数理解析研究所講究録
第 1851 巻 2013 年 87-98

PARTIAL STATIONARY REFLECTION PRINCIPLES

TOSHIMICHI USUBA

Toshimichi Usuba (薄葉 季路)
Institute for Advanced Research,
Nagoya University

1. Introduction

Throughout this paper, κ denotes a regular uncountable cardinal and λ a cardinal $\geq \kappa^+$, unless otherwise specified.

Partial stationary reflection on $\mathcal{P}_{\omega_1}\omega_2$ was introduced by H. Sakai [2]. First we extend his notion to arbitrary κ and λ.

Definition 1.1. Let S^* be a stationary subset of $\mathcal{P}_\kappa\lambda$. For a stationary set $T \subseteq \mathcal{P}_{\kappa^+}\lambda$, we say that $\mathrm{RP}(S^*, T)$ *holds* if for every stationary subset $S \subseteq S^*$ there exists $X \in T$ such that $\kappa \subseteq X$ and $S \cap \mathcal{P}_\kappa X$ is stationary in $\mathcal{P}_\kappa X$. $\mathrm{RP}(S^*)$ means $\mathrm{RP}(S^*, \mathcal{P}_{\kappa^+}\lambda)$.

It is known that total stationary reflection $\mathrm{RP}(\mathcal{P}_\kappa\lambda)$ is a large cardinal property (e.g., see Velicikovic [3]), but Sakai [2] showed that partial stationary reflection on $\mathcal{P}_{\omega_1}\omega_2$ is not:

Fact 1.2 ([2]). *Suppose CH. If $\square_{\omega_1}$ holds, then there are a stationary set $S^* \subseteq \mathcal{P}_{\omega_1}\omega_2$ and a σ-Baire, ω_2-c.c. poset $\mathbb{P}$ such that $\mathbb{P}$ forces* $\mathrm{RP}(S^*)$.

In this paper, we generalize his result as follows:

Theorem 1.3. *Suppose $\kappa^{<\kappa} = \kappa$. Let $T \subseteq \mathcal{P}_{\kappa^+}\lambda$ be a stationary set such that $\forall X \in T\,(\kappa \subseteq X)$. Then there exists a κ-closed, κ^+-c.c. poset which forces the following statements:*

(1) T is stationary.

(2) There exists a stationary set $S^* \subseteq \mathcal{P}_\kappa\lambda$ such that

(a) $\forall X \in T\,(S^* \cap \mathcal{P}_\kappa X$ contains a club in $\mathcal{P}_\kappa X)$,

(b) $\mathrm{RP}(S^*, T)$ *holds.*

This theorem shows that, even $\kappa > \omega_1$ and $\lambda > \omega_2$, our partial stationary reflection is not a large cardinal property.

Next we consider a natural strengthening of partial stationary reflection, *simultaneous partial stationary reflection.*

Definition 1.4. For stationary sets $S_0^*, S_1^* \subseteq \mathcal{P}_\kappa\lambda$ and $T \subseteq \mathcal{P}_{\kappa^+}\lambda$, we say that $\mathrm{RP}^2(S_0^*, S_1^*, T)$ *holds* if for every stationary subsets $S_0 \subseteq S_0^*$ and $S_1 \subseteq S_1^*$ in $\mathcal{P}_\kappa\lambda$, there exists $X \in T$ such that $\kappa \subseteq X$ and both $S_0 \cap \mathcal{P}_\kappa X$ and $S_1 \cap \mathcal{P}_\kappa X$ are stationary in $\mathcal{P}_\kappa X$. $\mathrm{RP}^2(S_0^*, S_1^*)$ means $\mathrm{RP}^2(S_0^*, S_1^*, \mathcal{P}_{\kappa^+}\lambda)$.

We prove that our simultaneous partial stationary reflection is a large cardinal property by showing the following:

Definition 1.5. For a regular uncountable cardinal μ, $\square(\mu)$ *holds* if there exists a sequence $\langle C_\xi : \xi < \mu \rangle$ satisfying the following:

(1) for all $\xi < \mu$, C_ξ is club in ξ and for all $\eta \in \lim(C_\xi)$, $C_\eta = C_\xi \cap \eta$,
(2) for all club C in μ, there exists $\xi \in \lim(C)$ such that $C \cap \xi \neq C_\xi$.

Such an sequence $\langle C_\xi : \xi < \mu \rangle$ is called a $\square(\mu)$*-sequence.*

Theorem 1.6. *Suppose* $\mathrm{RP}^2(S_0^*, S_1^*)$ *holds for some stationary* $S_0^*, S_1^* \subseteq \mathcal{P}_\kappa\lambda$. *Then for every regular* μ *with* $\kappa^+ \leq \mu \leq \lambda$, $\square(\mu)$ *fails.*

We also prove the following:

Theorem 1.7. *For every stationary* $S_0^*, S_1^* \subseteq \mathcal{P}_\kappa\lambda$ *and regular* μ *with* $\kappa^+ \leq \mu \leq \lambda$, $\mathrm{RP}^2(S_0^*, S_1^*, \{X \in \mathcal{P}_{\kappa^+}\lambda : \mathrm{cf}(X \cap \mu) < \kappa\})$ *fails, where* $\mathrm{cf}(X) = \mathrm{cf}(\mathrm{ot}(X))$.

Todorcevic showed that $\mathrm{RP}(\mathcal{P}_{\omega_1}\omega_2)$ implies that $2^\omega \leq \omega_2$. However we prove the following, which shows that our partial stationary reflection does not affect the size of the continuum:

Theorem 1.8. *(1) Suppose* $\mathrm{RP}(S^*)$ *for some stationary* $S^* \subseteq \mathcal{P}_\kappa\lambda$. *Then every* κ*-c.c. forcing preserves* $\mathrm{RP}(S^*)$.

(2) Suppose PFA^{++}. *Let* $\lambda \geq \omega_2$. *Then every c.c.c. forcing notion forces* $\mathrm{RP}^2(\mathcal{P}^V_{\omega_1}\lambda, \mathcal{P}^V_{\omega_1}\lambda)$.

2. Preliminaries

For a set X of ordinals, let $\mathrm{cf}(X) = \mathrm{cf}(\mathrm{ot}(X))$.

For regular cardinals $\nu < \mu$, let $E^\mu_\nu = \{\alpha < \mu : \mathrm{cf}(\alpha) = \nu\}$ and $E^\mu_{<\nu} = \{\alpha < \mu : \mathrm{cf}(\alpha) < \nu\}$.

