below (i.e., it cannot be that f_n is generic over $M[f_k : k \geq n+1]$ for all n), it is possible that the f_n are partially generic (i.e., f_n is generic over any $M[f_k : k \in F]$ where F is finite) and this makes the whole construction work (see, e.g., [Br3]). However, if $\kappa > \omega$ it is simply impossible to have $f_n \in \kappa^\kappa$ with $f_{n+1} <^* f_n$ for this would immediately yield a decreasing sequence of ordinals.

On the other hand, Roitman's Problem (Problem 5) does have a positive answer for uncountable regular κ: using club guessing, Blass, Hyttinen, and Zhang [BHZ] proved that $\mathfrak{d}_\kappa = \kappa^+$ implies $\mathfrak{a}_\kappa = \kappa^+$. They also obtained a number of results on the κ-versions of the cardinals $\mathfrak{a}_\mathfrak{e}$, $\mathfrak{a}_\mathfrak{p}$, and $\mathfrak{a}_\mathfrak{g}$ discussed in Section 2. However the following is open:

Problem 23. (Jech and Veličković, see [BHZ, Question 1.13]) *Is $\mathfrak{a}_\kappa$ consistently smaller than the κ-versions of $\mathfrak{a}_\mathfrak{e}$, $\mathfrak{a}_\mathfrak{p}$, or $\mathfrak{a}_\mathfrak{g}$?*

We turn to singular κ. The most basic observation is $\mathfrak{a}_\kappa \leq \mathfrak{a}_{cf(\kappa)}$. (Simply write κ as a union of disjoint sets A_α ($\alpha < cf(\kappa)$) of size $< \kappa$ with $|A_\alpha| \to \kappa$; then, given a mad family $\mathcal{A}$ on $cf(\kappa)$, $\mathcal{A}^* = \{\bigcup_{\alpha \in A} A_\alpha : A \in \mathcal{A}\}$ is easily seen to be mad on κ.)

[5] It is an open problem whether the inaccessible is needed for the Ramsey property.

Now let $(\kappa_\alpha : \alpha < cf(\kappa))$ be a strictly increasing sequence of regular cardinals with supremum κ. Define $\mathfrak{b}(\prod_{\alpha<cf(\kappa)} \kappa_\alpha, \leq^*)$ to be the least size of an unbounded family in the partial order of functions in $\prod_{\alpha<cf(\kappa)} \kappa_\alpha$ ordered by eventual dominance. A diagonal argument shows that $\mathfrak{b}(\prod_{\alpha<cf(\kappa)} \kappa_\alpha, \leq^*) > \kappa$. Denote by $\mathfrak{b}_\kappa$ the supremum of $\mathfrak{b}(\prod_{\alpha<cf(\kappa)} \kappa_\alpha, \leq^*)$ over all possible choices of the sequence $(\kappa_\alpha : \alpha < cf(\kappa))$. This is a pcf type cardinal. Notice that $\mathfrak{b}_{cf(\kappa)} < \mathfrak{b}_\kappa$ (e.g., under GCH), $\mathfrak{b}_{cf(\kappa)} = \mathfrak{b}_\kappa$ (add κ^+ many $cf(\kappa)$-Hechler functions to a model GCH, where λ-Hechler forcing is the natural $< \lambda$-closed and λ^+-cc generalization of Hechler forcing to an arbitrary regular cardinal λ), and $\mathfrak{b}_{cf(\kappa)} > \mathfrak{b}_\kappa$ (add more than κ^+ many $cf(\kappa)$-Hechler functions to a model GCH) are all consistent. Kojman, Kubiś, and Shelah [KKS] proved $\mathfrak{a}_\kappa \geq \min\{\mathfrak{b}_\kappa, \mathfrak{b}_{cf(\kappa)}\}$. They asked

Problem 24. (Kojman, Kubiś, and Shelah [KKS, Problem 2.10]) *Is it consistent that $\mathfrak{a}_{\aleph_\omega} = \aleph_\omega$?*

Using iterations along templates, the author [Br4] proved that $\mathfrak{a} = \aleph_\omega$ is consistent. In his model $\mathfrak{b} = \aleph_2$ and $\mathfrak{b}_{\aleph_\omega} = \aleph_{\omega+1}$. It is unclear, however, what the value of $\mathfrak{a}_{\aleph_\omega}$ is. Assuming the existence of a measurable cardinal, the author [Br7] obtained a template model for $\mathfrak{a}_{\aleph_\omega} < \mathfrak{a}$. However, the following strikingly simple question is still open:

Problem 25. [Br7] *Is $\mathfrak{b} \neq \mathfrak{a}_{\aleph_\omega}$ consistent?*

The real problem seems to be to which extent "robust" mad families can be constructed on $\aleph_\omega$, that is, mad families which are basically distinct from those induced by mad families on ω. Versions of this problem are:

Problem 26. [Br7]

1. *Is every mad family on $\aleph_\omega$ ccc-destructible? Or are there ccc-indestructible mad families?*
2. *Can pcf theory or a guessing principle be used to construct,* directly, *a mad family on $\aleph_\omega$? Or is* $\max \text{pcf}\, \aleph_n < \mathfrak{a}_{\aleph_\omega}$ *($\mathfrak{b}_{\aleph_\omega} < \mathfrak{a}_{\aleph_\omega}$) consistent?*

We do not know whether we can distinguish the almost disjointness number of two distinct singular cardinals κ and λ of countable cofinality:

Problem 27. [Br7] *Is $\mathfrak{a}_\kappa \neq \mathfrak{a}_\lambda$ consistent?*

Neither do we know whether the almost disjointness number of a singular cardinal of uncountable cofinality can be distinguished from the one of its cofinality:

Problem 28. [Br7] *Is $\mathfrak{a}_{\aleph_{\omega_1}} < \mathfrak{a}_{\aleph_1}$ consistent?*

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