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A MODEL THEORETIC REFLECTION PRINCIPLE REVISITED

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ABSTRACT. We sketch a simplified construction of a model in which Chang's conjecture for triples $(\omega_3, \omega_2, \omega_1) \twoheadrightarrow (\omega_2, \omega_1, \omega)$ holds.

1. INTRODUCTION

Suppose that $\mathcal{N} = (N; R, \cdots)$ is a structure for a countable first-order language with a distinguished unary relation symbol interpreted by $R \subset N$. For a pair of cardinals $\nu > \nu'$ we say that $\mathcal{N}$ is of type (ν, ν') if $|N| = \nu$ and $|R| = \nu'$. For such pairs (ν, ν') and (μ, μ') with $\nu > \mu$ and $\nu' > \mu'$ define

$$(\nu, \nu') \twoheadrightarrow (\mu, \mu') \text{ holds iff for every } \mathcal{N} \text{ of type } (\nu, \nu') \text{ there is } \mathcal{M} \prec \mathcal{N} \text{ of type } (\mu, \mu').$$

Clearly the statement strengthens the downward Löwenheim–Skolem theorem. Following [11], the statement $(\omega_2, \omega_1) \twoheadrightarrow (\omega_1, \omega)$ is now called Chang's conjecture.

In [10] Silver introduced a variation of the Levy collapse (now called the Silver collapse) and established the consistency of Chang's conjecture:

Theorem (Silver). *If an ω_1-Erdős cardinal exists, then there is a forcing extension in which $(\omega_2, \omega_1) \twoheadrightarrow (\omega_1, \omega)$ holds.*

Silver's argument (see [5]) required Martin's Axiom to hold in some intermediate model. This was later removed by Shelah [7], who proved further that the Levy collapse forces Chang's conjecture to hold.

What about $(\omega_3, \omega_2, \omega_1) \twoheadrightarrow (\omega_2, \omega_1, \omega)$? The meaning of the statement should be evident: This time we consider structures with *two* distinguished unary relations. It is easy to see that $(\omega_3, \omega_2, \omega_1) \twoheadrightarrow (\omega_2, \omega_1, \omega)$ implies $(\omega_3, \omega_2) \twoheadrightarrow (\omega_2, \omega_1)$. Foreman and Magidor [3] showed that under PFA the analogue of Shelah's result fails for $(\omega_3, \omega_2) \twoheadrightarrow (\omega_2, \omega_1)$.

The consistency of $(\omega_3, \omega_2) \twoheadrightarrow (\omega_2, \omega_1)$ was established in effect by Kunen [6]:

Theorem (Kunen). *Let $\mu < \kappa$ be regular cardinals with κ huge. Then there is a forcing extension in which $\kappa = \mu^+$ and $(\mu^{++}, \mu^+) \twoheadrightarrow (\mu^+, \mu)$ holds.*

We refer the reader to [2] for a comprehensive survey of Kunen's method. Extending the method, Foreman [1] established the consistency of Chang's conjecture for triples:

Theorem (Foreman). *If a 2-huge cardinal exists, then there is a forcing extension in which $(\omega_3, \omega_2, \omega_1) \twoheadrightarrow (\omega_2, \omega_1, \omega)$ holds.*

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It is still unknown whether $(\omega_4, \omega_3, \omega_2, \omega_1) \twoheadrightarrow (\omega_3, \omega_2, \omega_1, \omega)$ is consistent.

In this paper we sketch a new proof of Foreman's theorem that is simpler than the original one. A novel element of our proof, which can be found in §3, is that we can identify the term forcing of a slight modification of the Silver collapse. §4 illustrates our approach with a new model of Chang's conjecture (for pairs). In §5 we construct the first two stages of iterated forcing toward a model of Chang's conjecture for triples. §6 is devoted to getting a master condition for this forcing. Finally in §7 we prove that the forcing followed by the modified Silver collapse gives the desired model.

2. Preliminaries

Our notation should be standard. We refer the reader to [4] for background material. Throughout the paper κ denotes a regular cardinal, and R the class of regular cardinals.

Note that $(\nu, \nu') \twoheadrightarrow (\mu, \mu')$ holds iff for every $f : {}^{<\omega}\nu \to \nu$ there is $x \in [\nu]^\mu$ closed under f such that $|x \cap \nu'| = \mu'$. Similarly $(\nu, \nu', \nu'') \twoheadrightarrow (\mu, \mu', \mu'')$ holds iff for every $f : {}^{<\omega}\nu \to \nu$ there is $x \in [\nu]^\mu$ closed under f such that $|x \cap \nu'| = \mu'$ and $|x \cap \nu''| = \mu''$.

Let P and R be posets. We say that a map $\pi : P \to R$ is a projection if

(1) π is order-preserving, i.e. $p' \leq_P p$ implies $\pi(p') \leq_R \pi(p)$,
(2) $\pi(1_P) = 1_R$ and
(3) $r' \leq_R \pi(p)$ implies that there is $p^* \leq_P p$ with $\pi(p^*) \leq_R r'$.

Suppose $\pi : P \to R$ is a projection. Then if D is dense open in R, $\pi^{-1}(D)$ is dense in P. So if $\bar{G} \subset P$ is generic, $\pi``\bar{G}$ generates a generic filter over R, which we denote by $\pi[\bar{G}]$. Furthermore $\operatorname{ran}\pi$ is dense in R and the map $r \mapsto \sum\{p \in P : \pi(p) \leq r\}$ is a complete embedding of R into the completion of P (without the least element). Finally note that the class of projections is closed under taking the composite and the product.

Suppose further $\dot{S}$ is an R-name for a poset. The term forcing $T(R, \dot{S})$ is the set (of representatives under the canonical identification from)

$$\{\dot{s} \in V^R : \Vdash_R \dot{s} \in \dot{S}\}$$

ordered by $\dot{s}' \leq \dot{s}$ iff $\Vdash_R \dot{s}' \mathrel{\dot{\leq}} \dot{s}$. By

$$P \star_\pi \dot{S} \text{ or } P \star \dot{S}$$

we mean the set $P \times T(R, \dot{S})$ ordered by: $(p', \dot{s}') \leq (p, \dot{s})$ iff

$$p' \leq_P p \text{ and } \pi(p') \Vdash_R \dot{s}' \mathrel{\dot{\leq}} \dot{s}.$$

Note that the canonical map $\mathrm{pr} : P \star_\pi \dot{S} \to P$ is a projection and hence $P \star_\pi \dot{S}$ can be identified with an iteration of the form $P * \dot{Q}$. In particular $P \star_\pi \dot{S} = P * \dot{S}$ if $P = R$ and $\pi = \mathrm{id}$.

The following lemma is essentially due to Laver:

Lemma (Laver). *Suppose that $\pi : P \to R$ is a projection and $\dot{S}$ is an R-name for a poset. Then $\mathrm{id} : P \times T(R, \dot{S}) \to P \star_\pi \dot{S}$ is a projection.*

Note also that under the hypothesis of Laver's lemma $\pi \times \mathrm{id} : P \star_\pi \dot{S} \to R * \dot{S}$ is a projection.

Let I and J be sets of ordinals. First we fix our notation for products. Suppose that S_ξ is a poset for $\xi \in I \cup J$. By

$$\prod_{\xi \in I}^{\kappa} S_\xi \times \prod_{\xi \in J}^{\mathrm{E}} S_\xi$$

we denote the set of sequences q such that

- $\operatorname{dom} q \subset I \cup J$, $|\operatorname{dom} q \cap I| < \kappa$, $\operatorname{dom} q \cap J$ is Easton and
- $q(\xi) \in S_\xi$ for every $\xi \in \operatorname{dom} q$

ordered by: $q' \leq q$ iff

$$\operatorname{dom} q' \supset \operatorname{dom} q \text{ and } q'(\xi) \leq_\xi q(\xi) \text{ for every } \xi \in \operatorname{dom} q.$$

Here a set d is Easton if $d \subset \mathsf{R}$ and $\sup(d \cap \gamma) < \gamma$ for every $\gamma \in \mathsf{R}$. It is understood that $q(\xi) = 1_\xi$ unless $\xi \in \operatorname{dom} q$.

