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Finite Time Extinction of Historical Superprocess Related to Stable Measure

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安定測度に関連するヒストリカル超過程の有限時間消滅性

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We consider a class of historical superprocesses in the Dynkin sense, which is closely related to an another class of superprocesses (i.e., measure-valued branching Markov processes) associated with stable random measure. Our main concern has been the extinction property of superprocesses, and in this article we study, in particular, finite time extinction of the historical superprocesses associated with stable random measure. Since the key result is about the compact support property of superprocesses in question, our emphasis is especially placed on the compact support equivalent statement and the compact support property for those superprocesses related to stable random measure.

安定ランダム測度に付随して定まる超過程（すなわち, 測度値分枝マルコフ過程）に密接に関連するデインキンの意味でのヒストリカル超過程のあるクラスについて考察する. われわれの最近の最大の関心事は超過程の消滅性についてである. 特にこの報告集では安定ランダム測度に付随するヒストリカル超過程の有限時間消滅性について研究する. これを研究する上でキーとなる前段階の結果は問題としている超過程のコンパクト・サポート性に関するものであるので, われわれが取り扱っている安定ランダム測度に関連する超過程に対する, コンパクト・サポートの同値命題およびコンパクト・サポート性にとくに焦点を当てて報告する.

1. Historical Superprocess

The superprocesses with branching rate functional form a class of measure-valued branching Markov processes. We write $\langle \mu, f \rangle = \int f d\mu$ for measure μ. For simplicity, $M_F = M_F(\mathbb{R}^d)$ is the space of finite measures on $\mathbb{R}^d$. Define a second order elliptic differential operator $L = \frac{1}{2}\nabla \cdot a\nabla + b \cdot \nabla$, and $a = (a_{ij})$ is positive definite and assume that $a_{ij}, b_i \in C^{1,\varepsilon} = C^{1,\varepsilon}(\mathbb{R}^d)$. Here the space $C^{1,\varepsilon}$ is the totality of all Hölder continuous functions with index ε $(0 < \varepsilon \leqslant 1)$, having continuous first order derivatives. $\Xi = \{\xi, \Pi_{s,a}, s \geq 0, a \in \mathbb{R}^d\}$ is a corresponding L-diffusion. The transition probability of the L-diffusion is allowed to possess its density, which is denoted by $p(t,x,y)$. Moreover, CAF stands for continuous additive functional. When we write C_b as the set of bounded continuous functions on $\mathbb{R}^d$, then C_b^+ is the set of positive members in C_b. The symbol $\mathbb{C} = C(\mathbb{R}_+, \mathbb{R}^d)$ denotes the space of continuous paths on $\mathbb{R}^d$ with uniform convergence topology. To each $w \in \mathbb{C}$ and $t > 0$, we write $w^t \in \mathbb{C}$ as the stopped path of w. We denote by $\mathbb{C}^t$ the totality of all these paths stopped at time t. To every $w \in \mathbb{C}$ we associate the corresponding stopped path trajectory $\tilde{w}$ defined by $\tilde{w}_t = w^t$ for $t \geq 0$. Let K be a positive CAF of ξ. $\tilde{\mathbb{X}} = \{\tilde{X}, \tilde{\mathbb{P}}_{s,\mu}, s \geq 0,\ \mu \in M_F(\mathbb{C}^s)\}$ is said to be a Dynkin's *historical superprocess* [13,4] if $\tilde{X} = \{\tilde{X}_t\}$ is a time-inhomogeneous Markov process with state $\tilde{X}_t \in M_F(\mathbb{C}^t)$, $t \geq s$, with transition Laplace functional

$$\tilde{\mathbb{P}}_{s,\mu} e^{-\langle \tilde{X}_t, \varphi \rangle} = e^{-\langle \mu, v(s,t) \rangle}, \quad 0 \leqslant s \leqslant t, \quad \mu \in M_F(\mathbb{C}^s), \quad \varphi \in C_b^+(\mathbb{C}), \tag{1}$$

where the function v is uniquely determined by the log-Laplace type equation

$$\tilde{\Pi}_{s,w_s} \varphi(\tilde{\xi}_t) = v(s, w_s) + \tilde{\Pi}_{s,w_s} \int_s^t v^2(r, \tilde{\xi}_r) K(dr), \quad 0 \leqslant s \leqslant t, \quad w_s \in \mathbb{C}^s. \tag{2}$$

