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SOME RESULTS IN THE EXTENSION WITH A COHERENT SUSLIN TREE, PART II

TADATOSHI MIYAMOTO AND TERUYUKI YORIOKA

ABSTRACT. It is proved the following: (**Theorem 1**) $\mathsf{PFA}(S)$ implies MRP. (**Theorem 2**) For any coherent Suslin tree S and an S-name $\underset{\sim}{\vec{C}}$ for a ladder system on ω_1, there exists an almost strongly proper forcing notion (which will be defined in §3) which generically adds an S-name for a club on ω_1 which violates the statement that $\underset{\sim}{\vec{C}}$ is a weak club guessing ladder systems.

1. INTRODUCTION

At first, we introduce two problems. Both are introduced by Stevo Todorčević in 1980s.

Todorčević–Veličković showed that Martin's Axiom is equivalent to some Ramsey theoretic statement [22, 20]. From the view point of this type of Ramsey theory, Todorčević introduced several weak forms of its Ramsey theoretic statement [20]. One of them is $\mathcal{K}_2$: Every ccc forcing has the Knaster property (every uncountable set has a pairwise-compatible uncountable subset). It is well known that $\mathsf{MA}_{\aleph_1}$ implies $\mathcal{K}_2$. It is still unknown whether $\mathcal{K}_2$ implies $\mathsf{MA}_{\aleph_1}$.

Todorčević introduced other Ramsey theoretic axiom, so called the P-ideal Dichotomy PID [19, 1, 21]. One of the motivations of introducing PID is the S-space problem: Is every regular hereditarily separable space hereditarily Lindelöf? (See e.g. [4, 13, 14].) An S-space means a regular hereditarily separable space which has non-Lindelöf subspace, so the S-space problem sometimes is stated the following: Are there no S-spaces? Todorčević proved that if PID holds and the pseudo-intersection number $\mathfrak{p}$ is greater than $\aleph_1$, then there are no S-spaces. It is also unknown whether $\mathfrak{p} > \aleph_1$ holds whenever PID holds and there

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are no S-spaces (that is, under PID, whether $\mathfrak{p} > \aleph_1$ is equivalent that there are no S-spaces).

Stevo Todorčević introduced the weak form of the Proper Forcing Axiom, so called $\mathsf{PFA}(S)$, to attack two problems simultaneously [23]. $\mathsf{PFA}(S)$ is the statement that there exists a coherent Suslin tree S such that the forcing axiom holds for every proper forcing which preserves S to be Suslin, that is, for every proper forcing $\mathbb{P}$ which preserves S to be Suslin and $\aleph_1$-many dense subsets D_α, $\alpha \in \omega_1$, of $\mathbb{P}$, there exists a filter on $\mathbb{P}$ which intersects all the D_α. Since the preservation of a Suslin tree by the proper forcing is closed under countable support iteration (due to the first author [9]), it is consistent relative to some large cardinal assumption that $\mathsf{PFA}(S)$ holds.

We note that a Suslin tree forces that $\mathfrak{p} = \aleph_1$ [3, Lemma 2], and hence, a Suslin tree forces that $\mathsf{MA}_{\aleph_1}$ fails. Todorčević proved that a coherent Suslin tree S, which witnesses $\mathsf{PFA}(S)$, forces several consequences from PFA, i.e. the Open Coloring Axiom (due to Todorčević), the failure of square axioms, and PID. Therefore, under $\mathsf{PFA}(S)$, if a coherent Suslin tree S, which witnesses $\mathsf{PFA}(S)$, forces $\mathcal{K}_2$, then the first problem above is answered negatively, and if such an S forces that there are no S-spaces, then the second problem above is answered negatively. There are some partial results on above two problems [7, 24] and [23, 26].

It is also interesting what happens in the extension with a coherent Suslin tree which witnesses $\mathsf{PFA}(S)$, assuming $\mathsf{PFA}(S)$. There are some researches on $\mathsf{PFA}(S)$, e.g. [23, 5, 6, 12, 16, 17, 18, 25, 26]. In this note, we consider the following three consequences from PFA; MRP, measuring [(1)], and weak club guessing ladder systems [10, 2, 11]. We will mention the definition of MRP in §2 and weak club guessing ladder system in §3. We note that MRP implies measuring, and measuring implies that there are no weak club guessing ladder system.

The first author proved the following.