Next we introduce a generalization of $P \star \dot{S}$ to which Laver's lemma can be generalized. Suppose that $\pi_\xi : P \to R_\xi$ is a projection and $\dot{S}_\xi$ is an R_ξ-name for a poset for $\xi \in I \cup J$. By

$$P \star \left(\prod_{\xi \in I}^{\kappa} \dot{S}_\xi \times \prod_{\xi \in J}^{\mathrm{E}} \dot{S}_\xi \right)$$

we mean the set of pairs of $p \in P$ and a sequence q such that

- $\operatorname{dom} q \subset I \cup J$, $|\operatorname{dom} q \cap I| < \kappa$, $\operatorname{dom} q \cap J$ is Easton and
- $q(\xi)$ is an R_ξ-name and $\Vdash_\xi q(\xi) \in \dot{S}_\xi$ for every $\xi \in \operatorname{dom} q$

ordered by: $(p', q') \leq (p, q)$ iff

$$p' \leq_P p, \ \operatorname{dom} q' \supset \operatorname{dom} q \text{ and } \pi_\xi(p') \Vdash_\xi q'(\xi) \mathrel{\dot{\leq}_\xi} q(\xi) \text{ for every } \xi \in \operatorname{dom} q.$$

Here $\Vdash_\xi$ denotes the forcing relation associated with R_ξ. Note that as sets

$$P \star \left(\prod_{\xi \in I}^{\kappa} \dot{S}_\xi \times \prod_{\xi \in J}^{\mathrm{E}} \dot{S}_\xi \right) = P \times \left(\prod_{\xi \in I}^{\kappa} T(R_\xi, \dot{S}_\xi) \times \prod_{\xi \in J}^{\mathrm{E}} T(R_\xi, \dot{S}_\xi) \right).$$

Here is the generalization of Laver's lemma:

Lemma 1. *Let I and J be sets of ordinals. Suppose that $\pi_\xi : P \to R_\xi$ is a projection and $\dot{S}_\xi$ is an R_ξ-name for a poset for $\xi \in I \cup J$. Then*

$$\mathrm{id} : P \times \left(\prod_{\xi \in I}^{\kappa} T(R_\xi, \dot{S}_\xi) \times \prod_{\xi \in J}^{\mathrm{E}} T(R_\xi, \dot{S}_\xi) \right) \to P \star \left(\prod_{\xi \in I}^{\kappa} \dot{S}_\xi \times \prod_{\xi \in J}^{\mathrm{E}} \dot{S}_\xi \right)$$

is a projection.

Under the hypothesis of Lemma 1 the canonical map from $P \star \left(\prod_{\xi \in I}^{\kappa} \dot{S}_\xi \times \prod_{\xi \in J}^{\mathrm{E}} \dot{S}_\xi \right)$ to P is a projection. Suppose further $I' \subset I$, $J' \subset J$ and $\Vdash_\xi \dot{\varphi}_\xi : \dot{S}_\xi \to \dot{S}'_\xi$ is a projection for $\xi \in I' \cup J'$. Then we can define a projection

$$\dot{\varphi}_\xi : T(R_\xi, \dot{S}_\xi) \to T(R_\xi, \dot{S}'_\xi)$$

(by abuse of notation) naturally, and the map

$$\mathrm{id} \times \prod_{\xi \in I'} \dot\varphi_\xi \times \prod_{\xi \in J'} \dot\varphi_\xi : P \star \left(\prod_{\xi \in I}^{\kappa} \dot S_\xi \times \prod_{\xi \in J}^{\mathrm{E}} \dot S_\xi \right) \to P \star \left(\prod_{\xi \in I'}^{\kappa} \dot S'_\xi \times \prod_{\xi \in J'}^{\mathrm{E}} \dot S'_\xi \right)$$

is a projection.

Suppose that $j : V \to M$ is an elementary embedding. Let $\rho : j(P) \to P$ be a projection. We say that $p^* \in j(P)$ is a master condition for j and ρ if $\bar p \leq j(\rho(\bar p))$ for every $\bar p \leq p^*$. In what follows we suppress the mention of j, which should be clear from the context. If $\bar G \subset j(P)$ is generic and contains a master condition for ρ, then $(j \circ \rho)``\bar G \subset \bar G$ and hence $j``\rho[\bar G] \subset \bar G$, which allows us to extend j to $j : V[\rho[\bar G]] \to M[\bar G]$ in $V[\bar G]$.

Suppose further $\pi : P \to R$ is a projection and $\dot S$ is an R-name for a poset. Let $\varphi : j(P) \to P \star \dot S$ be a projection. Then we can define projections

$$\varphi^+ : j(P \star \dot S) \to P \star \dot S \text{ and } \varphi^- : j(P) \to P$$

by composing the projections from

$$j(P \star \dot S) \xrightarrow{j(\mathrm{pr})} j(P) \xrightarrow{\varphi} P \star \dot S \xrightarrow{\mathrm{pr}} P.$$

Note that $(1_{j(P)}, \dot s^*)$ is a master condition for $\varphi^+ : j(P \star \dot S) \to P \star \dot S$ iff $(\bar p, \dot s^*) \leq j(\varphi(\bar p))$ for every $\bar p \in j(P)$.

Foreman [1] proved a lemma that enables us to transfer a master condition for a projection

$$j(R * \dot S) \to R * \dot S$$

to a master condition for a projection

$$j(P \star \dot S) \to P \star \dot S.$$

Let us restate Foreman's lemma in terms of projections:

Lemma (Foreman). *Suppose*

- $j : V \to M$, $\pi : P \to R$ *is a projection and* $\dot S$ *is an* R*-name for a poset,*
- *the following diagram of projections commutes:*

$$\begin{array}{ccc} j(P) & \xrightarrow{j(\pi)} & j(R) \\ {\scriptstyle \varphi_\star}\downarrow & & \downarrow{\scriptstyle \varphi_*} \\ P \star \dot S & \xrightarrow[\pi \times \mathrm{id}]{} & R * \dot S, \end{array}$$

- $1_{j(P)}$ *is a master condition for* $\varphi_\star^- : j(P) \to P$,
- $(1_{j(R)}, \dot s^*)$ *is a master condition for* $\varphi_*^+ : j(R * \dot S) \to R * \dot S$.

Then $(1_{j(P)}, \dot s^*)$ *is a master condition for* $\varphi_\star^+ : j(P \star \dot S) \to P \star \dot S$.

3. The main lemmas

In this section we introduce a slight modification of the Silver collapse. Results of this section should be valid for other canonical collapses if they are suitably modified (see [9]).

Throughout the section let $\kappa < \lambda$ be regular cardinals with λ inaccessible. Define $S(\kappa, \lambda)$ to be the set of functions $s : \delta \times d \to \lambda$ such that

- $\delta < \kappa$, $d \subset [\kappa, \lambda)$ is a set of κ-closed cardinals of size $\leq \kappa$ and

- $s(\alpha, \gamma) < \gamma$ for every $(\alpha, \gamma) \in \delta \times d$.

Here a cardinal γ is κ-closed if $\gamma^{<\kappa} = \gamma$. $S(\kappa, \lambda)$ is ordered by reverse inclusion: $s' \leq s$ iff $s' \supset s$.

$S(\kappa, \lambda)$ shares nice properties of the original Silver collapse: It is κ-directed closed and a subset of V_λ, has the λ-cc and forces $\lambda = \kappa^+$. Furthermore we can identify the term forcing of $S(\kappa, \lambda)$ defined in the extensions by a small poset.

Suppose that R has the κ-cc. Then it is easy to check that

$$D\left(R, \dot{S}(\kappa, \lambda)\right) = \left\{\dot{s} \in T\left(R, \dot{S}(\kappa, \lambda)\right) : \exists \delta < \kappa \exists d \subset [\kappa, \lambda) \Vdash \operatorname{dom} \dot{s} = \delta \times d\right\}$$

is dense in $T\left(R, \dot{S}(\kappa, \lambda)\right)$.