2. Superprocess Related to Random Measure

Suppose that $p > d$, and let $\phi_p(x) = (1+|x|^2)^{-p/2}$ be the reference function. $C = C(\mathbb{R}^d)$ denotes the space of continuous functions on $\mathbb{R}^d$, and define $C_p = \{f \in C : |f| \leqslant C_f \cdot \phi_p, \exists C_f > 0\}$. We denote by $M_p = M_p(\mathbb{R}^d)$ the set of non-negative measures μ on $\mathbb{R}^d$, satisfying $\langle \mu, \phi_p \rangle = \int \phi_p(x)\mu(dx) < \infty$. It is called the space of p-tempered measures. When $\{\xi_t, \Pi_{s,a}\}$ is an L-diffusion, then we define the continuous additive functional K_η of ξ by $K_\eta = \langle \eta, \delta_x(\xi_r) \rangle dr$ for $\eta \in M_p$. For some $q > 0$, we write $K \in \mathbb{K}^q$ [14] if a continuous additive functional K is in the Dynkin class with index q. Then $X^\eta = \{X_t^\eta; t \geq 0\}$ is said to be a measure-valued diffusion with branching rate functional K_η if for the initial measure $\mu \in M_F$, X^η satisfies the Laplace functional of the form $\mathbb{P}_{s,\mu}^\eta e^{-\langle X_t^\eta, \varphi \rangle} = e^{-\langle \mu, v(s) \rangle}$, ($\varphi \in C_b^+$), where the function $v \geq 0$ is uniquely determined by $\Pi_{s,a}\varphi(\xi_t) = v(s,a) + \Pi_{s,a} \int_s^t v^2(r, \xi_r) K_\eta(dr)$, ($0 < s \leqslant t, a \in \mathbb{R}^d$). Assume that $d = 1$ and $0 < \nu < 1$. Let $\lambda \equiv \lambda(dx)$ be the Lebesgue measure on $\mathbb{R}$, and let $(\gamma, \mathbb{P})$ be the stable random measure on $\mathbb{R}$ with Laplace functional

$$-\log \mathbb{E} \exp\left\{-\int_{\mathbb{R}} \varphi(x)\gamma(dx)\right\} = \int \varphi^\nu(x)\lambda(dx), \quad \varphi \in C_b^+. \tag{3}$$

Note that $\mathbb{P}$-a.a ω realization, $\gamma(\omega)$ lies in M_p as far as the condition $p > \nu^{-1}$ is satisfied. We consider a positive CAF K_γ of ξ for $\mathbb{P}$-a.a. ω. So that, thanks to Dynkin's general formalism for superprocess with branching rate functional, there exists an (L, K_γ, μ)-superprocess X^γ when we adopt a p-tempered measure γ for CAF K_η instead of η, as far as $K_\gamma = K_\gamma(\omega; dr)$ may lie in the class $\mathbb{K}^q$.

THEOREM 1. *Let $K_\gamma \in \mathbb{K}^q$. For $\mu \in M_F$ with compact support, there exists an (L, K_γ, μ)-superprocess $\{X^\gamma, \mathbb{P}_{s,\mu}^\gamma, s \geq 0\}$ with branching rate functional K_γ.*

Note that when $d = 1, a = 1$ and $b = 0$, X^γ is called a stable catalytic SBM, and that this was initially constructed and investigated by Dawson-Fleischmann-Mueller [2].

Proof. Let $\ell \in \mathbb{N}$ and $I(k) = (-k, k) \subset \mathbb{R}$, and define $E_\ell = \cup_{n=1}^\ell \{n\} \times I(n)$. When τ_n is the first hitting time of ξ starting at x to the boundary $\pm n$, then the Markov process Y_t^ξ in E_ℓ is defined by $Y_t^\xi = (\{n\}, \xi_t)$ for $0 \leqslant t \leqslant \tau_n$, and it dies out at time $t = \tau_\ell$. While, the new measure $\gamma_\ell(\{n\} \times (a,b))$ is given by $\gamma(I(n) \cap (a,b))$ for $n \leqslant \ell$, and forms a random measure on E_ℓ. By using this measure, we define K_{γ_ℓ} as

$$K_{\gamma_\ell}(Y^\xi, t) = \int \Lambda_{t,x}(Y^\xi)\gamma_\ell(dx) \tag{4}$$

with the local time $\Lambda_{t,x}(Y^{\xi})$ for the process Y_t^{ξ}. Then se shall write $\hat{X}_t^{\ell}$ its resulting $(L, K_{\gamma_\ell}, \mu)$-superprocess. For an arbitrarily chosen $\mu \in M_F(\mathbb{R})$, the initial measure $\hat{X}_0^{\ell}$ for $\hat{X}_t^{\ell}$ is provided by

$$\hat{X}_0^{\ell}(\{n\} \times B) = \mu(B \cap \{[n-1, n) \cup (-n, 1-n]\}) \qquad \text{for} \quad n \geq 1. \tag{5}$$