Theorem 1. $\mathsf{PFA}(S)$ *implies* MRP.

Thus measuring holds under $\mathsf{PFA}(S)$. Justin Tatch Moore pointed out that a Suslin tree forcing preserves measuring, so as the corollary of the theorem, assuming $\mathsf{PFA}(S)$, in the extension with a coherent Suslin tree which witnesses $\mathsf{PFA}(S)$, measuring holds and so there are no weak club guessing ladder systems. Moore also pointed out that in the extension with a Suslin tree, MRP fails (because υ_{AC} (in e.g. [10])

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(1) The axiom "measuring" is defined in the introduction of [2].

fails there). So a Suslin tree forcing can separate MRP and measuring. In §2, we will give a proof of **Theorem 1**.

By the previous observation, it has already been known that assuming PFA(S), in the extension with a coherent Suslin tree which witnesses PFA(S), there are no weak club guessing ladder systems. In §3, we will prove it by introducing variation of Mitchell's strong properness.

In [15, Chapter IX, 2.6 Definition], Shelah introduced the strong properness as follows: A forcing notion $\mathbb{P}$ is called (Shelah's) strongly proper if for some (any) sufficiently large θ, a countable elementary submodel M of $H(\theta)$ with $\mathbb{P} \in M$, a countable sequence $\langle D_n; n \in \omega \rangle$ with $D_n \subseteq \mathbb{P} \cap M$ dense in $\mathbb{P} \cap M$ and $p \in \mathbb{P} \cap M$, there exists $q \leq_{\mathbb{P}} p$ such that for all $n \in \omega$, D_n is predense below q. In [8, Definition 2.3], Mitchell introduced a slightly different and stronger concept of the strong properness as follows: A forcing notion $\mathbb{P}$ is called (Mitchell's) strongly proper if for some (any) sufficiently large θ, a countable elementary submodel M of $H(\theta)$ with $\mathbb{P} \in M$ and $p \in \mathbb{P} \cap M$, there exists $q \leq_{\mathbb{P}} p$ such that every dense subset of $\mathbb{P} \cap M$ is predense below q. The first author proved that a strongly proper forcing notion preserves a Suslin tree [9, (1.5) Proposition]. In §3, we introduce a weak variation of Mitchell's strong properness, which is called the almost strong properness (**Definition 3.2**), and prove that an almost strongly proper forcing notion still preserves a Suslin tree (**Lemma 3.3**). Moreover, we will prove the following.

Theorem 2. *For any coherent Suslin tree S and an S-name $\underset{\sim}{\vec{C}}$ for a ladder system on ω_1, there exists an almost strongly proper forcing notion which generically adds an S-name for a club on ω_1 which violates the statement that $\underset{\sim}{\vec{C}}$ is a weak club guessing ladder systems.*

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2. A PROOF OF **Theorem 1**

Definition 2.1 (Moore). *Σ is called an open stationary set mapping when there are an uncountable set X and a regular cardinal θ with $X \in H(\theta)$ such that*

- dom(Σ) *is a club subset of the set of countable elementary submodels of $H(\theta)$,*
- *for every $M \in$* dom(Σ),
  - *$\Sigma(M)$ is an open subset of the space $[X]^{\aleph_0}$ equipped with the Ellentuck topology, and*

– *$\Sigma(M)$ is M-stationary, i.e. for any club subset E of $[X]^{\aleph_0}$, if $E \in M$, then $E \cap \Sigma(M) \cap M \neq \emptyset$.*

MRP *is the statement that for any open stationary set mapping Σ about X and θ, there exists a continuos $\in$-chain $\langle N_\nu; \nu \in \omega_1 \rangle$ in* $\mathrm{dom}(\Sigma)$ *such that for all limit ordinals $\nu \in \omega_1$, there exists $\nu_0 < \nu$ such that for any $\xi \in (\nu_0, \nu)$, $N_\xi \cap X \in \Sigma(N_\nu)$.*