The proofs of the following lemmatta are easy:

Lemma 2.1. *For a stationary $S \subseteq \mathcal{P}_\kappa\lambda$ and a κ-c.c. poset $\mathbb{P}$, $\mathbb{P}$ preserves the stationarity of S.*

Lemma 2.2. *For $S \subseteq \mathcal{P}_\kappa\lambda$, if $\{X \in \mathcal{P}_{\kappa^+}\lambda : S \cap \mathcal{P}_\kappa X$ is stationary in $\mathcal{P}_\kappa X\}$ is stationary in $\mathcal{P}_{\kappa^+}\lambda$, then S is stationary in $\mathcal{P}_\kappa\lambda$.*

Lemma 2.3. *For stationary sets $S^* \subseteq \mathcal{P}_\kappa\lambda$ and $T \subseteq \mathcal{P}_{\kappa^+}\lambda$, suppose* $\mathrm{RP}(S^*, T)$ *holds. Then for every stationary $S \subseteq S^*$, $\{X \in T : S \cap \mathcal{P}_\kappa X$ is stationary in $\mathcal{P}_\kappa X\}$ is stationary in $\mathcal{P}_{\kappa^+}\lambda$.*

We define club shootings into $\mathcal{P}_\kappa\lambda$, which was observed in [2].

Definition 2.4. For $S \subseteq \mathcal{P}_\kappa\lambda$, let $\mathbb{C}(S)$ be the poset which consists of all functions p such that:

(1) $|p| < \kappa$,
(2) $p : d(p) \times d(p) \to \kappa$ for some $d(p) \in \mathcal{P}_\kappa\lambda$, and
(3) $\forall x \subseteq d(p)\, (x \in S \Rightarrow x$ is not closed under $p)$.

For $p, q \in \mathbb{C}(S)$, $p \leq q \iff q \subseteq p$.

Let $\mathbb{C} = \mathbb{C}(\emptyset)$.

Lemma 2.5. *(1) $\mathbb{C}(S)$ satisfies the $(2^{<\kappa})^+$-c.c.*
(2) For every $x \in \mathcal{P}_\kappa\lambda$, $\{p \in \mathbb{C}(S) : x \subseteq d(p)\}$ is a dense open set in $\mathbb{C}(S)$.
(3) Whenever G is $(V, \mathbb{C}(S))$-generic, $\bigcup G$ is a function from $\lambda \times \lambda$ to κ, and every $x \in S$ is not closed under the function.

Proof. For (1), take $A \subseteq \mathbb{C}(S)$ with size $(2^{<\kappa})^+$. By Δ-system lemma, we can find $B \subseteq A$ and $a \in \mathcal{P}_\kappa\lambda$ such that $|B| = (2^{<\kappa})^+$ and $d(p) \cap d(q) = a$ for every distinct $p, q \in B$. Moreover we may assume that $p|a \times a = q|a \times a$ for every $p, q \in B$. We check that B is a pairwise compatible set.

Take $p, q \in B$. Pick $\alpha < \kappa$ with $\alpha > \sup(d(p) \cap \kappa) + 1, \sup(d(q) \cap \kappa) + 1$. Then define r as $\mathrm{dom}(r) = (d(p) \cup d(q)) \times (d(p) \cup d(q))$ and

$$r(\xi, \eta) = \begin{cases} p(\xi, \eta) & \text{if } \xi, \eta \in d(p). \\ q(\xi, \eta) & \text{if } \xi, \eta \in d(q). \\ \alpha & \text{otherwise.} \end{cases}$$

We have $r \leq p, q$. (2) follows from a similar argument, and (3) is straightforward.

□

3. The proof of Theorem 1.3

Suppose $\kappa^{<\kappa} = \kappa$. Fix a stationary set $T \subseteq \mathcal{P}_{\kappa^+}\lambda$ such that $\forall X \in T\,(\kappa \subseteq X)$.

We consider the following poset $\mathbb{P}_T$, which adds a new stationary subset S^* of $\mathcal{P}_\kappa\lambda$.

Definition 3.1. $\mathbb{P}_T$ is the set of all functions p satisfying the following:

(1) $|p| < \kappa$ and $\mathrm{dom}(p) \subseteq T$,

(2) for every $X \in \mathrm{dom}(p)$, $p(X)$ is a $\subseteq$-increasing continuous set $\{x_i : i \leq \gamma\}$ in $\mathcal{P}_\kappa X$ such that $\gamma < \kappa$ and $x_i \cap \kappa \in \kappa$ for all $i \leq \gamma$.

For $p \in \mathbb{P}_T$ and $X \in \mathrm{dom}(p)$, $\max(p(X))$ denotes the maximum element of $p(X)$. Let $u(p) = \bigcup\{p(X) : X \in \mathrm{dom}(P)\}$. Note that $u(p) \subseteq \mathcal{P}_\kappa\lambda$ and $|u(p)| < \kappa$. For $p, q \in \mathbb{P}_T$, define $p \leq q \iff$

(a) $\mathrm{dom}(p) \supseteq \mathrm{dom}(q)$,

(b) $\forall X \in \mathrm{dom}(q)(q(X) = \{x \in p(X) : x \subsetneq \max(q(X))\})$ (hence $u(p) \supseteq u(q)$),

(c) $\forall x \in u(p)\ (x \subseteq \bigcup u(q) \Rightarrow x \in u(q))$,

(d) $\forall X \in \mathrm{dom}(p) \setminus \mathrm{dom}(q)\ (\max(p(X)) \nsubseteq \bigcup u(q))$

(e) $\forall X \in \mathrm{dom}(q) \forall x \in p(X) \setminus q(X)\ (x \nsubseteq \bigcup u(q))$.

Lemma 3.2. *(1) $\mathbb{P}_T$ is κ-closed,*

(2) $\mathbb{P}_T$ satisfies the κ^+-c.c. (if $\kappa^{<\kappa} = \kappa$),

(3) for all $X \in T$ and $x \in \mathcal{P}_\kappa X$, $\{p \in \mathbb{P}_T : X \in \mathrm{dom}(p)$ and $x \subsetneq \max(p(X))\}$ is dense in $\mathbb{P}_T$.

Proof. (1). Let $\gamma < \kappa$ be a limit ordinal and $\langle p_i : i < \gamma \rangle$ be a decreasing sequence in $\mathbb{P}_T$. Then define the function p^* as the following manner:

(i) $\mathrm{dom}(p^*) = \bigcup_{i<\gamma} \mathrm{dom}(p_i)$,

(ii) for $X \in \mathrm{dom}(p^*)$, $p^*(X) = \bigcup\{p_i(X) : i < \gamma, X \in \mathrm{dom}(p_i)\} \cup \{\bigcup\{\max(p_i(X)) : i < \gamma, X \in \mathrm{dom}(p_i)\}\}$.