Notice that the law equivalence $\mathcal{L}(\hat{X}^m \upharpoonright E_\ell) = \mathcal{L}(\hat{X}^\ell)$ holds for any pair (ℓ, m) such that $m > \ell$. This means that the sequence $\{P_{X_0}^{L,\ell}\}_\ell$ of laws for $\hat{X}^\ell$ becomes a consistent family of probability measures, so that its projective limit may generate the probability law of an $\mathcal{M}(E_\infty)$-valued process $\hat{X}^\infty$. This allows us to possess an increasing sequence $\{Z_t^\ell(B)\}$ of $M_F(I(\ell))$-valued processes. The log-Laplace function

$$u^\ell(t,x) = -\log \mathbb{E}_{\delta_x}[\exp\{-\langle Z_t^\ell, \varphi\rangle\}] = -\log \mathbb{E}_{\delta_x} \prod_{n=1}^{\ell} e^{-\langle \hat{X}_t^\infty(\{n\}\times(\cdot)), \varphi\rangle} \tag{6}$$

for the process Z_t^ℓ satisfies $\Pi_{0,x}^\ell[\varphi(Y_t^\xi)] = u^\ell(t,x) + \Pi_{0,x}\int_0^t u^2(r, Y_{t-r}^\xi) K_{\gamma_\ell}(dr)$, where $(Y_t^\xi, \Pi_{s,a}^\ell)$ is a L-diffusion which just corresponds to the L-diffusion equation with Dirichlet boundary conditions on $I(\ell)$. Moreover, we observe

$$\mathbb{E}_\mu[Z_t^\ell(B)] = \langle \mu, 1_B \cdot \Pi_{0,x}^\ell 1(Y_t^\xi)\rangle = \int_{I(\ell)} \int p_\ell(t,x,y) 1_B(x) dy \mu(dx). \tag{7}$$

On this account, the limit procedure $X_t(dx) := \lim_{\ell\to\infty} Z_t^\ell(dx)$ defines the $M_F(\mathbb{R})$-valued process with initial measure μ. $\square$

3. Historical Superprocess Related to Random Measure

We shall show below the existence of the corresponding historical superprocess in the Dynkin sense. Let K_γ be a positive CAF of ξ lying in the Dynkin class $\mathbb{K}^q$. The historical superprocess $\tilde{\mathbb{X}}^\gamma = \{\tilde{X}^\gamma, \tilde{\mathbb{P}}_{s,\mu}^\gamma,\ s \geq 0,\ \mu \in M_F(\mathbb{C}^s)\}$ in the Dynkin sense is a time-inhomogeneous Markov process with state $\tilde{X}_t^\gamma \in M_F(\mathbb{C}^t)$, $t \geq s$, with transition Laplace functional

$$\tilde{\mathbb{P}}_{s,\mu}^\gamma \exp\left\{-\langle \tilde{X}_t^\gamma, \varphi\rangle\right\} = e^{-\langle \mu, v(s,t)\rangle}, \quad 0 \leqslant s \leqslant t, \mu \in M_F(\mathbb{C}^s), \text{ and } \varphi \in C_b^+(\mathbb{C}), \tag{8}$$

where the function v is uniquely determined by the log-Laplace type equation

$$\tilde{\Pi}_{s,w_s}\varphi(\tilde{\xi}_t) = v(s, w_s) + \tilde{\Pi}_{s,w_s}\int_s^t v^2(r, \tilde{\xi}_r)\tilde{K}_\gamma(\omega; dr), \quad 0 \leqslant s \leqslant t, \quad w_s \in \mathbb{C}^s. \tag{9}$$

The above process can be reformulated as follows. We shall adopt some notation and terminology from [12]. Let $(E_t, \mathcal{B}_t)$ be a measurable space that describes the state space of the underlying process ξ at time t (which can usually be imbedded isomorphically into a compact metrizable space $\mathcal{C}$), and $\hat{E}$ be the global state space given by the set of pairs $t \in \mathbb{R}_+$ and $x \in E_t$. The symbol $\mathcal{B}(\hat{E})$ denotes the σ-algebra in $\hat{E}$, generated by functions $f : \hat{E} \to \mathbb{R}$. Note that $\hat{E}(I) = \{(r,x) : r \in I, x \in E_r\} \in \mathcal{B}(\hat{E})$ for every interval I. The sample space W is a set of paths (or trajectories) $\xi_t(w) = w_t$ for each $w \in W$. Furthermore, $\mathcal{F}(I)$ is the σ-algebra generated by $\xi_t(w)$ for $t \in I$. Let $w(I)$ denote the restriction of $w \in W$ to I, and $W(I)$ be the image of W under this mapping. Moreover, $\tilde{\Xi} = (\xi_{\leqslant t}, \mathcal{F}_{\tilde{\Xi}}(I), \tilde{\Pi}_{r,w(\leqslant r)}) = (\xi(-\infty,t], \mathcal{F}_{\tilde{\Xi}}(I), \tilde{\Pi}_{r,w(-\infty,r]})$ is the historical process for $\xi = (\xi_t, \mathcal{F}(I), \Pi_{r,a})$. Under those circumstances, we can get the historical superprocess in question.