Proof of **Theorem 1**. Let X be an uncountable set, θ a regular cardinal such that $2^X \in H(\theta)$, and Σ an open stationary set mapping such that for any $K \in \mathrm{dom}(\Sigma)$, $\Sigma(K) \subseteq [X]^{\aleph_0}$. We always consider that the structure $H(\theta)$ equips its well-ordering as a predicate. For a subset Y of $H(\theta)$, we write the Skolem hull of Y on $H(\theta)$ as $\mathsf{sk}_{\mathsf{H}(\theta)}(\mathsf{Y})$ (by the well-ordering which is equipped with $H(\theta)$). We write $\mathbb{P}_\Sigma$ as the set of all continuous $\in$-increasing sequence $p : \alpha + 1 \to \mathrm{dom}(\Sigma)$, for some $\alpha \in \omega_1$, such that for every $\nu \in \mathsf{Lim} \cap (\alpha + 1)$, there exists $\nu_0 < \nu$ such that for every $\xi \in (\nu_0, \nu)$, if $\nu_0 \in p(\xi)$, then $p(\xi) \cap X \in \Sigma(p(\nu))$, ordered by end-extension. This is defined in [10], and it is proved that $\mathbb{P}_\Sigma$ is proper. So for a proof of **Theorem 1**, it suffices to show that $\mathbb{P}_\Sigma$ preserves any Suslin tree to be still Suslin.

Suppose that S is a Suslin tree, $\dot{A}$ is a $\mathbb{P}_\Sigma$-name for a maximal antichain through S, and $p \in \mathbb{P}_\Sigma$. We will show that there exists $q \leq_{\mathbb{P}_\Sigma} p$ such that

$$q \Vdash_{\mathbb{P}_\Sigma} \text{“ } \dot{A} \text{ is countable ”}.$$

To show this, we take a regular cardinal λ such that $H(\lambda)$ contains the set

$$\left\{X, \Sigma, H(\theta), S, \dot{A}, p, H(|\mathbb{P}_\Sigma|^+)\right\}$$

as a member, and take a countable elementary substructure M of $H(\lambda)$ which contains the above set as a member. We enumerate all dense subsets of $\mathbb{P}_\Sigma$ which belong to M by $\{D_i; i \in \omega\}$ and the set of all nodes in S of height $\omega_1 \cap M$ by $\{s_i; i \in \omega\}$. Let $\langle \delta_i; i \in \omega \rangle$ be an increasing sequence of ordinals converging to $\omega_1 \cap M$. By induction on $i \in \omega$, we will build $p_i \in \mathbb{P}_\Sigma$, such that

- $p_{i+1} \leq_{\mathbb{P}_\Sigma} p_i$ (and $p_0 := p$),
- $p_{i+1} \in D_i \cap M$,
- for every $K \in \mathrm{ran}(p_{i+1} \setminus p_0)$, $X \cap K \in \Sigma(H(\theta) \cap M)$,
- $\mathrm{dom}(p_{i+1}) \geq \delta_i$, and
- there exists $t <_S s_i$ such that $p_{i+1} \Vdash_{\mathbb{P}_\Sigma}$ “ $t \in \dot{A}$ ”.

After the construction, we let

$$q := \bigcup_{i \in \omega} p_i \cup \{\langle \omega_1 \cap M, H(\theta) \cap M \rangle\}.$$

Then by the above properties of p_i's, we note that q is a $(\mathbb{P}_\Sigma, M)$-generic condition and

$$q \Vdash_{\mathbb{P}_\Sigma} \text{“ } \dot{A} \subseteq S_{<\omega_1 \cap M} \text{ ”},$$

which finishes the proof.

In the rest of this section, we mention how to build p_{i+1} when p_i has been built.

Let E_i be the set of all subsets of X of the form $X \cap N^*$ where N^* is a countable elementary substructure of $H(|\mathbb{P}_\Sigma|^+)$ which contains the set

$$\left\{X, \Sigma, H(\theta), S, D_i, \dot{A}, \delta_i, p_i\right\}$$

as a member. We note that E_i contains a club on $[X]^{\aleph_0}$ and E_i belongs to M. Since $2^X \in H(\theta)$, E_i also belongs to $H(\theta)$, so does to $H(\theta) \cap M$. Therefore, since $\Sigma(H(\theta) \cap M)$ is $(H(\theta) \cap M)$-stationary , there exists a countable elementary submodel N_i^* of $H(|\mathbb{P}_\Sigma|^+)$ and a finite subset x_i of $X \cap N_i^*$ such that $X \cap N_i^*$ is in the set $E_i \cap \Sigma(H(\theta) \cap M) \cap M$, $[x_i, X \cap N_i^*] \subseteq \Sigma(H(\theta) \cap M)$. By the elementarity of M, there exists $N_i^{**} \in M$ such that N_i^{**} is a countable elementary substructure $H(|\mathbb{P}_\Sigma|^+)$ which contains the set

$$\left\{X, \Sigma, H(\theta), S, D_i, \dot{A}, \delta_i, p_i\right\}$$

as a member such that

$$X \cap N_i^{**} = X \cap N_i^*.$$

We define

$$p_i' := p_i \cup \left\{\langle \mathrm{dom}(p_i), \mathsf{sk}_{H(\theta)}(\{p_i\} \cup x_i)\rangle\right\}.$$