Since the p_i's are decreasing, it is easy to show that $p^* \in \mathbb{P}_T$. For $i < \gamma$, we show $p \leq p_i$. It is easily verified that the conditions (a) and (b) in the definition of the order are satisfied.

(c). Take $x \in u(p^*)$ such that $x \subseteq \bigcup u(p_i)$. Take $X \in \mathrm{dom}(p^*)$ such that $x \in p^*(X)$. If $x \neq \max(p^*(X))$, then $x \in p_j(X)$ for some $j > i$ with $X \in \mathrm{dom}(p_j)$. Since $p_j \leq p_i$, we have $x \in p_i(X)$. Next suppose $x = \max(p^*(X))$. Take $k < \gamma$ such that $i < k$ and $X \in \mathrm{dom}(p_k)$. Then $\max(p_k(X)) \subseteq \max(p^*(X)) = x \subseteq \bigcup u(p_i)$ holds. Hence $X \in \mathrm{dom}(p_i)$ by (d). For each $j \geq i$, $\max(p_j(X)) \subseteq \max(p^*(X)) = x \subseteq$

$\bigcup u(p_i)$ holds. Thus we have $\max(p_j(X)) \in p_i(X)$ by (e). Therefore $\{\max(p_j(X)) : i \leq j < \gamma\} \subseteq p_i(X)$, and we have $\max(p^*(X)) = \bigcup\{\max(p_j(X)) : i \leq j < \gamma\} \in p_i(X)$.

(d). Take $X \in \mathrm{dom}(p^*) \setminus \mathrm{dom}(p_i)$. Then there exists $j > i$ such that $X \in \mathrm{dom}(p_j)$. We know $\max(p_j(X)) \not\subseteq \bigcup u(p_i)$. Because $\max(p_j(X)) \subseteq \max(p^*(X))$, we know $\max(p^*(X)) \not\subseteq \bigcup u(p_i)$.

(e). Take $X \in \mathrm{dom}(p_i)$ and $x \in p^*(X) \setminus p_i(X)$. Then there exist $j \geq i$ and $y \in \mathrm{dom}(p_j)$ such that $y \subseteq x$ and $y \notin p_i(X)$. Hence $y \not\subseteq \bigcup u(p_i)$ and $x \not\subseteq \bigcup u(p_i)$.

(2). Take an arbitrary $A \subseteq \mathbb{P}_T$ with $|A| \geq \kappa^+$. We prove that A is not an antichain. By Δ-system lemma, we can find $r \in \mathcal{P}_\kappa T$, $s \in \mathcal{P}_\kappa \lambda$, and $B \subseteq A$ with $|B| \geq \kappa^+$ such that $\forall p, q \in B$ ($\mathrm{dom}(p) \cap \mathrm{dom}(q) = r$ and $\bigcup u(p) \cap \bigcup u(q) = s$). By our cardinal arithmetic assumption, there exists $C \subseteq B$ with $|C| \geq \kappa^+$ such that $\forall p, q \in B(\forall X \in r\,(p(X) = q(X))$ and $\mathcal{P}_\kappa s \cap u(p) = \mathcal{P}_\kappa s \cap u(q))$. We check that any two elements of C are pairwise compatible. Take $p, q \in C$. For each $X \in \mathrm{dom}(p) \cup \mathrm{dom}(q)$, fix $a_X \in \mathcal{P}_\kappa X$ such that $(\bigcup u(p) \cup \bigcup u(q)) \cap X \subsetneq a_X$. Define the function r as the following:

(i) $\mathrm{dom}(r) = \mathrm{dom}(p) \cup \mathrm{dom}(q)$,

(ii) $r(X) = p(X) \cup \{a_X\}$ if $X \in \mathrm{dom}(p)$, and $r(X) = q(X) \cup \{a_X\}$ if $X \in \mathrm{dom}(q)$.

This is well-defined because $p(X) = q(X)$ for all $X \in \mathrm{dom}(p) \cap \mathrm{dom}(q)$. We see that r is a lower bound of p and q. $r \in \mathbb{P}_T$ is easily verified. For $r \leq p$, the conditions (a) and (b) are clear.

(c). Take $x \in u(r)$ such that $x \subseteq \bigcup u(p)$. Then $x \neq a_X$ for all $X \in \mathrm{dom}(p) \cup \mathrm{dom}(q)$. Hence $x \in u(p) \cup u(q)$. If $x \in u(p)$ then we have done. Assume $x \in u(q)$. Then $x \subseteq \bigcup u(q)$. Since $x \subseteq \bigcup u(p)$, we have $x \subseteq \bigcup u(p) \cap \bigcup u(q) = s$ and $x \in \mathcal{P}_\kappa s$. Because $\mathcal{P}_\kappa s \cap u(p) = \mathcal{P}_\kappa s \cap u(q)$, we have $x \in \mathcal{P}_\kappa s \cap u(p)$ and $x \in u(p)$.

(d). Take $X \in \mathrm{dom}(r) \setminus \mathrm{dom}(p)$. Then $\max(r(X)) = a_X \supsetneq \bigcup u(p) \cap X$, thus $\max(r(X)) \not\subseteq \bigcup u(p)$.

(e). Take $X \in \mathrm{dom}(p)$ and $x \in r(X) \setminus p(X)$. By the definition of $r(X)$, we have $r(X) = p(X) \cup \{a_X\}$. Hence $x = a_X \not\subseteq \bigcup u(p)$.

$r \leq q$ can be proved by the same argument.

(3). Take $X \in T$, $x \in \mathcal{P}_\kappa X$ and $q \in \mathbb{P}$. Take $x^* \in \mathcal{P}_\kappa X$ such that $\bigcup u(q) \cap X \subsetneq x^*$. Define p as $\mathrm{dom}(p) = \mathrm{dom}(q) \cup \{X\}$, $p|\mathrm{dom}(q) = q$ and $p(X) = \{x^*\}$ if $X \notin \mathrm{dom}(q)$, and $q(X) \cup \{x^*\}$ if $X \in \mathrm{dom}(q)$. Then $p \leq q$ can be verified. □

Note that the following: For $\gamma < \kappa$ and a decreasing sequence $\langle p_i : i < \gamma \rangle$ in $\mathbb{P}_T$, let p^* be a lower bound of the p_i's as constructed in the proof of (1) above. Then p^* is the largest lower bound of the p_i's and $\bigcup u(p^*) = \bigcup_{i<\gamma}(\bigcup u(p_i))$.

Definition 3.3. For a canonical name of $(V, \mathbb{P}_T)$-generic filter $\dot{G}$, let $\dot{S}^*$ be a $\mathbb{P}_T$-name such that

$$\Vdash_{\mathbb{T}} \text{“}\dot{S}^* = \bigcup\{u(p) : p \in \dot{G}\}\text{”}.$$

The following are easily verified by the definition of $\mathbb{P}_T$.