THOREM 2. *Let $\tilde{\Xi}$ be a historical process, $\tilde{K}_\gamma = \tilde{K}_\gamma(\omega)$ be its CAF associated to stable random measure γ with properties:*
(a) *For every $q > 0$, $r < t$ and $x \in E_r$, $\tilde{\Pi}_{r,x(\leqslant r)} e^{q\tilde{K}_\gamma(\omega;(r,t))} < \infty$.* (b) *For every $t_0 < t$, there exists a positive constant C such that $\tilde{\Pi}_{r,x(\leqslant r)} \tilde{K}_\gamma(\omega;(r,t)) \leqslant C$ holds for $r \in [t_0,t)$, $x \in E_r$. Put $\psi^t(x,z) = b^t(x)z^2 = 1 \times z^2$. Then there exists a Markov process $M^\gamma = (M_t^\gamma, \mathcal{G}(I), P_{r,\mu}^\gamma)$ on the space $\mathcal{M}_{\leqslant t} = M_F(\mathbb{C}^t)$ of all finite measures on $(W, \mathcal{F}_{\leqslant t}^*) = (W, \mathcal{F}^*(-\infty,t])$ with the universal completion $\mathcal{F}_{\leqslant t}^*$ of σ-algebra, such that for every $t \in \mathbb{R}_+$ and $\varphi \in \mathcal{F}_{\leqslant t}^*$,*

$$P_{r,\mu}^\gamma \exp\{-\langle M_t^\gamma, \varphi\rangle\} = e^{-\langle \mu, v(r,\cdot)\rangle}, \quad 0 \leqslant r \leqslant t, \quad \mu \in \mathcal{M}_{\leqslant r}, \tag{10}$$

where $v^r(w_{\leqslant r}) = v(r, w(-\infty,r])$ is a progressive function determined uniquely by the equations

$$\begin{aligned} &v^r(x_{\leqslant r}) + \tilde{\Pi}_{r,x(\leqslant r)} \int_r^t \psi^s(\xi_{\leqslant s}, v^s(\xi_{\leqslant s})) \tilde{K}_\gamma(\omega; ds) = \tilde{\Pi}_{r,x(\leqslant r)} \varphi(\xi_{\leqslant t}) \quad \text{for} \quad r \leqslant t \\ &v^r(x_{\leqslant r}) = 0 \quad \text{for} \quad r > t. \end{aligned} \tag{11}$$

Proof. As stated in Theorem 2, set $\psi = \psi^t(x,z)$ as special branching mechanism. The historical superprocess $M^\gamma = (M_t^\gamma, \mathcal{G}(I), P_{r,\mu}^\gamma)$ with parameters $(\tilde{\Xi}, \tilde{K}_\gamma, \psi)$ can be obtained from the superprocess X^γ with the almost same parameters (Ξ, K_γ, ψ) by the direct construction. First of all we define the finite-dimensional distributions of the random measure M_t^γ as $\mu_{t_1 t_2 \cdots t_n}(A_1 \times A_2 \times \cdots \times A_n) = M_t^\gamma(\{w(t_1) \in A_1, w(t_2) \in A_2, \ldots, w(t_n) \in A_n\})$ for time partition $\Delta = \{t_k\}$ with $t_1 < t_2 < \cdots < t_n \leqslant t$ and $A_1 \in \mathcal{B}_{t_1}$, $A_2 \in \mathcal{B}_{t_2}$, $\ldots$, $A_n \in \mathcal{B}_{t_n}$. Actually, this $\mu_{t_1 \cdots t_n}$ determines uniquely the probability distribution on $\mathcal{B}(E_{t_1} \times \cdots \times E_{t_n})$. To this end we replace $X_{t_1}^\gamma$ by its restriction $\hat{X}_{t_1}^\gamma$ $(= X_{t_1}^\gamma \restriction A_1)$ to A_1 and run the superprocess during the time interval $[t_1, t_2]$ starting from $\hat{X}_{t_1}^\gamma$. Moreover we can proceed analogously until getting a $Z \in \mathcal{M}_t$ and then take $Z(E_t)$ as the value for $Q_{\pi_\eta/\beta}^\beta \exp\{-\langle \beta Y_t, f\rangle\} = e^{-\langle \eta, v^t(\beta)\rangle}$, where π_μ is the Poisson random measure on $(\hat{E}, \mathcal{B}(\hat{E}))$ with intensity μ, $\langle \eta, v\rangle = \int_{\hat{E}} v(r,x)\eta(dr,dx)$, $\beta > 0$, and Y_t is a counting measure. Then we construct a measure M_t^γ on $\mathcal{M}_{\leqslant t}$ by applying the Kolmogorov extension theorem to the family $\{\mu_{t_1 \cdots t_n}\}$. Indeed, if $\{\mu_{t_1 \cdots t_n}\}$ satisfies the consistency condition:

$$\begin{aligned} &\mu_{t_1 \cdots t_{k-1} t_{k+1} \cdots t_n}(A_1 \times \cdots \times A_{k-1} \times \check{A}_k \times A_{k+1} \times \cdots \times A_n) \\ &\qquad = \mu_{t_1 \cdots t_{k-1} t_k t_{k+1} \cdots t_n}(A_1 \times \cdots \times A_{k-1} \times E_{t_k} \times A_{k+1} \times \cdots \times A_n) \end{aligned} \tag{12}$$

for $k = 1, 2, \ldots, n$ and $A_k \in \mathcal{B}_{t_k}$ $(k = 1, 2, \ldots, n)$, where the symbol $\vee$ means exclusion of the number or item crowned with $\vee$ from the set $N = \{1, 2, \ldots, n\}$, then the Kolmogorov extension theorem guarantees that there exists a unique probability measure P on $(\hat{\Omega}, \mathcal{B}(\hat{\Omega}))$ such that the finite-dimensional distribution of $M_t^\gamma \in \mathcal{M}_{\leqslant t}$ is equal to $\{\mu_{t_1 \cdots t_n}\}$. Here $\hat{\Omega}$ is given by $\hat{\Omega} = (\mathcal{M}_{\leqslant t})^{[0,\infty)} = \{\omega | \omega(\cdot) : [0,\infty) \to \mathcal{M}_{\leqslant t}\}$. As a matter of fact, the historical superprocess can be obtained from a branching particle system by the limit procedure applied to the special process $\mathcal{Y} = \{\mathcal{Y}_t\}$. In fact, as a function of t, $\mathcal{Y}_t$ is a measure-valued process in functional spaces $W_{\leqslant t} = W(-\infty, t]$, called historical path space. Moreover, note that the complete picture of a branching particle system is given by the random tree composed of the paths of all particles. The construction of $\mathcal{Y}_t$ goes almost similarly as in [13], hence omitted. In this way, as a function of t, an integer-valued measure $\mathcal{Y}_t$ on $W_{\leqslant t}$ is constructed as a measure-valued process in functional space $W_{\leqslant t}$. Lastly some comments on progressivity of transition probablity should be mentinoed. Indeed, a natural question is to ask whether that kind of progressivity for the underlying Markov process $\Xi = \{\xi\}$ implies an anlogous condition for the historical process $\tilde{\Xi}$. Here the condition in question is as follows: "The transition probabilities are progressive, i.e. the function $f^t(x) = 1_{\{t<u\}}\Pi_{t,x}(\xi_u \in B)$ is progressive for every $u \in \mathbb{R}_+$ and $B \in \mathcal{B}_u$." Note that the above condition is satisfied even for the historical process $\tilde{\Xi}$ as far as it may be valid for the underlying process ξ. □

4. Compact Support Property

Let $\text{supp}(\mu)$ be the closed support of a measure $\mu \in M_F(\mathbb{R})$, and let $\text{Gsupp}(X)$ be the global support of a measure-valued process $X_t(dx)$, which is defined as the closure of the union of $\text{supp}(X_t)$ for all $t \geq 0$. We consider the following boundary value problem (BVP)

$$Lv(x) = v^2(x)\frac{\gamma(dx)}{dx} \qquad \text{for} \quad x \in \overset{\circ}{I}_0 = (a, b) \tag{13}$$

The solution is a continuous convex function on the interval $I_0 = [a, b]$, and for every $x, h \in \mathbb{R}$ satisfying $a \leqslant h \leqslant x + h \leqslant b$, we have

$$v(x+h) = v(h) + v'(h+0)x + 2\int_h^{x+h} ds \int_h^s v^2(t)\gamma(dt). \tag{14}$$