We note that $p_i' \in \mathbb{P}_\Sigma \cap N_i^{**}$ and $p_i' \leq_{\mathbb{P}_\Sigma} p_i$. We note that $\mathbb{P}_\Sigma$ is in N_i^{**}. Next, we take a condition $p_i'' \in D_i \cap N_i^{**}$ such that $p_i'' \leq_{\mathbb{P}_\Sigma} p_i'$ and $\mathrm{dom}(p_i'') \geq \delta_i$. We note that $p_i'' \in M$, because $N_i^{**} \subseteq M$. At last, since the set

$$\left\{u \in S;\ \text{there are } t <_S u \text{ and } r \leq_{\mathbb{P}_\Sigma} p_i'' \text{ such that } r \Vdash_{\mathbb{P}_\Sigma} \text{“ } t \in \dot{A} \text{ ”}\right\}$$

is a dense open subset of S in the model N_i^{**} and the set

$$\{u \in S \cap N_i^{**}; u <_S s_i \restriction (\omega_1 \cap N_i^{**})\}$$

is a (S, N_i^{**})-generic filter, by the elementarity of N_i^{**}, we can take $t <_S s_i \restriction (\omega_1 \cap N_i^{**})$ and $p_{i+1} \in \mathbb{P}_\Sigma \cap N_i^{**}$ such that $p_{i+1} \leq_{\mathbb{P}_\Sigma} p_i''$ and

$$p_{i+1} \Vdash_{\mathbb{P}_\Sigma} \text{“ } t \in \dot{A} \text{ ”}.$$

Then p_{i+1} belongs to M, and by the property of N_i^* and N_i^{**}, we note that for every $K \in \operatorname{ran}(p_{i+1} \setminus p_i)$, since

$$x_i \subseteq X \cap \mathsf{sk}_{H(\theta)}(\{p_i\} \cup x_i) = X \cap p_{i+1}(\operatorname{dom}(p_i)) \\ \subseteq X \cap K \subseteq X \cap N_i^{**} = X \cap N_i^*,$$

$X \cap K \in \Sigma(H(\theta) \cap M)$ holds. This finishes the construction, and the proof. □

3. A proof of **Theorem 2**

Definition 3.1 (Shelah). *A sequence $\vec{C} = \langle C_\xi; \xi \in \omega_1 \cap \mathsf{Lim}\rangle$ is called a ladder system if for any $\xi \in \omega_1 \cap \mathsf{Lim}$, C_ξ is a cofinal subset of ξ and is of order type ω.*

A ladder system $\langle C_\xi; \xi \in \omega_1 \cap \mathsf{Lim}\rangle$ is called weak club guessing if for any club E on ω_1, there exists $\xi \in \omega_1 \cap \mathsf{Lim}$ such that $C_\xi \cap E$ is cofinal (i.e infinite) in ξ.

Definition 3.2. *A forcing notion $\mathbb{P}$ is almost strongly proper if for any sufficiently large regular cardinal θ with $\mathbb{P} \in H(\theta)$, any sufficiently large regular cardinal λ with $H(\lambda) \in H(\theta)$, any countable elementary substructure N of $H(\lambda)$ with $\mathbb{P}, H(\theta) \in N$, and any $p \in \mathbb{P} \cap N$, there exists an extension q of p in $\mathbb{P}$ such that for any subset E of $\mathbb{P}$, if the set*

$$\Big\{M \in N; \mathbb{P} \in M \prec H(\theta) \text{ countable} \\ \& \ E \cap M \text{ is a dense subset of } \mathbb{P} \cap M\Big\}$$

is cofinal with respect to $N \cap H(\theta)$ (i.e. for every $a \in N \cap H(\theta)$, there exists M in the above set), E is predense below q.