Lemma 3.4. *(1)* $\Vdash_{\mathbb{P}_T}$ “$\forall X \in T\,(\dot{S}^* \cap \mathcal{P}_\kappa X$ *contains a club in* $\mathcal{P}_\kappa X)$”,
(2) for all $p \in \mathbb{P}_T$, $p \Vdash_{\mathbb{P}_T}$ “ $\{y \in \dot{S}^* : y \subseteq \bigcup u(p)\} = u(p)$ ”.

Now fix a name $\dot{S}$ such that

$$\Vdash_{\mathbb{P}_T} \text{“}\dot{S} \subseteq \dot{S}^* \text{ and } \forall X \in T\ (\mathcal{P}_\kappa X \cap \dot{S} \text{ is non-stationary in } \mathcal{P}_\kappa X)\text{”}.$$

We see that $\mathbb{P}_T * \mathbb{C}(\dot{S})$ has good properties.

For each $X \in T$, fix a name $\dot{g}_X$ such that

$$\Vdash_{\mathbb{P}_T} \text{“ } \dot{g} : [X]^{<\omega} \to X \text{ and } \forall x \in \mathcal{P}_\kappa X\ (x \text{ is closed under } \dot{g}_X \Rightarrow x \notin \dot{S})\text{”}.$$

Let $\dot{\mathbb{Q}}$ be a name such that $\Vdash$“$\dot{\mathbb{Q}} = \mathbb{C}(\dot{S})$”. We prove that $\mathbb{P}_T * \dot{\mathbb{Q}}$ has a κ-closed dense subset.

Lemma 3.5. *Let* $D = \{p \in \mathbb{P}_T : \forall X \in \text{dom}(p)\,(p \Vdash_{\mathbb{P}_T}$ “$\max(p(X))$ *is closed under* $\dot{g}_X$”$)\}$. *Then D is dense in* $\mathbb{P}_T$.

Proof. Take $p \in \mathbb{P}_T$. We want to find $q \in D$ such that $q \leq p$. We take a decreasing sequence p_i $(i < \omega)$ in $\mathbb{P}_T$ by induction on $i < \omega$. Let $p_0 = p$. Suppose p_i is defined. By the κ-closedness of $\mathbb{P}_T$, we can choose $p' \leq p_i$ and $a \in \mathcal{P}_\kappa\lambda$ such that $p' \Vdash$“ $\dot{g}_X$“$[\max(p_i(X))]^{<\omega} \subseteq a \cap X$” for all $X \in \text{dom}(p_i)$. Then choose $p_{i+1} \leq p'$ such that $a \cap X \subseteq \max(p_{i+1}(X))$ for all $X \in \text{dom}(p_i)$.

Finally let q be the greatest lower bound of the p_i's. By our construction, it is easy to see that $q \in D$. □

Lemma 3.6. *Let D be as in Lemma 3.5. Let* $D' = \{\langle p, q\rangle \in \mathbb{P}_T * \dot{\mathbb{Q}} : p \in D,$ $q = \check{r}$ *for some* $r \in \mathbb{C}$ *and* $d(r) = \bigcup(u(p))\}$. *Then D' is a κ-closed dense subset in* $\mathbb{P}_T * \dot{\mathbb{Q}}$.

Proof. Density: Take $\langle p, \dot{q}\rangle \in \mathbb{P}_T * \dot{\mathbb{Q}}$. Take $p' \in D$ and r such that $p' \Vdash$“$\check{r} = \dot{q}$” and $\bigcup u(p') \supseteq d(r)$. Now define r' as the following:

(1) $r' : \bigcup u(p') \times \bigcup u(p') \to \kappa$,

(2) for $a \in \bigcup u(p') \times \bigcup u(p')$, if $a \in d(r) \times d(r)$ the $r'(a) = r(a)$, otherwise $r'(a) = \sup(\bigcup(u(p') \cap \kappa)) + 1$.

It is easy to show that $p' \Vdash$"$\check{r}' \in \mathbb{C}(\dot{S})$" and $\langle p', \check{r}' \rangle \leq \langle p, \dot{q} \rangle$.

Next we prove D' is κ-closed. Let $\gamma < \kappa$ and $\langle p_i, \check{q}_i \rangle$ $(i < \gamma)$ be a decreasing sequence in D'. We show that this sequence has a lower bound. Let $p^* \in \mathbb{P}_T$ be the greatest lower bound of the p_i's. Note that for all $X \in \mathrm{dom}(p^*)$, $p^* \Vdash_{\mathbb{P}_T}$"$\max(p^*(X))$ is closed under $\dot{g}_X$".

Let $q^* = \bigcup_{i<\gamma} q_i$. q^* is a function with the domain $d(q^*) \times d(q^*)$, where $d(q^*) = \bigcup_{i<\gamma} d(q_i)$. Notice that $d(q^*) = \bigcup_{i<\gamma} d(q_i) = \bigcup_{i<\gamma} \bigcup u(p_i) = \bigcup u(p^*)$. We complete the proof by showing the following claim.

Claim 3.7. *$p^* \Vdash$ "$q^* \in \mathbb{C}(\dot{S})$".*

Proof. Take a $(V, \mathbb{P}_T)$-generic G with $p^* \in G$ and work in $V[G]$. First note that $\{x \in S^* : x \subseteq \bigcup u(p^*)\} = u(p^*)$. To show that $q^* \in \mathbb{C}(S)$, take $x \subseteq d(q^*)$ with $x \in S$. We check that x is not closed under q^*. Since $x \subseteq d(q^*) = \bigcup u(p^*)$ and $x \in S \subseteq S^*$, we have $x \in u(p^*)$. Hence there exists $X \in \mathrm{dom}(p^*)$ such that $x \in p^*(X)$. Because $\max(p^*(X))$ is closed under g_X, we know $\max(p^*(X)) \notin S$. Thus $x \neq \max(p^*(X))$ and $x \in p_i(X)$ for some $i < \gamma$ with $X \in \mathrm{dom}(p_i)$. Then $x \subseteq \bigcup u(p_i) = d(q_i)$. Since q_i is a condition, x is not closed under q_i, and not closed under q^*. □[Claim]

□

Note that, in fact, D' is κ-directed closed.