Lemma 2. *Suppose that R has the κ-cc and size $\leq \kappa$. Then $S(\kappa, \lambda)$ is isomorphic to $D\left(R, \dot{S}(\kappa, \lambda)\right)$.*

In what follows we say that

$$\text{the isomorphism } i : S(\kappa, \lambda) \to D\left(R, \dot{S}(\kappa, \lambda)\right) \text{ or}$$
$$\text{the dense embedding } i : S(\kappa, \lambda) \to T\left(R, \dot{S}(\kappa, \lambda)\right)$$

of Lemma 2 are based on the list σ of R-names. Note that the construction of i from σ is canonical in the following sense: Suppose $\lambda' > \lambda$ is inaccessible and

$$i' : S(\kappa, \lambda') \to T\left(R, \dot{S}(\kappa, \lambda')\right)$$

is a dense embedding based on a list that end-extends σ. Then for $s \in S(\kappa, \lambda')$ we have

$$\Vdash_R i'(s)|(\kappa \times \lambda) = i(s|(\kappa \times \lambda)).$$

Suppose that $\pi : P \to R$ is a projection. Then we can form the poset $P \star \dot{S}(\kappa, \lambda)^R$, where $\dot{S}(\kappa, \lambda)^R$ is an R-name for the modified Silver collapse. By Laver's lemma and Lemma 2 we have

Proposition 3. *Suppose that $\pi : P \to R$ is a projection and R has the κ-cc and size $\leq \kappa$. Then there is a dense embedding $i : S(\kappa, \lambda) \to T\left(R, \dot{S}(\kappa, \lambda)\right)$, which induces a projection* $\operatorname{id} \times i : P \times S(\kappa, \lambda) \to P \star \dot{S}(\kappa, \lambda)^R$.

The following corollary of Proposition 3, which was proved in [8], suffices for the application in §4: Suppose that P has the κ-cc and size $\leq \kappa$. Then there is a projection of the form $\operatorname{id} \times i : P \times S(\kappa, \lambda) \to P * \dot{S}(\kappa, \lambda)$.

By Lemmas 1 and 2 we have

Proposition 4. *Let I and J be sets of ordinals. Suppose that $\kappa_\xi \in [\kappa, \lambda) \cap \mathsf{R}$, $\pi_\xi : P \to R_\xi$ is a projection and R_ξ has the κ-cc and size $\leq \kappa$ for $\xi \in I \cup J$. Then there is a dense embedding $i_\xi : S(\kappa_\xi, \lambda) \to T\left(R_\xi, \dot{S}(\kappa_\xi, \lambda)\right)$ for $\xi \in I \cup J$, which* *induces a projection*

$$\mathrm{id} \times \prod_{\xi \in I} i_\xi \times \prod_{\xi \in J} i_\xi : P \times \left(\prod_{\xi \in I}^{\kappa} S(\kappa_\xi, \lambda) \times \prod_{\xi \in J}^{\mathrm{E}} S(\kappa_\xi, \lambda) \right)$$
$$\to P \star \left(\prod_{\xi \in I}^{\kappa} \dot{S}(\kappa_\xi, \lambda)^{R_\xi} \times \prod_{\xi \in J}^{\mathrm{E}} \dot{S}(\kappa_\xi, \lambda)^{R_\xi} \right).$$

In what follows we write $\dot{S}(\kappa_\xi, \lambda)$ for $\dot{S}(\kappa_\xi, \lambda)^P$ in case $R_\xi = P$ and $\pi_\xi = \mathrm{id}$.
In §5 we need a commutative diagram of projections of the following form:

$$\begin{array}{ccc} (P \star \dot{S}) \times S(\kappa, \lambda) & \xrightarrow{\mathrm{id} \times i} & (P \star \dot{S}) \star \dot{S}(\kappa, \lambda)^P \\ {\scriptstyle \mathrm{id}} \downarrow & & \downarrow {\scriptstyle \mathrm{id} \times k} \\ (P \star \dot{S}) \times S(\kappa, \lambda) & \xrightarrow[\mathrm{id} \times i^*]{} & (P \star \dot{S}) * \dot{S}(\kappa, \lambda), \end{array}$$

where

$$\dot{S} = \prod_{\xi \in I}^{\kappa} \dot{S}(\kappa_\xi, \lambda)^{R_\xi} \times \prod_{\xi \in J}^{\mathrm{E}} \dot{S}(\kappa_\xi, \lambda)^{R_\xi}.$$

Since $P \star \dot{S}$ can be identified with an iteration of the form $P * \dot{Q}$, the following lemma should suffice:

Lemma 5. *Suppose that $P * \dot{Q}$ has the κ-cc and size $\leq \kappa$. Let*

$$i : S(\kappa, \lambda) \to D\left(P, \dot{S}(\kappa, \lambda)\right)$$

be an isomorphism based on a list σ of P-names. Then there are an isomorphism

$$i^* : S(\kappa, \lambda) \to D\left(P * \dot{Q}, \dot{S}(\kappa, \lambda)\right)$$

based on a list of $P * \dot{Q}$-names and an isomorphism

$$k : D\left(P, \dot{S}(\kappa, \lambda)\right) \to D\left(P * \dot{Q}, \dot{S}(\kappa, \lambda)\right)$$

such that the following diagram of projections commutes:

$$\begin{array}{ccc} (P * \dot{Q}) \times S(\kappa, \lambda) & \xrightarrow{\mathrm{id} \times i} & (P * \dot{Q}) \star \dot{S}(\kappa, \lambda)^P \\ {\scriptstyle \mathrm{id}} \downarrow & & \downarrow {\scriptstyle \mathrm{id} \times k} \\ (P * \dot{Q}) \times S(\kappa, \lambda) & \xrightarrow[\mathrm{id} \times i^*]{} & (P * \dot{Q}) * \dot{S}(\kappa, \lambda). \end{array}$$

4. A new model of Chang's conjecture

This section presents a forcing notion for Chang's conjecture (for pairs) that is simpler than Kunen's:

Theorem 6. *Let $\mu < \kappa < \lambda$ be regular cardinals with κ huge and λ its target. Then*

$$\left(\prod_{\gamma \in [\mu, \kappa) \cap \mathsf{R}}^{\mu} S(\gamma, \kappa) \right) * \dot{S}(\kappa, \lambda)$$

forces that $\kappa = \mu^+$, $\lambda = \mu^{++}$ and $(\mu^{++}, \mu^+) \twoheadrightarrow (\mu^+, \mu)$ holds.

From Kunen's argument [6] Foreman [1] isolated a sufficient condition for getting a master condition for a projection

$$j\left(P * \dot{S}(\kappa,\lambda)\right) \to P * \dot{S}(\kappa,\lambda).$$

Let us restate Kunen's lemma in terms of projections:

Lemma (Kunen). *Suppose*

- $j : V \to M$ *witnesses that* κ *is huge with target* λ *and* $P \subset V_\kappa$ *has the* κ*-cc,*
- $\varphi : j(P) \to P * \dot{S}(\kappa,\lambda)$ *is a projection,*
- $1_{j(P)}$ *is a master condition for* $\varphi^- : j(P) \to P$.

Then there is a master condition $(1_{j(P)}, \dot{s}^*)$ *for* $\varphi^+ : j\left(P * \dot{S}(\kappa,\lambda)\right) \to P * \dot{S}(\kappa,\lambda)$.

Proof of Theorem 6. Let $j : V \to M$ witness that κ is huge with target λ. Let

$$P = \prod_{\gamma \in [\mu,\kappa)\cap \mathsf{R}}^{\mu} S(\gamma,\kappa).$$

It is easy to see that $P \subset V_\kappa$ is μ-closed. By standard arguments (or see [8]) P has the κ-cc. Having $S(\mu,\kappa)$ as a complete suborder, P forces $\kappa = \mu^+$. Thus $P * \dot{S}(\kappa,\lambda)$ forces $\kappa = \mu^+$ and $\lambda = \mu^{++}$.