THEOREM 3. *Assume that* $\text{supp}(X_0) \subset I_0 \subset I(\ell)$.
(a) *There exist sequences* $\{\alpha_n\}_n$, $\{\beta_n\}_n$ *such that* $\alpha_n > 0$, $\alpha_n \nearrow \infty$, $\beta_n > 0$, *and* $\beta_n \nearrow \infty$, *satisfying that for each* $n \in \mathbb{N}$, *the BVP* (13) *has a unique solution* $v(x, \alpha, \beta)$ *with* $\alpha = \alpha_n$ *and* $\beta = \beta_n$.
(b) *For any sequence of functions* $u(x, \alpha_n, \beta_n)$ *satisfying the conditions of* (a), *we have*

$$\begin{aligned} \mathbb{P}^{\gamma}_{X_0}\{\,\text{Gsupp}(X) \subset I_0\,\} &= \mathbb{P}^{\gamma}_{X_0}\{\,\text{supp}(X_t) \cap I_0^c = \emptyset, \forall t \geq 0\,\} \\ &= \lim_{n\to\infty} \exp\left\{-\int_a^b u(x, \alpha_n, \beta_n) X_0(dx)\right\}. \end{aligned} \tag{15}$$

Proof. Let let $\psi_n \in C^+$ such that $\psi_n \nearrow 1_{T(\ell)} \cdot 1_{I_0^c}$. According to [17], the occupation time processes $\hat{Z}_t = \int_0^t Z_s^\ell ds$ and $\hat{X}_t = \int_0^t X_s ds$ are well defined. Let us take $X_0 \in M_F(\mathbb{R})$ (resp. $\psi \in C^+(\mathbb{R})$) satisfying $\text{supp}(X_0) \subset I(\ell)$ (resp. $\text{supp}(\psi) \subset I(\ell)$) respectively. Let $\theta > 0$. Here we need the following lemma.

LEMMA 4. *The occupation time process $\hat{Z}^\ell_t$ satisfies the Laplace functional*

$$\mathbb{P}^\gamma_{X_0}[\exp\{-\theta\langle \hat{Z}^\ell_t, \psi\rangle\}] = \exp\{-\langle X_0, v^\ell(t, \theta\psi)\rangle\} \tag{16}$$

where the function $v^\ell(t, x, \theta\psi)$ is a solution of the following log-Laplace equation

$$\theta\Pi^\ell_{0,x}\int_0^t \psi(Y^\ell_{t-s})ds = v(t,x) + \Pi^\ell_{0,x}\int_0^t v^2(s, Y^\ell_{t-s})K^{\gamma_\ell}(ds), \quad \text{for} \quad x \in I(\ell). \tag{17}$$

Proof of Lemma 4. It goes almost similarly as in the proof of Theorem 3.1 in [16], hence omitted. □

Remark 5. (a) Note that $u(t,x) = 0$ holds for $x \in I(\ell)^c$.
(b) the solution $v^\ell(t, x, \theta\psi)$ satisfies the following estimate

$$v^\ell(t, x, \theta\psi) \leqslant \sup_{t,x}\Pi^\ell_{0,x}\int_0^t \theta\cdot\psi(Y^\ell_{t-s})ds < \infty.$$

We may employ Lemma 4 together with the passage to limit $t \to \infty$, to derive

$$\mathbb{P}^\gamma_{X_0}[\exp\left\{-\theta\int_0^\infty \hat{Z}^\ell_s(I^c_0)ds\right\}] = \exp\{-\langle X_0, \lim_{t\to\infty}(\lim_{n\to\infty} v^\ell(t, x, \theta\psi_n))\rangle\}. \tag{18}$$

For simplicity, we put $\lim_{n\to\infty} v^\ell(t, x, \theta\psi_n) = \Phi(t, x, \theta)$ and $\lim_{t\to\infty}\Phi(t, x, \theta) = \Psi(x, \theta)$.

LEMMA 6. *Let $\phi \in C(\mathbb{R})$ having* $\mathrm{supp}(\phi) \subset I(\ell)$, *and take $h > 0$ such that $0 < h \ll 1$. Then we have uniformly in h*

$$\lim_{t\to\infty}\frac{1}{h}\left\{\int \Phi(t+h, x, \theta)\phi(x)dx - \int \Phi(t, x, \theta)\phi(x)dx\right\} = 0. \tag{19}$$

Proof of Lemma 6. We have only to show it for positive $\phi \geq 0$, because of the monotone property of $\Phi(t, x, \theta)$ in t. When we take the above-mentioned property into consideration, then Lemma 4 yields to

$$\begin{aligned}
0 &\leqslant \lim_{t\to\infty}\frac{1}{h}\left\{\int \Phi(t+h, x, \theta)\phi(x)dx - \int \Phi(t, x\theta)\phi(x)dx\right\} \\
&\leqslant \lim_{t\to\infty}\frac{1}{h}\left\{\int_t^{t+h}\int \Pi^\ell_{0,x}\theta 1_{I^c_0}(Y^\ell_s)\phi(x)ds - \Pi^\ell_{0,x}\int_t^{t+h}\Phi^2(\theta,, Y^\ell_s, \theta)K^{\gamma_\ell}(ds)\right. \\
&+ \left.\int \Pi^\ell_{0,x}\int_0^t\{\Phi^2(t-s, Y^\ell_s, \theta) - \Phi^2(t+h-s, Y^\ell_s; \theta)\}K^{\gamma_\ell}(ds)\phi(x)dx\right\} \\
&\leqslant \lim_{t\to\infty} 2\ell\theta\cdot\sup_x\left\{\phi(x)\int p_\ell(t, x, y)dy\right\} = 0.
\end{aligned} \tag{20}$$