By an iteration of the above forcing, we can prove Theorem 1.3. Let $\langle \mathbb{P}_\xi, \dot{\mathbb{Q}}_\eta : \xi < \zeta, \eta < \zeta \rangle$ be a $< \kappa$-support iteration such that for every $\xi < \zeta$,

(1) $\dot{\mathbb{Q}}_0 = \mathbb{P}_T$,

(2) $\mathbb{P}_\xi$ satisfies the κ^+-c.c. and has a κ-closed dense subset,

(3) for $\xi > 0$ there exists $\mathbb{P}_\xi$-name $\dot{S}_\xi$ such that

$$\Vdash_\xi \text{“}\dot{S}_\xi \subseteq \dot{S}^* \text{ and } \forall X \in T\, (\mathcal{P}_\kappa X \cap \dot{S}_\xi \text{ is non-stationary in } \mathcal{P}_\kappa X)\text{”},$$

(4) for every $X \in T$, $\dot{g}^\xi_X$ is a $\mathbb{P}_\xi$-name such that

$$\Vdash_\xi \text{“}\dot{g}^\xi_X : [X]^{<\omega} \to X \text{ and } \forall x \in \mathcal{P}_\kappa X\ (x \in \dot{S}_\xi \Rightarrow x \text{ is not closed under } \dot{g}^\xi_X)\text{”},$$

(5) $\Vdash_\xi$"$\dot{\mathbb{Q}}_\xi = \mathbb{C}(\dot{S}_\xi)$" for $\xi > 0$,

(6) let D_ξ is the set of all $p \in \mathbb{P}_\xi$ such that

(a) $\forall \eta \in \mathrm{supp}(p) \setminus \{0\}$ ($p(\eta) = \check{r}$ for some $r \in \mathbb{C}$),
(b) for all $X \in \mathrm{dom}(p(0))$ and $\eta \in \mathrm{supp}(p) \setminus \{0\}$ ($p|\eta \Vdash_\eta$"$\max(p(0)(X))$ is closed under $\dot{g}^{\eta}_X$",
(c) $\bigcup(u(p(0)) = d(p(\eta))$ for all $\eta \in \mathrm{supp}(p) \setminus \{0\}$.

Then D_ξ is a κ-closed dense set in $\mathbb{P}_\xi$.

Let $\mathbb{P}_\zeta$ and D_ζ be as intended. We can check that D_ζ is a κ-closed dense set in $\mathbb{P}_\zeta$, and $\mathbb{P}_\zeta$ has the κ^+-c.c.

By a standard book keeping method, we can destroy the stationarity of all non-reflecting subset of S^* by an iteration above. By κ^+-c.c., T remains stationary in $\mathcal{P}_{\kappa^+}\lambda$ in the generic extension. Thus S^* is stationary in $\mathcal{P}_\kappa\lambda$, and $\mathrm{RP}(S^*, T)$ holds.

4. Proof of Theorems 1.6 and 1.7

Proposition 4.1. *Let μ be a regular cardinal with $\kappa^+ \le \mu \le \lambda$. Let $T = \{X \in \mathcal{P}_{\kappa^+}\lambda : \kappa \subseteq X,\ \mathrm{cf}(X \cap \mu) < \kappa\}$. Then for every stationary sets $S^*_0, S^*_1 \subseteq \mathcal{P}_\kappa\lambda$, $\mathrm{RP}^2(S^*_0, S^*_1, T)$ fails.*

Proof. Suppose not. For each $\xi \in E^\mu_{<\kappa}$, fix an increasing sequence $\langle \gamma^\xi_i : i < \mathrm{cf}(\xi)\rangle$ with limit ξ. For $n < 2$, $i < \kappa$, and $\delta < \mu$, let

$$S_{n,i,\delta} = \{x \in S^*_n : \delta = \min(x \setminus \gamma_i^{\sup(x\cap\mu)})\}.$$

Claim 4.2. *(1) For every $\xi < \mu$, there exist $i < \kappa$ and $\delta < \mu$ such that $\delta > \xi$ and $S_{0,i,\delta}$ is stationary.*
(2) For every $i < \kappa$ and $\delta < \mu$, if $S_{0,i,\delta}$ is stationary then $S_{1,i,\delta}$ is stationary.
(3) For every $i < \kappa$ and $\delta_0, \delta_1 < \mu$, if S_{0,i,δ_0} and S_{1,i,δ_1} are stationary then $\delta_0 = \delta_1$.

Proof. (1). Let $T' = \{X \in T : S^*_0 \cap \mathcal{P}_\kappa X$ is stationary, $\xi \in X\}$. T' is stationary in $\mathcal{P}_{\kappa^+}\lambda$. Take $X \in T'$. Then $\mathrm{cf}(X \cap \mu) < \kappa \subseteq X$ and $\sup(X \cap \mu) > \xi$, hence there exists $i \in X$ such that $\gamma_i^{\sup(X\cap\mu)} > \xi$. By applying Fodor's lemma to T', there exists $i < \kappa$ such that $T'' = \{x \in T' : \gamma_i^{\sup(X\cap\mu)} > \xi\}$ is stationary in $\mathcal{P}_{\kappa^+}\lambda$. For $X \in T''$ let $\delta_X = \min(X \setminus \gamma_i^{\sup(X\cap\mu)})$. By Fodor's lemma again, there is $\delta < \mu$ such that $T^* = \{X \in T'' : \gamma_i^{\sup(X\cap\mu)} > \xi, \delta = \min(X \setminus \gamma_i^{\sup(X\cap\mu)})\}$ is stationary in $\mathcal{P}_{\kappa^+}\lambda$.

Pick $X \in T^*$. Since $\mathrm{cf}(X \cap \mu) < \kappa$, the set $D_X = \{x \in \mathcal{P}_\kappa X : \sup(x \cap \mu) = \sup(X \cap \mu), \delta \in x\}$ contains a club in $\mathcal{P}_\kappa X$. Clearly $x \in S_{0,i,\delta}$ for each $x \in D_X \cap S^*_0$. This means that $S_{0,i,\delta}$ is stationary in $\mathcal{P}_\kappa\lambda$.

(2). By $\mathrm{RP}^2(S^*_0, S^*_1)$, $T' = \{X \in T : \delta \in X, S_{0,i,\delta} \cap \mathcal{P}_\kappa X,\ S^*_1 \cap \mathcal{P}_\kappa X$ are stationary $\}$ is stationary in $\mathcal{P}_{\kappa^+}\lambda$. Fix $X \in T'$. Since $S_{0,i,\delta} \cap \mathcal{P}_\kappa X$ is stationary in $\mathcal{P}_\kappa X$ and

$\mathrm{cf}(X\cap\mu)<\kappa$, we have that $\delta=\min(X\setminus\gamma_i^{\sup(X\cap\mu)})$. By the same argument as (1), we have that $S_{1,i,\delta}$ is stationary in $\mathcal{P}_\kappa\lambda$.