Such a q is called almost strongly $(N, \mathbb{P})$-generic.

We note that Mitchell's strongly proper forcing notion is almost strongly proper, and an almost strongly proper forcing notion is proper.

Lemma 3.3. *An almost strongly proper forcing notion preserves a Suslin tree.*

Proof. Let $\mathbb{P}$ be an almost strongly proper forcing notion, and T a Suslin tree. Suppose that $p \in \mathbb{P}$ and $\dot{A}$ be an S-name for a maximal antichain in S.

We take sufficiently large regular cardinals θ and λ as in **Definition 3.2** and a countable elementary substructure N of $H(\lambda)$ which contains

$\mathbb{P}$, $H(\theta)$, S, p and $\dot{A}$ as members. Let q be an almost strongly $(N, \mathbb{P})$-generic condition of $\mathbb{P}$ which extends p in $\mathbb{P}$. We will show that

$$q \Vdash_{\mathbb{P}} \text{“} \forall t \in T_{\omega_1 \cap N} \exists s \in \dot{A} \text{ such that } s <_T t \text{”}.$$

Then it follows that

$$q \Vdash_{\mathbb{P}} \text{“} \dot{A} \text{ is countable ”},$$

which finishes the proof.

Let $t \in T_{\omega_1 \cap N}$, $z \leq_{\mathbb{P}} q$, and define the set

$$E_t := \left\{ x \in \mathbb{P}; \exists s <_T t \text{ such that } x \Vdash_{\mathbb{P}} \text{“ } s \in \dot{A} \text{ ”} \right\}.$$

At first, we show that the set

$$\Big\{ M \in N; \mathbb{P} \in M \prec H(\theta) \text{ countable} \\ \& \ E_t \cap M \text{ is a dense subset of } \mathbb{P} \cap M \Big\}$$

is cofinal with respect to $N \cap H(\theta)$. To show this, let $a \in N \cap H(\theta)$. We take a countable elementary substructure $M \in N$ of $H(\theta)$ which contains $\mathbb{P}$, T, $\dot{A}$ and a as members. We show that $E_t \cap M$ is a dense subset of $\mathbb{P} \cap M$. Let $x \in \mathbb{P} \cap M$. Since $\{r \in T \cap M; r <_T t\}$ is an (M, T)-generic filter and the set

$$\left\{ r \in T; \exists y \leq_{\mathbb{P}} x \, \exists s <_T r \text{ such that } y \Vdash_{\mathbb{P}} \text{“ } s \in \dot{A} \text{ ”} \right\}$$

is dense in T and is a member of M, there exists r in this set such that $r \in M$ and $r <_T t$. Then there exists $y \in \mathbb{P} \cap M$ and $s <_T r$ such that $y \leq_{\mathbb{P}} x$ and

$$y \Vdash_{\mathbb{P}} \text{“ } s \in \dot{A} \text{ ”}.$$

Then $y \in E_t \cap M$.

Therefore, since q is almost strongly $(N, \mathbb{P})$-generic and $z \leq_{\mathbb{P}} q$, E_t is predense below q, hence there exists $z' \in E_t$ which extends z (this can be done because E_t is an open subset of $\mathbb{P}$). This finishes the proof. □

Let $\mathcal{F}$ be the set of all functions f such that

- $\mathrm{dom}(f)$ is a club subset of ω_1,
- f is strictly increasing and continuous, and
- for any $\xi \in \mathrm{dom}(f) \cap \mathsf{Lim}$, $f(\xi)$ is a limit of limit ordinals

and let $\mathcal{PF}$ be the set of all finite partial subfunctions of members of $\mathcal{F}$.

Proof of **Theorem 2**. S denotes a coherent Suslin tree. Suppose that $\underset{\sim}{\vec{C}} = \left\langle \dot{C}_\xi; \xi \in \omega_1 \cap \mathsf{Lim} \right\rangle$ is an S-name for a ladder system on ω_1. We will define an almost strongly proper forcing which generically adds an S-name for a club on ω_1 which witnesses that $\left\langle \dot{C}_\xi; \xi \in \omega_1 \cap \mathsf{Lim} \right\rangle$ is not a weak club guessing sequence in the extension with S.