□

LEMMA 7. *Let $\{T^\ell_t; t \geq 0\}$ be the semigroup of the killed L-diffusion process. Then the following identity is valid, namely, for $\phi \in C^2$,*

$$\begin{aligned}
&\lim_{t\to\infty}\frac{1}{h}\int\{\Phi(t+h, x, \theta) - \Phi(t, x, \theta)\}\phi(x)dx \\
&= \int \frac{1}{h}(T^\ell_h - I)\phi(x)\cdot\Psi(x, \theta)dx \\
&+ \frac{1}{h}\left\{\int_0^h\int \Pi^\ell_{0,x}\theta 1_{I^c_0}(Y^\xi_{h-s})\cdot\phi(x)dxds - \int\int_0^h \Pi^\ell_{0,x}\Psi^2(\theta, Y^\xi_{h-s})K^{\gamma_\ell}(ds)\phi(x)dx\right\}.
\end{aligned} \tag{21}$$

Proof of Lemma 7. It is due to a simple computation. In fact, the result yields immediately from the expression (17) and the monotone convergence theorem. □

Choose $\phi \in C^2$ with $\mathrm{supp}(\phi) \subset I(\ell)$, and by the passage to limit $h \to 0$ in (21), we can derive

$$\langle \Psi(\cdot,\theta), L^*\phi\rangle + \langle \theta 1_{I_0^c}, \phi\rangle = \langle \gamma_\ell, \Psi^2(\cdot,\theta)\phi\rangle. \tag{22}$$

Recall the distribution theory [21]. Consider the second derivative of a distribution Φ on $\mathbb{R}$, and if the second derivative Φ'' in the distribution sense is a locally finite measure, (it does not matter whether it is a signed or nonnegative measure, though), then the distribution Φ is a continuous function of bounded variation on every finite interval. Moreover, if its second derivative Φ'' is a nonnegative measure, then it is a continuous and convex function and its first derivative Φ' exists in the usual sense except possibly at a countable set of points, and it is an increasing function having left and right limits at every point. On this account, thanks to Schwartz' argument, we can deduce at once from (22) that the second distribution derivative of Ψ is a possibly signed measure and also that the left and right limits of the first derivative v' of $v(x) = \Psi(x,\theta)$ satisfy

$$\frac{dv}{dx}(x \pm 0) = 2\int_{x_0}^{x\pm 0} v^2(y)\gamma_\ell(dy) - 2\theta \int_{x_0}^{x\pm 0} 1_{I_0^c}(y)dy + (\text{a constant}) \tag{23}$$

as far as $x \in I(\ell)$. Integration of (23) again leads to (14). Applications of Chebyshev's inequality and the Borel-Cantelli lemma verifies

$$\mathbb{P}^\gamma_{\delta_a}\left\{\int_0^t Z_s^\ell((a-1,a))ds > 0\right\} = 1 \quad \text{for any} \quad t > 0, \tag{24}$$

so that we obtain $\lim_{\theta\to\infty}\Phi(t,a,\theta) = \lim_{\theta\to\infty}\Phi(t,b,\theta) = \infty$. Therefore it follows from the above immediately that $\lim_{\theta\to\infty}\Psi(a,\theta) = \lim_{\theta\to\infty}\Psi(b,\theta) = \infty$. On the other hand, note that the map $t \mapsto Z_t^\ell$ is right continuous. In addition to that, it is known from [1] that the map $\mu \mapsto \mathrm{supp}(\mu)$ is lower semicontinuous. Combining the above two results together, we can verify that the event $\mathrm{supp}(Z_t^\ell) \cap I_0^c = \emptyset$ is measurable for any $t \geq 0$. Hence, it turns out to be that the event $\mathrm{supp}(X_t) \cap I_0^c$ is also measurable, because the set $\mathrm{supp}(X_t) \cap I_0^c = \emptyset$ is expressed as $\cap_{\ell=1}^\infty \mathrm{supp}(Z_t^\ell) \cap I_0^c = \emptyset$ for any $t \geq 0$. Therefore we readily obtain

$$\begin{aligned}
&\mathbb{P}^\gamma_{X_0}\{\mathrm{supp}(X_t) \cap I_0^c = \emptyset, \quad \forall t \geq 0\} \qquad (25)\\
&= \lim_{\ell\to\infty}\mathbb{P}^\gamma_{X_0}\{\mathrm{supp}(Z_t^\ell)\cap I_0^c = \emptyset, \forall t \geq 0\} = \lim_{\ell\to\infty}\mathbb{P}^\gamma_{X_0}\left\{\int_0^\infty Z_s^\ell(I_0^c)ds = 0\right\}\\
&= \lim_{\ell\to\infty}\lim_{\theta\to\infty}\exp\{-\langle X_0, \Psi(\cdot,\theta)\rangle\} = \lim_{n\to\infty}\exp\left\{-\int_a^b v(x,\alpha_n,\beta_n)X_0(dx)\right\},
\end{aligned}$$

since we made use of right continuity in the second equality and the third equality yields directly from (18). □