(3). Let $X\in T$ be such that $\delta_0,\delta_1\in X$ and $S_{0,i,\delta_0}\cap\mathcal{P}_\kappa X$, $S_{1,i,\delta_1}\cap\mathcal{P}_\kappa X$ are stationary. Choose $x_0\in S_{0,i,\delta_0}\cap\mathcal{P}_\kappa X$ and $x_1\in S_{1,i,\delta_1}\cap\mathcal{P}_\kappa X$ such that $\sup(x_0\cap\mu)=\sup(x_1\cap\mu)=\sup(X\cap\mu)$ and $\delta_0,\delta_1\in x_0\cap x_1$. By the minimality of δ_0, we have $\delta_0\leq\delta_1$. Similarly we know $\delta_1\leq\delta_0$. Therefore $\delta_0=\delta_1$. □[Claim]

Hence we have that if $S_{0,i,\delta}$ and $S_{0,i,\delta'}$ are stationary, then $\delta=\delta'$.

For each $i<\kappa$, define $\delta_i<\mu$ as follows: if $S_{0,i,\delta}$ is stationary for some $\delta<\mu$, then let δ_i be a (unique) $\delta<\mu$ such that $S_{0,i,\delta}$ is stationary. If there is no such δ, then let $\delta_i=0$. Since $\mu=\mathrm{cf}(\mu)>\kappa$, we know $\sup_{i<\kappa}\delta_i<\mu$. But this contradicts (1) of the claim. □

Proposition 4.3. *Let $S_0^*,S_1^*\subseteq\mathcal{P}_\kappa\lambda$ be stationary and suppose $\mathrm{RP}^2(S_0^*,S_1^*)$ holds. Then for every regular μ with $\kappa^+\leq\mu\leq\lambda$, $\square(\mu)$ fails.*

Proof. We prove only the case $\mu=\lambda$. Other cases follow from similar arguments.

Toward the contradiction, suppose $\square(\lambda)$ holds. Let $\langle C_\xi:\xi<\lambda\rangle$ be a $\square(\lambda)$-sequence.

Let $T=\{X\in\mathcal{P}_{\kappa^+}\lambda:\mathrm{cf}(X)=\kappa\subseteq X\}$. We assumed $\mathrm{RP}^2(S_0^*,S_1^*)$, but by the previous proposition, in fact $\mathrm{RP}^2(S_0^*,S_1^*,T)$ holds.

For each $\alpha<\lambda$ and $n<2$, let

$$S_{n,\alpha}=\{x\in S_n^*:C_{\sup(x)}\cap\sup(x\cap\alpha)=C_\alpha\cap\sup(x\cap\alpha)\}.$$

Let $A_n=\{\alpha<\lambda:S_{n,\alpha}\text{ is stationary}\}$.

Claim 4.4. *For each $n<2$, A_n is unbounded in λ.*

Proof. Fix $n<2$. By shrinking S_n^* by a club in $\mathcal{P}_\kappa\lambda$, we may assume that the following:

(1) For all $x\in S_n^*$ and $\alpha\in x$, if $x\cap\alpha$ is bounded in α then $\mathrm{cf}(\alpha)\geq\kappa$.

(2) For all $x\in S_n^*$ and $\alpha\in x\cap E^\lambda_{\geq\kappa}$, $\sup(x\cap\alpha)\in\lim(C_\alpha)$ holds.

Let $T'=\{X\in T:S_n^*\cap\mathcal{P}_\kappa X\text{ is stationary}\}$. Then T' is stationary in $\mathcal{P}_{\kappa^+}\lambda$. To show that A_n is unbounded, take $\xi<\lambda$. Fix $X\in T'$ with $\sup(X)>\xi$. Since $\mathrm{cf}(X)=\kappa$, the set $\{\beta<\sup(X):\beta\in\lim(C_{\sup X})\}$ contains a club in $\sup(X)$. Note that $C_{\sup(X)}\cap\beta=C_\beta$ for each β from the club. Hence we know $S_X=\{x\in S_n^*\cap\mathcal{P}_\kappa X:C_{\sup(x)}=C_{\sup(X)}\cap\sup(x)\}$ is stationary in $\mathcal{P}_\kappa X$. Since $\mathrm{cf}(\sup(X))=\kappa$, $\lim(X)\cap\lim(C_{\sup(X)})$ is unbounded in $\sup(X)$. Take $\beta\in\lim(X)\cap\lim(C_{\sup(X)})$

with $\beta > \xi$ and $\mathrm{cf}(\beta) < \kappa$. Note that $\{x \in \mathcal{P}_\kappa X : x \cap \beta \text{ is unbounded in } \beta\}$ contains a club. Since $\beta \in \lim(C_{\sup(X)})$, $C_{\sup(X)} \cap \beta = C_\beta$ holds. For each $x \in S_X$ such that $x \cap \beta$ is unbounded in β and $\sup(x) > \beta$, let $\beta_x = \min(x \setminus \beta)$.

Case 1. $\beta_x = \beta$. Then $C_{\beta_x} \cap \sup(x \cap \beta_x) = C_\beta = C_{\sup(X)} \cap \beta = C_{\sup(x)} \cap \beta = C_{\sup(x)} \cap \sup(x \cap \beta_x)$.

Case 2. $\beta_x > \beta$. Then $\sup(x \cap \beta_x) = \beta$ and $\beta = \sup(x \cap \beta) \in \lim(C_{\beta_x})$, hence $C_{\beta_x} \cap \beta = C_\beta = C_{\sup(X)} \cap \beta = C_{\sup(x)} \cap \beta = C_{\sup(x)} \cap \sup(x \cap \beta_x)$.

Hence for each $x \in S_X$ such that $x \cap \beta$ is unbounded in β and $\sup(x) > \beta$, we have $C_{\sup(x)} \cap \sup(x \cap \beta_x) = C_{\beta_x} \cap \sup(x \cap \beta_x)$. By applying Fodor's lemma to S_X, we can find $\beta_X \in X$ such that $\{x \in S_X : \beta_X = \beta_x\}$ is stationary. Thus $\{x \in S^* \cap \mathcal{P}_\kappa X : C_{\sup(x)} \cap \sup(x \cap \beta_X) = C_{\beta_X} \cap \sup(x \cap \beta_X)\}$ is stationary.