To define our forcing notion, we take a club E on ω_1 such that for each $\delta \in E$, any node of S_δ decides the value of $\dot{C}_\xi$ for every $\xi < \delta$. (This can be done because S is ccc.) For each $s \in S$ and $\xi \in \sup(E \cap \mathsf{lv}\,(s))$, the value of $\dot{C}_\xi$ in the extension by s is denoted by $\mathrm{val}(\dot{C}_\xi, s)$, i.e.

$$s \Vdash_S \text{“ } \dot{C}_\xi = \mathrm{val}(\dot{C}_\xi, s) \text{ ”}.$$

We define $\mathbb{Q}$ which consists of a finite function p such that

- $\mathrm{dom}(p)$ is a finite subset of S,
- for each $s \in \mathrm{dom}(p)$, $p(s)$ is the triple $\langle p_0^s, p_1^s, p_2^s \rangle$ such that
  - p_0^s is in $\mathcal{PF}$ such that both $\mathrm{dom}(p_0^s)$ and $\mathrm{ran}(p_0^s)$ are subsets of $\sup(E \cap \mathsf{lv}\,(s))$,
  - p_1^s is a finite partial regressive [(2)] function from $\omega_1 \cap \mathsf{Lim}$ into ω_1 such that $\mathrm{dom}(p_1^s)$ is a subset of $\sup(E \cap \mathsf{lv}\,(s))$,
  - p_2^s is a finite subset of $\mathsf{lv}\,(()\, s)$,
  - $\displaystyle\bigcup_{\substack{t \in \mathrm{dom}(p) \\ t \leq_S s}} p_0^t$ is still in $\mathcal{PF}$ and
  - $\displaystyle\bigcup_{\substack{t \in \mathrm{dom}(p) \\ t \leq_S s}} p_1^t$ is still a function,
- $\mathrm{dom}(p)$ is closed under $\wedge$,
- for any s and t in $\mathrm{dom}(p)$, if s and t are incomparable in S, then
  - $p_0^{s \wedge t} \supseteq (p_0^s \restriction \sup(E \cap \mathsf{lv}\,(s \wedge t))) \cup (p_0^t \restriction \sup(E \cap \mathsf{lv}\,(s \wedge t)))$, and
  - $p_1^{s \wedge t} \supseteq (p_1^s \restriction \sup(E \cap \mathsf{lv}\,(s \wedge t))) \cup (p_1^t \restriction \sup(E \cap \mathsf{lv}\,(s \wedge t)))$,
- for each $s \in \mathrm{dom}(p)$,

$$\left(\mathrm{ran}\left(\bigcup_{\substack{t \in \mathrm{dom}(p) \\ t \leq_S s}} p_0^t \right) \right) \cap \left(\bigcup_{\substack{t \in \mathrm{dom}(p) \\ t \leq_S s}} p_2^t \right) = \emptyset,$$

  and

(2) A function f on ordinals is called regressive if for every $\xi \in \mathrm{dom}(f)$, $f(\xi) < \xi$.

- for each $s \in \mathrm{dom}(p)$ and $\xi \in \mathrm{dom}\left(\bigcup_{\substack{t \in \mathrm{dom}(p) \\ t \leq_S s}} p_1^t\right)$,

$$s \Vdash_S \text{“ } \mathrm{ran}\left(\bigcup_{\substack{t \in \mathrm{dom}(p) \\ t \leq_S s}} p_0^t\right) \cap \dot{C}_\xi \subseteq \left(\bigcup_{\substack{t \in \mathrm{dom}(p) \\ t \leq_S s}} p_1^t\right)(\xi) \text{ ”},$$

and for each p and q in $\mathbb{Q}$, we define $q \leq_{\mathbb{Q}} p$ iff $\mathrm{dom}(q) \supseteq \mathrm{dom}(p)$ and for every $s \in \mathrm{dom}(p)$, $q_0^s \supseteq p_0^s$ and $q_1^s \supseteq p_1^s$.

By the definition and the genericity argument, if $\mathbb{Q}$ is proper and preserves S to be Suslin, we note that

$$\Vdash_{\mathbb{Q}} \text{“ } \Vdash_S \text{“ } \mathrm{ran}\left(\bigcup_{\substack{p \in \dot{G}_{\mathbb{Q}} \\ s \in \mathrm{dom}(p) \cap \dot{G}_S}} p_0^s\right) \text{ witnesses that}$$

$$\left\langle \dot{C}_\xi; \xi \in \omega_1 \cap \mathsf{Lim} \right\rangle \text{ is not a weak club guesseing sequence, ” ”}$$

where $\dot{G}_{\mathbb{Q}}$ means the canonical $\mathbb{Q}$-name for its generic, and $\dot{G}_S$ means the canonical S-name for its generic.