It remains to prove that $P * \dot{S}(\kappa,\lambda)$ forces $(\lambda,\kappa) \twoheadrightarrow (\kappa,\mu)$ to hold. Since ${}^\lambda M \subset M$, we have

$$j(P) = \prod_{\gamma \in [\mu,\lambda)\cap \mathsf{R}}^{\mu} S(\gamma,\lambda),$$

which can be identified with

$$\prod_{\gamma \in [\mu,\kappa)\cap \mathsf{R}}^{\mu} S(\gamma,\lambda) \times \prod_{\gamma \in [\kappa,\lambda)\cap \mathsf{R}}^{\mu} S(\gamma,\lambda).$$

Let $\varphi : j(P) \to P * \dot{S}(\kappa,\lambda)$ be the projection identified with the composite of the following:

$$\begin{array}{ccc}
\prod_{\gamma \in [\mu,\kappa)\cap \mathsf{R}}^{\mu} S(\gamma,\lambda) \times \prod_{\gamma \in [\kappa,\lambda)\cap \mathsf{R}}^{\mu} S(\gamma,\lambda) & & \\
\Big\downarrow \left(\prod_{\gamma \in [\mu,\kappa)\cap \mathsf{R}} \mathrm{rs}^\gamma_\kappa\right) \times \mathrm{pr}_\kappa & & \\
\left(\prod_{\gamma \in [\mu,\kappa)\cap \mathsf{R}}^{\mu} S(\gamma,\kappa)\right) \times S(\kappa,\lambda) & \xrightarrow{\mathrm{id} \times i} & \left(\prod_{\gamma \in [\mu,\kappa)\cap \mathsf{R}}^{\mu} S(\gamma,\kappa)\right) * \dot{S}(\kappa,\lambda).
\end{array}$$

Here

$$\mathrm{rs}^\gamma_\kappa : S(\gamma,\lambda) \to S(\gamma,\kappa) \qquad s \mapsto s|(\gamma \times \kappa)$$

$$\mathrm{pr}_\kappa : \prod_{\gamma \in [\kappa,\lambda)\cap \mathsf{R}}^{\mu} S(\gamma,\lambda) \to S(\kappa,\lambda) \qquad q \mapsto q(\kappa)$$

and $\mathrm{id} \times i$ is as in Proposition 3.

Note that $1_{j(P)}$ is a master condition for $\varphi^- : j(P) \to P$. To see this, let $\bar{p} \in j(P)$ and $\varphi(\bar{p}) = (p, \dot{s})$. Since $p \in V_\kappa$, we have $\bar{p} \le p = j(p)$, as desired. By Kunen's lemma we get a master condition $(1_{j(P)}, \dot{s}^*)$ for $\varphi^+ : j\left(P * \dot{S}(\kappa,\lambda)\right) \to P * \dot{S}(\kappa,\lambda)$.

Let $\bar{G} \subset j\left(P * \dot{S}(\kappa,\lambda)\right)$ be V-generic with $(1_{j(P)}, \dot{s}^*) \in \bar{G}$. Then $G = \varphi^+[\bar{G}]$ is V-generic over $P * \dot{S}(\kappa,\lambda)$ and we can extend j to $j : V[G] \to M[\bar{G}]$ in $V[\bar{G}]$. We claim that $(\lambda,\kappa) \twoheadrightarrow (\kappa,\mu)$ holds in $V[\bar{G}]$. Fix $f : {}^{<\omega}\lambda \to \lambda$ in $V[G]$. Then $j``\lambda$ witnesses that in $M[\bar{G}]$ there is $x \in [j(\lambda)]^{j(\kappa)}$ closed under $j(f)$ such that $|x \cap j(\kappa)| = |\kappa| = \mu = j(\mu)$. By elementarity there is $x \in [\lambda]^{\kappa}$ closed under f such that $|x \cap \kappa| = \mu$ in $V[G]$, as desired. □

Remark 1. Just like Kunen's forcing, the poset of Theorem 6 forces that κ carries a κ^+-saturated filter. See [8] for a proof.

In [9] we introduced a poset $E(\mu,\kappa)$ that collapses a Mahlo cardinal κ to μ^+, and proved that under the hypothesis of Theorem 6 the iteration $E(\mu,\kappa) * \dot{E}(\kappa,\lambda)$ forces κ to carry a κ^+-saturated filter. We do not know, however, whether $(\mu^{++},\mu^+) \twoheadrightarrow (\mu^+,\mu)$ holds in the model.

5. The main forcing

Throughout this section we fix a regular cardinal μ. Let M denote the class of Mahlo cardinals $> \mu$ together with μ. For $\gamma \in \mathsf{M}$ we define a poset $P(\gamma)$ as follows: First let $P(\mu)$ be the trivial poset. If $\mu < \gamma \in \mathsf{M}$, define

$$P(\gamma) = \prod^{\mathrm{E}}_{\xi \in [\mu,\gamma)\cap \mathsf{M}} \; \prod^{\xi}_{\zeta \in [\mu,\xi]\cap \mathsf{M}} S(\xi,\gamma).$$

It is easy to see that $P(\gamma) \subset V_\gamma$ is μ-closed. The following lemma should also be standard.

Lemma 7. *$P(\gamma)$ has the γ-cc for every $\gamma \in \mathsf{M}$.*

For the rest of this section we further fix a huge cardinal $\kappa > \mu$. By recursion on γ we define for each pair of $\alpha \le \gamma$ from $[\mu,\kappa] \cap \mathsf{M}$

a poset $R(\alpha,\gamma) \subset V_\gamma$ and a projection $\pi_{\alpha\gamma} : P(\gamma) \to R(\alpha,\gamma)$.

First we fix a list of $P(\gamma)$-names for ordinals for each $\gamma \in (\mu,\kappa] \cap \mathsf{M}$. Unless otherwise stated, dense embeddings are based on these lists.

If $\alpha \in \{\mu,\gamma\}$, let $R(\alpha,\gamma) = P(\gamma)$ and $\pi_{\alpha\gamma} = \mathrm{id}$. Suppose next $\alpha \in (\mu,\gamma) \cap \mathsf{M}$. First define

$$R(\alpha,\gamma) = P(\alpha) \star \left(\prod^{\alpha}_{\zeta \in [\mu,\alpha]\cap \mathsf{M}} \dot{S}(\alpha,\gamma)^{R(\zeta,\alpha)} \times \prod^{\mathrm{E}}_{\xi \in (\alpha,\gamma)\cap \mathsf{M}} \dot{S}(\xi,\gamma) \right).$$

Next we define $\pi_{\alpha\gamma} : P(\gamma) \to R(\alpha,\gamma)$. We may assume $\pi_{\zeta\gamma} : P(\gamma) \to R(\zeta,\gamma)$ has been defined for $\zeta \in [\mu,\alpha) \cap \mathsf{M}$ as well. In the course of defining $\pi_{\alpha\gamma}$ we define dense embeddings

$$i^{\zeta}_{\alpha\gamma} : S(\alpha,\gamma) \to T\left(R(\zeta,\alpha), \dot{S}(\alpha,\gamma)\right) \qquad \text{for } \zeta \in [\mu,\alpha] \cap \mathsf{M}$$

$$i^{\xi}_{\alpha\gamma} : S(\xi,\gamma) \to T\left(P(\alpha), \dot{S}(\xi,\gamma)\right) \qquad \text{for } \xi \in (\alpha,\gamma) \cap \mathsf{M}$$

$$k^{\zeta}_{\alpha\gamma} : D\left(P(\zeta), \dot{S}(\alpha,\gamma)\right) \to T\left(R(\zeta,\alpha), \dot{S}(\alpha,\gamma)\right) \qquad \text{for } \zeta \in (\mu,\alpha) \cap \mathsf{M}.$$