By applying Fodor's lemma to T', we have $\beta_* < \lambda$ such that $\{X \in T' : \beta_* = \beta_X\}$ is stationary. Then S_{n,β_*} is stationary and $\beta_* > \xi$. □[Claim]

Claim 4.5. *For each $\alpha \in A_0$ and $\beta \in A_1$ with $\alpha < \beta$, $C_\alpha = C_\beta \cap \alpha$ holds.*

Proof. Let $T^* = \{X \in T : S_{0,\alpha} \cap \mathcal{P}_\kappa X,\ S_{1,\beta} \cap \mathcal{P}_\kappa X \text{ are stationary in } \mathcal{P}_\kappa X\}$. Take $X \in T^*$. Since $D_X = \{x \in \mathcal{P}_\kappa X : C_{\sup(X)} \cap \sup(x) = C_{\sup(x)}\}$ contains a club in $\mathcal{P}_\kappa X$, $D_X \cap S_{0,\alpha}$ is stationary in $\mathcal{P}_\kappa X$. For $x \in C_X \cap S_{0,\alpha}$, $C_\alpha \cap \sup(x \cap \alpha) = C_{\sup(x)} \cap \sup(x \cap \alpha) = C_{\sup(X)} \cap \sup(x \cap \alpha)$ holds. Since $\{\sup(x \cap \alpha) : x \in C_X \cap S_{0,\alpha}\}$ is unbounded in $\sup(X \cap \alpha)$, we have $C_{\sup(X)} \cap \sup(X \cap \alpha) = C_\alpha \cap \sup(X \cap \alpha)$. Similarly, we have $C_\beta \cap \sup(X \cap \beta) = C_{\sup(X)} \cap \sup(X \cap \beta)$. Therefore we have $C_\alpha \cap \sup(X \cap \alpha) = C_\beta \cap \sup(X \cap \alpha)$.

Because $\{\sup(X \cap \alpha) : X \in T^*\}$ is unbounded in α, we have $C_\alpha = C_\beta \cap \alpha$. □[Claim]

Now, let $C = \{C_\beta : \beta \in A_0\}$. Since A_0 is unbounded, C is unbounded. Furthermore, $C_\alpha = C_\beta \cap \alpha$ for all $\alpha < \beta \in A$; For $\alpha, \beta \in A_0$ with $\alpha < \beta$, choose $\gamma \in A_1$ with $\beta < \gamma$. Then $C_\alpha = C_\gamma \cap \alpha$ and $C_\beta = C_\gamma \cap \alpha$. Thus $C_\alpha = C_\beta \cap \alpha$. Hence C forms a club in λ. Take $\alpha \in \lim(C)$. Then there exists $\beta \in A_0$ such that $C \cap \alpha = C_\beta \cap \alpha$. Since $\alpha \in \lim(C)$, we know $\alpha \in \lim(C_\beta)$ and $C_\alpha = C_\beta \cap \alpha = C \cap \alpha$. Thus $\forall \alpha \in \lim(C)\ (C \cap \alpha = C_\alpha)$, this is a contradiction. □

Baumgartner[1] showed that if a weakly compact cardinal κ is collapsed to ω_2 by Levy-collapse with countable conditions, then $\mathrm{RP}(\mathcal{P}_{\omega_1}\omega_2)$ holds, and it is known that in fact $\mathrm{RP}^2(\mathcal{P}_{\omega_1}\omega_2, \mathcal{P}_{\omega_1}\omega_2)$ holds in the generic extension. Conversely, Velicikovic [3]showed that if $\mathrm{RP}(\mathcal{P}_{\omega_1}\omega_2)$ holds, then ω_2 is weakly compact in L. Consequently, we have the following equiconsistency:

Corollary 4.6. *The following are equiconsistent:*

(1) *ZFC + "there exists a weakly compact cardinal".*
(2) *ZFC + "*$\mathrm{RP}(\mathcal{P}_{\omega_1}\omega_2)$ *holds".*
(3) *ZFC + "*$\mathrm{RP}^2(\mathcal{P}_{\omega_1}\omega_2, \mathcal{P}_{\omega_1}\omega_2)$ *holds".*
(4) *ZFC + "*$\mathrm{RP}^2(S_0^*, S_1^*)$ *holds for some stationary sets* $S_0^*, S_1^* \subseteq \mathcal{P}_{\omega_1}\omega_2$*".*

5. Proof of Theorem 1.8

Proposition 5.1. *Suppose* $\mathrm{RP}(S^*)$ *for some stationary* $S^* \subseteq \mathcal{P}_\kappa\lambda$*. Then every* κ*-c.c. forcing preserves* $\mathrm{RP}(S^*)$*.*

Proof. First note that every κ-c.c. forcing preserves the stationarity of S^*.

Let $\mathbb{P}$ be a poset which satisfies the κ-c.c. Let $\dot{S}$ be a $\mathbb{P}$-name such that $\Vdash$"$\dot{S} \subseteq S^*$ is stationary". It is enough to show that there are some $p \in \mathbb{P}$ and $X \subseteq \mathcal{P}_{\kappa^+}\lambda$ such that $p \Vdash$"$\dot{S} \cap \mathcal{P}_\kappa X$ is stationary in $\mathcal{P}_\kappa X$".

Let $S' = \{x \in S^* : \exists p \in \mathbb{P}\, (p \Vdash$"$x \in \dot{S}$"$)\}$. It is easy to check that S' is a stationary subset of S^*. By $\mathrm{RP}(S^*)$, there is $X \in \mathcal{P}_{\kappa^+}\lambda$ such that $|X| = \kappa \subseteq X$ and $S' \cap \mathcal{P}_\kappa X$ is stationary in $\mathcal{P}_\kappa X$. We see that $p \Vdash$"$\dot{S} \cap \mathcal{P}_\kappa X$ is stationary" for some $p \in \mathbb{P}$. Suppose to the contrary that $\Vdash$"$\dot{S} \cap \mathcal{P}_\kappa X$ is non-stationary". Since $|X| = \kappa$ and $\mathbb{P}$ satisfies the κ-c.c., we can find a club $C \subseteq \mathcal{P}_\kappa X$ such that $\Vdash$"$\dot{S} \cap C = \emptyset$". $S' \cap \mathcal{P}_\kappa X$ is stationary, hence there is $x \in S' \cap C$. Pick $p \in \mathbb{P}$ with $p \Vdash$"$x \in \dot{S}$". Then $p \Vdash$"$x \in \dot{S} \cap C$", this is a contradiction. □

Recall that PFA^{++} is the assertion that for every proper forcing notion $\mathbb{P}$, every dense subsets D_i $(i < \omega_1)$ of $\mathbb{P}$, and every $\mathbb{P}$-names $\dot{S}_i$ $(i < \omega_1)$ for stationary subsets of ω_1, there is a filter F on $\mathbb{P}$ such that:

(1) $D_i \cap F \neq \emptyset$ for every $i < \omega_1$.
(2) $S_i = \{\alpha < \omega_1 : \exists p \in F\, (p \Vdash_{\mathbb{P}}$"$\alpha \in \dot{S}_i$"$)\}$ is stationary in ω_1 for $i < \omega_1$.