In the rest of this section, we show that $\mathbb{Q}$ is almost strongly proper.

Let θ be a regular cardinal which is large enough for $\mathbb{Q}$, and λ a regular cardinal such that $H(\theta)$ and its Skolem function are in $H(\lambda)$. We will show that for any countable elementary substructure N of $H(\lambda)$ which contains the set

$$\left\{S, \left\langle \dot{C}_\xi; \xi \in \omega_1 \cap \mathsf{Lim} \right\rangle, E, H(\theta), \text{a fixed Skolem function of } H(\theta)\right\}$$

as a member, and any $p \in \mathbb{Q} \cap N$, there exists an extension p' of p in $\mathbb{Q}$ which is almost strongly $(\mathbb{Q}, N)$generic.

Suppose that N and p are as above. We take an extension p' of p in $\mathbb{Q}$ such that

- for each $s \in \mathrm{dom}(p)$, there exists $t \in \mathrm{dom}(p')$ such that $\mathsf{lv}(t) \geq \omega_1 \cap N$ and $(p')_0^t = p_0^s \cup \{\langle \omega_1 \cap N, \omega_1 \cap N \rangle\}$, and
- for each $t \in \mathrm{dom}(p') \setminus \mathrm{dom}(p)$, $\{\langle \omega_1 \cap N, \omega_1 \cap N \rangle\} \subseteq (p')_0^t$.

We will show that p' is almost strongly $(\mathbb{Q}, N)$-generic.

Let q be an extension of p' in $\mathbb{Q}$, and E a subset of $\mathbb{Q}$ such that the set

$$\Big\{M \in N; \mathbb{Q} \in M \prec H(\theta) \text{ countable} \\ \& \ E \cap M \text{ is a dense subset of } \mathbb{Q} \cap M\Big\}$$

is cofinal with respect to $N \cap H(\theta)$. By extending q if necessary, we may assume that

- let X_q be the set of maximal nodes of $\mathrm{dom}(q)$, and then X_q satisfies that
  - $X_q \cap N = \emptyset$,
  - every member of X_q decides the value of $\dot{C}_{\omega_1 \cap N}$, and
  - for any $s \in \mathrm{dom}(q)$ and $t \in X_q$, if $s <_S t$, then $q_0^s \subseteq q_0^t$ and $q_1^s \subseteq q_1^t$.

We note that for every $t \in X_q$, $q_0^t \restriction N = q_0^t \cap N$ and $q_1^t \restriction N = q_1^t \cap N$ (because of the definition of $\mathbb{Q}$ and the fact that $E \in N$), and so $q \cap N \in \mathbb{Q} \cap N$. By the coherency of S, we can take $\gamma \in \omega_1 \cap N$ such that

- for every $t, t' \in X_q$, if $t \neq t'$, then
$$\{\xi \in \omega_1 \cap N; t(\xi) \neq t'(\xi)\} \subseteq \gamma,$$
  and
- for every $\xi \in \mathrm{dom}(q_1^{t_i^q}) \setminus \Big(N \cup \{\omega_1 \cap N\}\Big)$ with $q_1^{t_i^q}(\xi) \in N$, $t_i^q \restriction \gamma$ decides the value of $\dot{C}_\xi \cap N$ [(3)].

Let $\{t_i^q; i \in n\}$ be the $<_{\mathrm{lex}}$-increasing enumeration of X_q. Let

$$I := \Big\{i \in n; \omega_1 \cap N \in \mathrm{dom}(q_1^{t_i^q})\Big\}.$$

By our assumption, we can take a countable elementary substructure M of $H(\theta)$ such that $M \in N$, M contains the objects S, E, $q \cap N$, γ,

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(3)To do this, we use the following general remark: For any countable elementary submodel M of $H(\kappa)$ (κ is a regular cardinal greater than $\aleph_1$), $Z \in \mathcal{P}(S) \cap M$ and $v \in Z \setminus M$, there exists $w \in Z \cap M$ such that $w <_S v$. Because the set

$$\{x \in S; \text{ either } x \in Z \text{ or } \{y \in S; x \leq_S y\} \cap Z = \emptyset\}$$

belongs to M, is dense in S, and hence, it contains v as a member (because the set $\{w \in S \cap M; w <_S v\}$ is a (S, M)-generic filter).