Define $\pi_{\alpha\gamma}$ by composing the following projections:

$$P(\gamma) \xrightarrow{\psi_{\alpha\gamma}} P(\alpha) \times \left(\prod_{\zeta\in[\mu,\alpha]\cap\mathrm{M}}^{\alpha} S(\alpha,\gamma) \times \prod_{\xi\in(\alpha,\gamma)\cap\mathrm{M}}^{\mathrm{E}} S(\xi,\gamma) \right)$$

$$\downarrow \varphi_{\alpha\gamma}$$

$$P(\alpha) \star \left(\prod_{\zeta\in[\mu,\alpha]\cap\mathrm{M}}^{\alpha} \dot{S}(\alpha,\gamma)^{R(\zeta,\alpha)} \times \prod_{\xi\in(\alpha,\gamma)\cap\mathrm{M}}^{\mathrm{E}} \dot{S}(\xi,\gamma) \right).$$

Here $\psi_{\alpha\gamma}$ is defined as follows: First we identify

$$P(\gamma) = \prod_{\xi\in[\mu,\gamma)\cap\mathrm{M}}^{\mathrm{E}} \prod_{\zeta\in[\mu,\xi]\cap\mathrm{M}}^{\xi} S(\xi,\gamma)$$

with

$$\left(\prod_{\xi\in[\mu,\alpha)\cap\mathrm{M}}^{\mathrm{E}} \prod_{\zeta\in[\mu,\xi]\cap\mathrm{M}}^{\xi} S(\xi,\gamma) \right) \times \left(\prod_{\zeta\in[\mu,\alpha]\cap\mathrm{M}}^{\alpha} S(\alpha,\gamma) \right) \times \left(\prod_{\xi\in(\alpha,\gamma)\cap\mathrm{M}}^{\mathrm{E}} \prod_{\zeta\in[\mu,\xi]\cap\mathrm{M}}^{\xi} S(\xi,\gamma) \right).$$

Let $\psi_{\alpha\gamma}$ be the projection identified with

$$\left(\prod_{\xi\in[\mu,\alpha)\cap\mathrm{M}} \prod_{\zeta\in[\mu,\xi]\cap\mathrm{M}} \mathrm{rs}^{\xi}_{\alpha} \right) \times \mathrm{id} \times \left(\prod_{\xi\in(\alpha,\gamma)\cap\mathrm{M}} \mathrm{pr}^{\xi}_{\alpha} \right),$$

where

$$\mathrm{rs}^{\xi}_{\alpha} : S(\xi,\gamma) \to S(\xi,\alpha) \qquad s \mapsto s|(\xi\times\alpha)$$

$$\mathrm{pr}^{\xi}_{\alpha} : \prod_{\zeta\in[\mu,\xi]\cap\mathrm{M}}^{\xi} S(\xi,\gamma) \to S(\xi,\gamma) \qquad q \mapsto q(\alpha).$$

In brief, $\psi_{\alpha\gamma}$ sends $p \in P(\gamma)$ to the pair of

$$\langle\langle p(\xi)(\zeta)|(\xi\times\alpha) : \zeta \in \operatorname{dom} p(\xi)\rangle : \xi \in \operatorname{dom} p \cap [\mu,\alpha)\rangle$$
$$\text{and } p(\alpha)^\frown\langle p(\xi)(\alpha) : \xi \in \operatorname{dom} p \cap (\alpha,\gamma)\rangle.$$

Next define

$$\varphi_{\alpha\gamma} = \mathrm{id} \times \left(\prod_{\zeta\in[\mu,\alpha]\cap\mathrm{M}} i^{\zeta}_{\alpha\gamma} \times \prod_{\xi\in(\alpha,\gamma)\cap\mathrm{M}} i^{\xi}_{\alpha\gamma} \right),$$

where

$$i^{\zeta}_{\alpha\gamma} : S(\alpha,\gamma) \to T\left(R(\zeta,\alpha), \dot{S}(\alpha,\gamma)\right) \qquad \text{for } \zeta \in [\mu,\alpha] \cap \mathrm{M}$$

$$i^{\xi}_{\alpha\gamma} : S(\xi,\gamma) \to T\left(P(\alpha), \dot{S}(\xi,\gamma)\right) \qquad \text{for } \xi \in (\alpha,\gamma) \cap \mathrm{M}$$

are dense embeddings based on some lists of corresponding names. First by Lemma 2 we get a dense embedding

$$i^{\zeta}_{\alpha\gamma} : S(\alpha,\gamma) \to T\left(P(\alpha), \dot{S}(\alpha,\gamma)\right)$$

for $\zeta \in \{\mu, \alpha\}$, and a dense embedding $i^{\xi}_{\alpha\gamma}$ for $\xi \in (\alpha, \gamma) \cap \mathsf{M}$, each of which is based on the prefixed list of $P(\alpha)$-names. Suppose next $\zeta \in (\mu, \alpha) \cap \mathsf{M}$. Then a dense embedding

$$i^{\alpha}_{\zeta\gamma} : S(\alpha, \gamma) \to T\left(P(\zeta), \dot{S}(\alpha, \gamma)\right)$$

has been defined based on the prefixed list of $P(\zeta)$-names. By Lemma 5 with $P = P(\zeta)$ and $P * \dot{Q} = R(\zeta, \alpha)$ we get a dense embedding $i^{\zeta}_{\alpha\gamma}$ based on a list of $R(\zeta, \alpha)$-names, and a dense embedding $k^{\zeta}_{\alpha\gamma}$ such that the following diagram of projections commutes:

$$\begin{array}{ccc} R(\zeta,\alpha) \times S(\alpha,\gamma) & \xrightarrow{\text{id} \times i^{\alpha}_{\zeta\gamma}} & R(\zeta,\alpha) \star \dot{S}(\alpha,\gamma)^{P(\zeta)} \\ \text{id}\downarrow & & \downarrow \text{id} \times k^{\zeta}_{\alpha\gamma} \\ R(\zeta,\alpha) \times S(\alpha,\gamma) & \xrightarrow[\text{id} \times i^{\zeta}_{\alpha\gamma}]{} & R(\zeta,\alpha) * \dot{S}(\alpha,\gamma). \end{array}$$

This completes the description of the recursion.

Let $j : V \to M$ witness that κ is huge with target λ. In §7 we will force with the poset $j(R)(\kappa, \lambda)$. Since ${}^{\lambda}M \subset M$, we have

$$j(R)(\kappa,\lambda) = \left(j(P)(\kappa) \star \left(\prod^{\kappa}_{\zeta \in [\mu,\kappa] \cap \mathsf{M}} \dot{S}(\kappa,\lambda)^{j(R)(\zeta,\kappa)} \times \prod^{\mathrm{E}}_{\xi \in (\kappa,\lambda) \cap \mathsf{M}} \dot{S}(\xi,\lambda) \right) \right)^{M}$$

$$= P(\kappa) \star \left(\prod^{\kappa}_{\zeta \in [\mu,\kappa] \cap \mathsf{M}} \dot{S}(\kappa,\lambda)^{R(\zeta,\kappa)} \times \prod^{\mathrm{E}}_{\xi \in (\kappa,\lambda) \cap \mathsf{M}} \dot{S}(\xi,\lambda) \right).$$

Remark 2. It may seem more natural to define

$$R(\alpha,\gamma) = P(\alpha) \star \prod^{\alpha}_{\zeta \in [\mu,\alpha] \cap \mathsf{M}} \dot{S}(\alpha,\gamma)^{R(\zeta,\alpha)}$$

and projections

$$P(\gamma) \longrightarrow P(\alpha) \times \prod^{\alpha}_{\zeta \in [\mu,\alpha] \cap \mathsf{M}} S(\alpha,\gamma) \longrightarrow R(\alpha,\gamma)$$

suitably, or define (without changing the definition of $R(\alpha, \gamma)$)

$$P(\gamma) = \prod^{\mathrm{E}}_{\eta \in [\mu,\gamma) \cap \mathsf{M}} \left(\prod^{\eta}_{\zeta \in [\mu,\eta] \cap \mathsf{M}} S(\eta,\gamma) \times \prod^{\mathrm{E}}_{\xi \in (\eta,\gamma) \cap \mathsf{M}} S(\xi,\gamma) \right),$$

and projections

$$P(\gamma) \longrightarrow P(\alpha) \times \left(\prod^{\alpha}_{\zeta \in [\mu,\alpha] \cap \mathsf{M}} S(\alpha,\gamma) \times \prod^{\mathrm{E}}_{\xi \in (\alpha,\gamma) \cap \mathsf{M}} S(\xi,\gamma) \right) \longrightarrow R(\alpha,\gamma)$$

suitably. These alternatives would not work for some reason to be mentioned in §6.