Proposition 5.2. *Suppose* PFA^{++}*. Let* $\lambda \geq \omega_2$*. Then every c.c.c. forcing notion forces* $\mathrm{RP}^2(\mathcal{P}^V_{\omega_1}\lambda, \mathcal{P}^V_{\omega_1}\lambda)$*.*

Proof. Let $\mathbb{P}$ be a poset which satisfies the c.c.c. Let $\dot{S}_0$, $\dot{S}_1$ be $\mathbb{P}$-names so that $\Vdash$"$\dot{S}_0, \dot{S}_1 \subseteq \mathcal{P}^V_{\omega_1}\lambda$ are stationary". We will find $p \in \mathbb{P}$ and $X \in \mathcal{P}_{\omega_2}\lambda$ such that $p \Vdash$"$\dot{S}_0 \cap \mathcal{P}_{\omega_1}X$, $\dot{S}_1 \cap \mathcal{P}_{\omega_1}X$ are stationary".

Let $\dot{\mathbb{Q}}$ be a $\mathbb{P}$-name for a σ-closed poset which adds a bijection from ω_1 to λ. We know that $\Vdash_{\mathbb{P}*\dot{\mathbb{Q}}}$"$\dot{S}_0$, $\dot{S}_1$ remain stationary". Fix a $\mathbb{P}*\dot{\mathbb{Q}}$-name $\dot{\pi}$ for a bijection from ω_1 to λ. Let $\dot{E}_0$, $\dot{E}_1$ be $\mathbb{P}*\dot{\mathbb{Q}}$-names such that $\Vdash_{\mathbb{P}*\dot{\mathbb{Q}}}$"$\dot{E}_i = \{\alpha < \omega_1 : \dot{\pi}$"$\alpha \in \dot{S}_i, \dot{\pi}$"$\alpha \cap \omega_1 = \alpha\}$" for $i = 0, 1$. We know $\Vdash_{\mathbb{P}*\dot{\mathbb{Q}}}$"$\dot{E}_i$ is stationary in ω_1".

Now fix a sufficiently large regular cardinal θ and take $M \prec H_\theta$ such that $|M| = \omega_1 \subseteq M$ and M contains all relevant objects.

$\mathbb{P} * \dot{\mathbb{Q}}$ is proper, hence we can apply PFA^{++} to $\mathbb{P} * \dot{\mathbb{Q}}$ and $\dot{E}_i$. By PFA^{++} we can find a filter F on $\mathbb{P} * \dot{\mathbb{Q}}$ such that:

(1) $F \cap D \neq \emptyset$ for all dense $D \in M$ in $\mathbb{P} * \dot{\mathbb{Q}}$.

(2) $E_i = \{\alpha < \omega_1 : \exists p \in F\, (p \Vdash_{\mathbb{P}*\dot{\mathbb{Q}}} \text{“}\alpha \in \dot{E}_i\text{”})\}$ is stationary in ω_1 for $i = 0, 1$.

Let $X = \{\beta < \lambda : \exists p \in F \exists \alpha < \omega_1\, (p \Vdash_{\mathbb{P}*\dot{\mathbb{Q}}} \text{“}\dot{\pi}(\alpha) = \beta\text{”})\}$. We can check that $|X| = \omega_1 \subseteq X$.

Since $\dot{S}_0$, $\dot{S}_1$ are names for subsets of $\mathcal{P}^V_{\omega_1}\lambda$, for each $\alpha \in E_i$, we can find $x \in \mathcal{P}_{\omega_1}\lambda$ and $p \in F$ such that $x \cap \omega_1 = \alpha$ and $p \Vdash_{\mathbb{P}*\dot{\mathbb{Q}}} \text{“}\dot{\pi}\text{“}\alpha = x\text{”}$. Moreover it is easy to see that $x \in \mathcal{P}_{\omega_1}X$.

For $i < 2$ and $\alpha \in E_i$, take $x_{i,\alpha} \in \mathcal{P}_{\omega_1}X$ such that there is $p \in F$ with $p \Vdash_{\mathbb{P}*\dot{\mathbb{Q}}} \text{“}\dot{\pi}\text{“}\alpha = x_{i,\alpha}\text{”}$. Let $S_i = \{x_{i,\alpha} : \alpha \in E_i\}$. The following are easy to check for $i < 2$:

(1) $x_{i,\alpha} \subseteq x_{i,\beta}$ holds for $\alpha, \beta \in E_i$ with $\alpha < \beta$.

(2) If $\alpha \in \lim(E_i) \cap E_i$, then $x_{i,\alpha} = \bigcup_{\beta \in E_i \cap \alpha} x_{i,\beta}$.

(3) $\bigcup S_i = X$.

Furthermore, since $E_i = \{x_{i,\alpha} \cap \omega_1 : \alpha \in E_i\}$ is stationary in ω_1, we can check that each S_i is stationary in $\mathcal{P}_{\omega_1}X$.

Now we see that $p \Vdash_{\mathbb{P}} \text{“}\dot{S}_0 \cap \mathcal{P}_{\omega_1}X$, $\dot{S}_1 \cap \mathcal{P}_{\omega_1}X$ are stationary” for some $p \in \mathbb{P}$. Suppose otherwise. Since $\mathbb{P}$ satisfies the c.c.c. and $|X| = \omega_1$, we an find a club C in $\mathcal{P}_{\omega_1}X$ such that $\Vdash_{\mathbb{P}} \text{“}C \cap \dot{S}_0 = \emptyset$ or $C \cap \dot{S}_1 = \emptyset\text{”}$.

Since S_0 and S_1 are stationary in $\mathcal{P}_{\omega_1}X$, we can find $x_0 \in S_0 \cap C$ and $x_1 \in S_1 \cap C$. Then there is $q \in F$ such that $q \Vdash_{\mathbb{P}*\dot{\mathbb{Q}}} \text{“}x_0 \in \dot{S}_0$ and $x_1 \in \dot{S}_1\text{”}$. Thus $q \Vdash_{\mathbb{P}*\dot{\mathbb{Q}}} \text{“}C \cap \dot{S}_0 \neq \emptyset$ and $C \cap \dot{S}_1 \neq \emptyset\text{”}$, this is a contradiction. □

References

[1] J. E. Baumgartner, *A new class of order types.* Ann. Math. Logic 9 (1976), no. 3, pp 187–222.

[2] H. Sakai, *Partial stationary reflection in $\mathcal{P}_{\omega_1}\omega_2$.* RIMS kokyuroku, Vol. 1595, pp 47–62.

[3] B. Velickovic, *Forcing axioms and stationary sets.* Adv. Math. 94 (1992), no. 2, pp 256–284.

(T. Usuba) Institute for Advanced Research, Nagoya University, Furo-cho, Chikusa-ku, Nagoya, 464-8601, JAPAN

E-mail address: usuba@math.cm.is.nagoya-u.ac.jp