We note that $\dot{C}_\xi \cap N$ can be considered as an S-name for a finite subset of ξ if $\xi \in \omega_1 \setminus N$.

$\left\langle \dot{C}_\xi; \xi \in \omega_1 \cap \mathsf{Lim} \right\rangle$, the tuples

$$\Bigg\langle \Bigg\langle \left\langle q_1^{t_i^q}(\xi), \mathrm{val}(\dot{C}_\xi, t_i^q) \cap N \right\rangle; \\ \xi \in \mathrm{dom}(q_1^{t_i^q}) \setminus \Big(N \cup \{\omega_1 \cap N\}\Big) \ \& \ q_1^{t_i^q}(\xi) \in N \Bigg\rangle, \\ q_1^{t_i^q}(\omega_1 \cap N), t_i^q \restriction \gamma \Bigg\rangle,$$

for each $i \in I$, and the tuples

$$\Bigg\langle \Bigg\langle \left\langle q_1^{t_i^q}(\xi), \mathrm{val}(\dot{C}_\xi, t_i^q) \cap N \right\rangle; \\ \xi \in \mathrm{dom}(q_1^{t_i^q}) \setminus \Big(N \cup \{\omega_1 \cap N\}\Big) \ \& \ q_1^{t_i^q}(\xi) \in N \Bigg\rangle, \\ t_i^q \restriction \gamma \Bigg\rangle,$$

for each $i \in n \setminus I$ as members, and $E \cap M$ is a dense subset of $\mathbb{Q} \cap M$.

We take an extension q' of q in $\mathbb{Q}$ such that

- $q' \setminus M = q \setminus M$,
- for each $i \in I$, there exists $\delta_i \in \omega_1 \cap M$ such that $t_i^{q'} \restriction \delta_i \in \mathrm{dom}(q')$ and $(q')_2^{t_i^{q'} \restriction \delta_i}$ includes the set

$$\bigcup \Bigg\{ \Big(\mathrm{val}(\dot{C}_\xi, t_i^q) \cap N\Big) \setminus q_1^{t_i^q}(\xi); \\ \xi \in \mathrm{dom}(q_1^{t_i^q}) \setminus \Big(N \cup \{\omega_1 \cap N\}\Big) \ \& \ q_1^{t_i^q}(\xi) \in N \Bigg\} \\ \cup \Big\{ \Big(\mathrm{val}(\dot{C}_{\omega_1 \cap N}, t_i^q) \cap M\Big) \setminus q_1^{t_i^q}(\omega_1 \cap N) \Big\}$$

as a subset, and

- for each $i \in n \setminus I$, there exists $\delta_i \in \omega_1 \cap M$ such that $t_i^{q'} \restriction \delta_i \in \mathrm{dom}(q')$ and $(q')_2^{t_i^{q'} \restriction \delta_i}$ includes

$$\bigcup \Big\{ \Big(\mathrm{val}(\dot{C}_\xi, t_i^q) \cap N\Big) \setminus q_1^{t_i^q}(\xi); \\ \xi \in \mathrm{dom}(q_1^{t_i^q}) \setminus \Big(N \cup \{\omega_1 \cap N\}\Big) \;\&\; q_1^{t_i^q}(\xi) \in N \Big\}.$$

We note that $q' \cap M \in \mathbb{Q} \cap M$. Since $E \cap M$ is dense in $\mathbb{Q} \cap M$, there exists $r \in E \cap M$ which extends $q' \cap M$ in $\mathbb{Q}$. Then by the choice of q', r is compatible with q in $\mathbb{Q}$. □

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TADATOSHI MIYAMOTO: NANZAN UNIVERSITY, 18 YAMAZATO-CHO, SHOWA-KU, NAGOYA, 466-8673, JAPAN. 宮元忠敏：南山大学経営学部経営学科.
E-mail address: miyamoto@nanzan-u.ac.jp

TERUYUKI YORIOKA: DEPARTMENT OF MATHEMATICS, SHIZUOKA UNIVERSITY, OHYA 836, SHIZUOKA, 422-8529, JAPAN. 依岡輝幸：静岡大学理学部.
E-mail address: styorio@ipc.shizuoka.ac.jp