When the initial measure μ has compact support, according to Theorem 3, X^γ has the compact support property, with the result that the range $\mathcal{R}(X)$ of X^γ is compact. As a corollary of Theorem 3, we can obtain immediately

THEOREM 8. *If we have* $\sup_{\alpha,\beta} \inf_{x\in I_0} v(x,\alpha,\beta) = +\infty$, *then*

$$\mathbb{P}^{\gamma}_{\mu}\{\text{Gsupp}(X) \quad \text{is compact}\} = 0. \tag{26}$$

5. Finite Time Extinction

THEOREM 9. (Main Result) *Suppose that* $p > 1/\nu$. *Let* $\mu \in M_F$ *with compact support. Suppose that the BVP* (13) *has a solution* u. *If the integral* $\int_a^b u(x,\alpha,\beta)X_0(dx)$ *vanishes, then the historical superprocess* $\tilde{X}^\gamma$ *with branching rate functional* $\tilde{K}_\gamma$ *dies out for finite time with pwobability one. That is to say,*

$$\mathbb{P} - \text{a.a.} \quad \gamma, \quad \tilde{\mathbb{P}}^{\gamma}_{0,\mu}(\tilde{X}^\gamma_t = 0, \quad \exists t > 0) = 1. \tag{27}$$

Proof. We want to show that $\lim_{t\to\infty} \tilde{\mathbb{P}}^{\gamma}_{0,\mu}(\tilde{X}^\gamma_t \neq 0) = 0$, $\mathbb{P}$-a.s. Moreover, we define $\mathbb{C}_K = \{w \in \mathbb{C} : |w_s| < K, \forall s \geq 0\}$ for $K \geq 1$. Theorem 3 guarantees the compact support property for the superprocesses. By the compact support property, we have

$$\lim_{K\to\infty} \inf_{t\geq 0} \tilde{\mathbb{P}}^{\gamma}_{0,\mu}\left(\text{supp}(\tilde{X}^\gamma_t) \subseteq \mathbb{C}_K\right) = 1, \quad \mathbb{P} - \text{a.a.}\omega. \tag{28}$$

The goal is to show that, $\mathbb{P}$-a.s., $\tilde{\mathbb{P}}^{\gamma}_{0,\mu}(\tilde{X}^\gamma_t \neq 0)$ vanishes for large t. Hence it suffices to show that, for $\forall K$: large

$$\lim_{t\to\infty} \tilde{\mathbb{P}}^{\gamma}_{0,\mu}(\tilde{X}^\gamma_t \neq 0 \quad \text{and} \quad \text{supp}(\tilde{X}^\gamma_t) \subset \mathbb{C}_K) = 0. \tag{29}$$

By emplying the periodic extension technique $\gamma \to \gamma^K$, it suffices to show finite time extinction of $\tilde{X}^{\gamma^K}$ with fixed periodic extension γ^K: i.e. $\lim_{t\to\infty} \tilde{\mathbb{P}}^{\gamma^K}_{0,\mu}(\tilde{X}^{\gamma^K}_t \neq 0) = 0$, for each fixed $K > 1$. As a matter of fact, we can show the above expression by using the comparison of extinction probabilities [2] and also by a similar technique on finite time extinction of catalytic branching process of [2]. There is another important key point, i.e., decomposition of initial measures. Suppose that the initial measure has a finite deconposition $\mu = \sum_i \mu_i$. If we can show finite time extinction for each initial measure $X^\gamma_0 = \mu_i$, then the branching property implies finite time extinction for $X^\gamma_0 = \mu$. Therefore it is very useful that the stable random measure γ admits a representation of sum of discrete points. After all, we obtain $\lim_{t\to\infty} \tilde{\mathbb{P}}^{\gamma}_{0,\mu}(\tilde{X}^\gamma_t \neq 0 \quad \text{and} \quad \text{supp}(\tilde{X}^\gamma_t) \subseteq \mathbb{C}_K) = 0$ for a fixed sample $\gamma(\omega)$, which means that the process $\tilde{X}^\gamma$ exhibits finite time extinction.

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