Now suppose that $\alpha < \gamma$ are both from $(\mu, \kappa] \cap \mathsf{M}$. In what follows we let

$$\dot{Q}(\alpha,\gamma) = \prod^{\alpha}_{\zeta \in [\mu,\alpha] \cap \mathsf{M}} \dot{S}(\alpha,\gamma)^{R(\zeta,\alpha)} \times \prod^{\mathrm{E}}_{\xi \in (\alpha,\gamma) \cap \mathsf{M}} \dot{S}(\xi,\gamma)^{P(\alpha)},$$

so that we have

$$R(\alpha,\gamma) = P(\alpha) \star \dot{Q}(\alpha,\gamma).$$

We also let

$$\bar{Q}(\alpha,\gamma) = \prod_{\zeta\in[\mu,\alpha]\cap \mathsf{M}}^{\alpha} S(\alpha,\gamma) \times \prod_{\xi\in(\alpha,\gamma)\cap \mathsf{M}}^{\mathrm{E}} S(\xi,\gamma).$$

Thus

$$\pi_{\alpha\gamma} : P(\gamma) \to P(\alpha) \star \dot{Q}(\alpha,\gamma)$$

is the composite of the following projections:

$$P(\gamma) \xrightarrow{\psi_{\alpha\gamma}} P(\alpha) \times \bar{Q}(\alpha,\gamma) \xrightarrow{\varphi_{\alpha\gamma}} P(\alpha) \star \dot{Q}(\alpha,\gamma).$$

We extend the convention and state e.g. that

$$j(\pi)_{\kappa\lambda} : j(P(\kappa)) \to P(\kappa) \star j(\dot{Q})(\kappa,\lambda)$$

is the composite of the following projections:

$$j(P(\kappa)) \xrightarrow{j(\psi)_{\kappa\lambda}} P(\kappa) \times j(\bar{Q})(\kappa,\lambda) \xrightarrow{j(\varphi)_{\kappa\lambda}} P(\kappa) \star j(\dot{Q})(\kappa,\lambda).$$

6. Master conditions

We keep the convention of §5: Let $\mu < \kappa < \lambda$ be regular cardinals and $j : V \to M$ witness that κ is huge with target λ. We need at least a master condition for a projection

$$j\left(P(\kappa) \star j(\dot{Q})(\kappa,\lambda)\right) \to P(\kappa) \star j(\dot{Q})(\kappa,\lambda).$$

To get one, we need a master condition for a projection

$$j\left(P(\kappa) \star \prod_{\alpha\in[\mu,\kappa]\cap \mathsf{M}}^{\kappa} \dot{S}(\kappa,\lambda)^{R(\alpha,\kappa)}\right) \to P(\kappa) \star \prod_{\alpha\in[\mu,\kappa]\cap \mathsf{M}}^{\kappa} \dot{S}(\kappa,\lambda)^{R(\alpha,\kappa)}.$$

The problem reduces to that of getting for each $\alpha \in (\mu,\kappa) \cap \mathsf{M}$ a master condition for a projection

$$j\left(P(\kappa) \star \dot{S}(\kappa,\lambda)^{P(\alpha)\star\dot{Q}(\alpha,\kappa)}\right) \to P(\kappa) \star \dot{S}(\kappa,\lambda)^{P(\alpha)\star\dot{Q}(\alpha,\kappa)},$$

which in turn reduces to getting a commutative diagram of the following form:

$$\begin{array}{ccc} j(P(\kappa)) & \xrightarrow{j(\pi_{\alpha\kappa})} & j\left(P(\alpha) \star \dot{Q}(\alpha,\kappa)\right) \\ \varphi_\star \big\downarrow & & \big\downarrow \varphi_\star \\ P(\kappa) \star \dot{S}(\kappa,\lambda)^{P(\alpha)\star\dot{Q}(\alpha,\kappa)} & \xrightarrow[\pi_{\alpha\kappa}\times \mathrm{id}]{} & \left(P(\alpha) \star \dot{Q}(\alpha,\kappa)\right) * \dot{S}(\kappa,\lambda). \end{array}$$

This is the reason why $\prod_{\xi\in(\kappa,\lambda)\cap \mathsf{M}}^{\mathrm{E}} \dot{S}(\xi,\lambda)$ appears as a component of $j(\dot{Q})(\kappa,\lambda)$, while $j(P(\kappa))$ does not (at least seemingly) have the corresponding component (see Remark 2). More specifically, our definition makes the first diagram in the proof of Lemma 8 commute.

Suppose $\alpha \in (\mu, \kappa) \cap \mathsf{M}$. For the following lemma, let

$$\dot{\mathrm{pr}}_\alpha : \prod_{\zeta \in [\mu,\kappa] \cap \mathsf{M}}^{\kappa} T\left(R(\zeta,\kappa), \dot{S}(\kappa,\lambda)\right) \times \prod_{\xi \in (\kappa,\lambda) \cap \mathsf{M}}^{\mathrm{E}} T\left(P(\kappa), \dot{S}(\xi,\lambda)\right) \to T\left(R(\alpha,\kappa), \dot{S}(\kappa,\lambda)\right)$$

be the projection to the α-th coordinate: $\dot{\mathrm{pr}}_\alpha(q) = q(\alpha)$. Define a map

$$\dot{\rho} : j\left(\prod_{\zeta \in [\mu,\alpha] \cap \mathsf{M}}^{\alpha} T\left(R(\zeta,\alpha), \dot{S}(\alpha,\kappa)\right) \times \prod_{\xi \in (\alpha,\kappa) \cap \mathsf{M}}^{\mathrm{E}} T\left(P(\alpha), \dot{S}(\xi,\kappa)\right)\right) \to$$
$$\prod_{\zeta \in [\mu,\alpha] \cap \mathsf{M}}^{\alpha} T\left(R(\zeta,\alpha), \dot{S}(\alpha,\kappa)\right) \times \prod_{\xi \in (\alpha,\kappa) \cap \mathsf{M}}^{\mathrm{E}} T\left(P(\alpha), \dot{S}(\xi,\kappa)\right) \times T\left(P(\alpha), \dot{S}(\kappa,\lambda)\right)$$

as follows: We first identify the domain of $\dot{\rho}$ with

$$\prod_{\zeta \in [\mu,\alpha] \cap \mathsf{M}}^{\alpha} T\left(R(\zeta,\alpha), \dot{S}(\alpha,\lambda)\right) \times \prod_{\xi \in (\alpha,\kappa) \cap \mathsf{M}}^{\mathrm{E}} T\left(P(\alpha), \dot{S}(\xi,\lambda)\right) \times \prod_{\xi \in [\kappa,\lambda) \cap \mathsf{M}}^{\mathrm{E}} T\left(P(\alpha), \dot{S}(\xi,\lambda)\right).$$

Let $\dot{\rho}$ be the projection identified with

$$\prod_{\zeta \in [\mu,\alpha] \cap \mathsf{M}} \dot{\mathrm{rs}}^{\zeta}_{\kappa} \times \prod_{\xi \in (\alpha,\kappa) \cap \mathsf{M}} \dot{\mathrm{rs}}^{\xi}_{\kappa} \times \mathrm{pr}_\kappa,$$

where

$$\dot{\mathrm{rs}}^{\zeta}_{\kappa} : T\left(R(\zeta,\alpha), \dot{S}(\alpha,\lambda)\right) \to T\left(R(\zeta,\alpha), \dot{S}(\alpha,\kappa)\right) \qquad \Vdash_{R(\zeta,\alpha)} \dot{\mathrm{rs}}^{\zeta}_{\kappa}(\dot{s}) = \dot{s}|(\alpha \times \kappa)$$
$$\dot{\mathrm{rs}}^{\xi}_{\kappa} : T\left(P(\alpha), \dot{S}(\xi,\lambda)\right) \to T\left(P(\alpha), \dot{S}(\xi,\kappa)\right) \qquad \Vdash_{P(\alpha)} \dot{\mathrm{rs}}^{\xi}_{\kappa}(\dot{s}) = \dot{s}|(\xi \times \kappa)$$
$$\mathrm{pr}_\kappa : \prod_{\xi \in [\kappa,\lambda) \cap \mathsf{M}}^{\mathrm{E}} T\left(P(\alpha), \dot{S}(\xi,\lambda)\right) \to T\left(P(\alpha), \dot{S}(\kappa,\lambda)\right) \qquad \mathrm{pr}_\kappa(q) = q(\kappa).$$

Finally note that $P(\alpha) \star \left(\dot{Q}(\alpha,\kappa) \times \dot{S}(\kappa,\lambda)\right)$ and $\left(P(\alpha) \star \dot{Q}(\alpha,\kappa)\right) \star \dot{S}(\kappa,\lambda)^{P(\alpha)}$ are canonically isomorphic, and the following diagram of projections commutes:

$$\begin{array}{ccc}
\left(P(\alpha) \star \dot{Q}(\alpha,\kappa)\right) \times S(\kappa,\lambda) & \xrightarrow{\mathrm{id} \times j(i)^{\kappa}_{\alpha\lambda}} & \left(P(\alpha) \star \dot{Q}(\alpha,\kappa)\right) \star \dot{S}(\kappa,\lambda)^{P(\alpha)} \\
\downarrow \mathrm{id} & & \downarrow \mathrm{id} \times j(k)^{\alpha}_{\kappa\lambda} \\
\left(P(\alpha) \star \dot{Q}(\alpha,\kappa)\right) \times S(\kappa,\lambda) & \xrightarrow[\mathrm{id} \times j(i)^{\alpha}_{\kappa\lambda}]{} & \left(P(\alpha) \star \dot{Q}(\alpha,\kappa)\right) * \dot{S}(\kappa,\lambda).
\end{array}$$

Lemma 8. *Suppose $\alpha \in (\mu, \kappa) \cap \mathsf{M}$. Then the following diagram commutes:*

$$\begin{array}{ccc}
j(P(\kappa)) & \xrightarrow{j(\pi_{\alpha\kappa})} & j\left(P(\alpha) \star \dot{Q}(\alpha,\kappa)\right) \\
{\scriptstyle j(\pi)_{\kappa\lambda}}\downarrow & & \downarrow{\scriptstyle \mathrm{id} \times \dot{p}} \\
P(\kappa) \star j(\dot{Q})(\kappa,\lambda) & & P(\alpha) \star \left(\dot{Q}(\alpha,\kappa) \times \dot{S}(\kappa,\lambda)\right) \\
{\scriptstyle \mathrm{id} \times \dot{\mathrm{pr}}_\alpha}\downarrow & & \downarrow{\scriptstyle \mathrm{id} \times j(k)^{\alpha}_{\kappa\lambda}} \\
P(\kappa) \star \dot{S}(\kappa,\lambda)^{P(\alpha)\star\dot{Q}(\alpha,\kappa)} & \xrightarrow[\pi_{\alpha\kappa} \times \mathrm{id}]{} & \left(P(\alpha) \star \dot{Q}(\alpha,\kappa)\right) * \dot{S}(\kappa,\lambda).
\end{array}$$

It remains to get a master condition for a projection

$$j\left(P(\kappa) \star \prod^{\mathrm{E}}_{\gamma \in (\kappa,\lambda) \cap \mathsf{M}} \dot{S}(\gamma,\lambda)\right) \to P(\kappa) \star \prod^{\mathrm{E}}_{\gamma \in (\kappa,\lambda) \cap \mathsf{M}} \dot{S}(\gamma,\lambda)$$

via a suitable extension of Kunen's lemma:

Lemma 9. *Suppose*

- $j : V \to M$ *witnesses that* κ *is huge with target* λ *and* $P \subset V_\kappa$ *has the* κ*-cc,*
- $\varphi : j(P) \to P \star \prod^{\mathrm{E}}_{\gamma \in (\kappa,\lambda) \cap \mathsf{M}} \dot{S}(\gamma,\lambda)$ *is a projection,*
- $1_{j(P)}$ *is a master condition for* $\varphi^- : j(P) \to P$.

Then there is a master condition $(1_{j(P)}, r^*)$ *for*

$$\varphi^+ : j\left(P \star \prod^{\mathrm{E}}_{\gamma \in (\kappa,\lambda) \cap \mathsf{M}} \dot{S}(\gamma,\lambda)\right) \to P \star \prod^{\mathrm{E}}_{\gamma \in (\kappa,\lambda) \cap \mathsf{M}} \dot{S}(\gamma,\lambda).$$

7. The main theorem

This section is devoted to a proof of the following theorem:

Theorem 10. *Let $\mu < \kappa$ be regular cardinals with κ 2-huge. Then there is a forcing extension in which $\kappa = \mu^+$ and $(\mu^{+++}, \mu^{++}, \mu^+) \twoheadrightarrow (\mu^{++}, \mu^+, \mu)$ holds.*

Proof. Let $j : V \to M$ witness that κ is 2-huge, $\lambda = j(\kappa)$ and $\theta = j(\lambda)$. Then ${}^\theta M \subset M$. Define

$$j(R)(\kappa,\lambda) = P(\kappa) \star j(\dot{Q})(\kappa,\lambda)$$

as in §5. We claim that forcing with $\left(P(\kappa) \star j(\dot{Q})(\kappa,\lambda)\right) * \dot{S}(\lambda,\theta)$ yields the required model.

First note that $P(\kappa) \subset V_\kappa$ is μ-closed and has the κ-cc by Lemma 7. Having $S(\mu,\kappa)$ as a complete suborder, $P(\kappa)$ forces $\kappa = \mu^+$. Since $P(\kappa)$ has the κ-cc and $j(\bar{Q})(\kappa,\lambda)$ as defined in §5 is κ-closed, $P(\kappa) \times j(\bar{Q})(\kappa,\lambda)$ forces $\kappa = \mu^+$ by Easton's lemma. Since there is a projection from $P(\kappa) \times j(\bar{Q})(\kappa,\lambda)$ to $P(\kappa) \star j(\dot{Q})(\kappa,\lambda)$, the latter forces $\kappa = \mu^+$ as well. Since there is a projection from $j(P(\kappa)) = P(\lambda)$ to $P(\kappa) \star j(\dot{Q})(\kappa,\lambda)$, the λ-cc of the former implies that of the latter. Having $P(\kappa) * \dot{S}(\kappa,\lambda)$ as a complete suborder, $P(\kappa) \star j(\dot{Q})(\kappa,\lambda)$ forces $\lambda = \kappa^+$. Thus $\left(P(\kappa) \star j(\dot{Q})(\kappa,\lambda)\right) * \dot{S}(\lambda,\theta)$ forces $\kappa = \mu^+$, $\lambda = \mu^{++}$ and $\theta = \mu^{+++}$.

It remains to prove that $(\theta, \lambda, \kappa) \twoheadrightarrow (\lambda, \kappa, \mu)$ holds in some forcing extension by $\left(P(\kappa) \star j(\dot{Q})(\kappa, \lambda)\right) * \dot{S}(\lambda, \theta)$. Since ${}^{\theta}M \subset M$, the sets of regular (resp. Mahlo) cardinals $\leq \theta$ are the same between V and M. Furthermore j sends the relevant posets as expected:

$$\left(P(\gamma) \star \dot{Q}(\gamma, \kappa)\right) * \dot{S}(\kappa, \lambda) \mapsto \left(P(\gamma) \star j(\dot{Q})(\gamma, \lambda)\right) * \dot{S}(\lambda, \theta),$$

$$P(\kappa) \star \prod_{\alpha \in [\mu, \kappa] \cap \mathsf{M}}^{\kappa} \dot{S}(\kappa, \lambda)^{R(\alpha, \kappa)} \mapsto P(\lambda) \star \prod_{\alpha \in [\mu, \lambda] \cap \mathsf{M}}^{\lambda} \dot{S}(\lambda, \theta)^{j(R)(\alpha, \lambda)},$$

$$P(\kappa) \star \prod_{\gamma \in (\kappa, \lambda) \cap \mathsf{M}}^{\mathrm{E}} \dot{S}(\gamma, \lambda) \mapsto P(\lambda) \star \prod_{\gamma \in (\lambda, \theta) \cap \mathsf{M}}^{\mathrm{E}} \dot{S}(\gamma, \theta).$$

Claim. *There is a master condition* $(1_{j(P(\kappa))}, q^{*\frown} r^*)$ *for*

$$j(\pi)^{+}_{\kappa\lambda} : j\left(P(\kappa) \star j(\dot{Q})(\kappa, \lambda)\right) \to P(\kappa) \star j(\dot{Q})(\kappa, \lambda).$$

Proof. Define

$$d^* = \bigcup \{ j(d) : d \cap \kappa \subset \mathsf{M} \text{ has size } < \kappa \wedge d - \kappa \subset (\kappa, \lambda) \cap \mathsf{M} \text{ is Easton}\}.$$

Then $d^* \cap \lambda = [\mu, \kappa] \cap \mathsf{M}$. Since λ is inaccessible, $|\{d \subset \lambda : d \text{ is Easton}\}| = \lambda$. Since each $j(d) - \lambda$ is an Easton subset of $(\lambda, \theta) \cap \mathsf{M}$, so is $d^* - \lambda$.

Let $\alpha \in [\mu, \kappa] \cap \mathsf{M}$. Define a projection

$$\varphi_{\star} : j(P(\kappa)) \to P(\kappa) \star \dot{S}(\kappa, \lambda)^{R(\alpha, \kappa)}$$

by composing the projections

$$j(P(\kappa)) \xrightarrow{j(\pi)_{\kappa\lambda}} P(\kappa) \star j(\dot{Q})(\kappa, \lambda) \xrightarrow{\mathrm{id} \times \dot{\mathrm{pr}}_{\alpha}} P(\kappa) \star \dot{S}(\kappa, \lambda)^{R(\alpha, \kappa)}.$$

Here $\dot{\mathrm{pr}}_{\alpha}$ denotes the projection to the α-th coordinate as defined for Lemma 8. We claim that there is a master condition $(1_{j(P(\kappa))}, q^*(\alpha))$ for

$$\varphi^{+}_{\star} : j\left(P(\kappa) \star \dot{S}(\kappa, \lambda)^{R(\alpha, \kappa)}\right) \to P(\kappa) \star \dot{S}(\kappa, \lambda)^{R(\alpha, \kappa)}.$$

If $\alpha \in \{\mu, \kappa\}$, then $R(\alpha, \kappa) = P(\kappa)$ and hence the claim follows from Kunen's lemma. Suppose next $\alpha \in (\mu, \kappa) \cap \mathsf{M}$. By Lemma 8 we get a commutative diagram of projections of the following form:

$$\begin{array}{ccc} j(P(\kappa)) & \xrightarrow{j(\pi_{\alpha\kappa})} & j\left(R(\alpha, \kappa)\right) \\ \varphi_{\star}\downarrow & & \downarrow\varphi_{*} \\ P(\kappa) \star \dot{S}(\kappa, \lambda)^{R(\alpha, \kappa)} & \xrightarrow[\pi_{\alpha\kappa} \times \mathrm{id}]{} & R(\alpha, \kappa) * \dot{S}(\kappa, \lambda). \end{array}$$

Also by Kunen's lemma we get a master condition $(1_{j(R(\alpha,\kappa))}, q^*(\alpha))$ for

$$\varphi^{+}_{*} : j\left(R(\alpha, \kappa) * \dot{S}(\kappa, \lambda)\right) \to R(\alpha, \kappa) * \dot{S}(\kappa, \lambda).$$

It is easy to check that $1_{j(P(\kappa))}$ is a master condition for $\varphi^{-}_{\star} : j(P(\kappa)) \to P(\kappa)$. Thus $(1_{j(P(\kappa))}, q^*(\alpha))$ is a master condition for $\varphi^{+}_{\star}$ by Foreman's lemma.

Now it is straightforward to check the hypothesis of Lemma 9 with $P = P(\kappa)$ and φ the composite of the projections

$$j(P(\kappa)) \xrightarrow{j(\pi)_{\kappa\lambda}} P(\kappa) \star j(\dot{Q})(\kappa,\lambda) \xrightarrow{\mathrm{id}\times\rho} P(\kappa) \star \prod^{\mathrm{E}}_{\gamma\in(\kappa,\lambda)\cap\mathsf{M}} \dot{S}(\gamma,\lambda).$$

Here ρ denotes the restriction to the upper coordinates: $\rho(q) = q|(\kappa,\lambda)$. Thus we get a master condition $(1_{j(P(\kappa))}, r^*)$ for

$$\varphi^+ : j\left(P(\kappa) \star \prod^{\mathrm{E}}_{\gamma\in(\kappa,\lambda)\cap\mathsf{M}} \dot{S}(\gamma,\lambda)\right) \to P(\kappa) \star \prod^{\mathrm{E}}_{\gamma\in(\kappa,\lambda)\cap\mathsf{M}} \dot{S}(\gamma,\lambda).$$

It is easy to check that $(1_{j(P(\kappa))}, \langle q^*(\alpha) : \alpha \in [\mu,\kappa] \cap \mathsf{M}\rangle^\frown r^*)$ is a master condition for $j(\pi)^+_{\kappa\lambda}$, as desired. □

Let $\bar{H} \subset P(\lambda) \star \dot{Q}(\lambda,\theta)$ be V-generic with $(1_{j(P(\kappa))}, q^{*\frown}r^*) \in \bar{H}$ and $\bar{G}$ be the projection of $\bar{H}$ to $P(\lambda)$. Then $H = j(\pi)^+_{\kappa\lambda}[\bar{G}]$ is V-generic over $P(\kappa) \star j(\dot{Q})(\kappa,\lambda)$, and we can extend j to $j : V[H] \to M[\bar{H}]$ in $V[\bar{H}]$. Recall that there is a projection from $P(\lambda) \star \dot{Q}(\lambda,\theta)$ to $P(\lambda) \star \dot{S}(\lambda,\theta)^{P(\kappa)\star j(\dot{Q})(\kappa,\lambda)}$, which is a dense subset of $P(\lambda) * \dot{S}(\lambda,\theta)^{P(\kappa)\star j(\dot{Q})(\kappa,\lambda)}$. Hence we get a $V[H]$-generic filter over $S(\lambda,\theta)^{V[H]}$ (say) K. Standard arguments show that $\bigcup j``K \in S(\theta, j(\theta))^{M[\bar{H}]}$. Let $\bar{K} \subset S(\theta, j(\theta))^{M[\bar{H}]}$ be $V[\bar{H}]$-generic. Then $j``K \subset \bar{K}$. Thus we can extend $j : V[H] \to M[\bar{H}]$ further to $j : V[H][K] \to M[\bar{H}][\bar{K}]$ in $V[\bar{H}][\bar{K}]$.

The rest of the proof is as in Theorem 6. Fix $f : {}^{<\omega}\theta \to \theta$ in $V[H][K]$. Then $j``\theta$ witnesses that in $M[\bar{H}][\bar{K}]$ there is $x \in [j(\theta)]^{j(\lambda)}$ closed under $j(f)$ such that $|x \cap j(\lambda)| = \lambda = j(\kappa)$ and $|x \cap j(\kappa)| = |\kappa| = \mu = j(\mu)$. By elementarity there is $x \in [\theta]^\lambda$ closed under f such that $|x \cap \lambda| = \kappa$ and $|x \cap \kappa| = \mu$ in $V[H][K]$, as desired. □

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