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The expected volume of Wiener sausage for Brownian bridge joining the origin to a point outside a parabolic region

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0. Introduction and Notation

This work is originally motivated by a study of Wiener sausage swept by a disc/sphere attached to a d-dimensional Brownian motion started at the origin. Our interest is in finding a correct asymptotic form of the expected volume of the sausage of length t as $t \to \infty$ under the conditional law given that the Brownian motion at time t be at a given site $\mathbf{x}$ which is outside a parabolic region so that $\mathbf{x}^2 > t$. It turns out that to this end one needs to estimate the harmonic measure of the circle/sphere for the heat operator (the space-time distribution of Brownian hitting), so we are concerned with asymptotic estimation of such a harmonic measure as well. It has the density which can be factored as the product of the hitting time density and the density for the site distribution conditioned on the time and I will give the exposition of the results under the following titles

1. Density of Hitting Time Distribution,
2. Density of Hitting Place Distribution,
3. Expected Volume of Wiener Sausage for Brownian Bridge.

The subjects 2 and 3 are inter-related: some result in 2 uses one from 3 and vice versa. The results of both 2 and 3 heavily depend on those from 1 and I will give manners of the dependence. Most of the statements advanced therein may translate into the corresponding ones to Brownian motion with constant drift and some of them will be presented in

4. Brownian Motion with Constant Drift.

I will also include a corresponding result concerning

5. Range of Pinned Random Walk.

We fix the radius $a > 0$ of the Euclidian ball $U = \{\mathbf{x} \in \mathbf{R}^d : |\mathbf{x}| < a\}$ $(d = 2, 3, \ldots)$. Let $P_{\mathbf{x}}$ be the probability law of a d-dimensional standard Brownian motion started at $\mathbf{x} \in \mathbf{R}^d$ and $E_{\mathbf{x}}$ the expectation under P_x. The following notation is used throughout.

$$\nu = \frac{d}{2} - 1 \quad (d = 1, 2, \ldots);\ \mathbf{e} = (1, 0, \ldots, 0);$$
$$\sigma = \inf\{t > 0 : |B_t| \le a\};$$
$$q^{(d)}(x, t) = \frac{d}{dt} P_{\mathbf{x}}[\sigma \le t] \quad (x = |\mathbf{x}| > a).$$
$$p_t^{(d)}(x) = (2\pi t)^{-d/2} e^{-x^2/2t}.$$
$$\Lambda_\nu(y) = \frac{(2\pi)^{\nu+1}}{2y^\nu K_\nu(y)} \quad (y > 0); \qquad \Lambda_\nu(0) = \lim_{y \downarrow 0} \Lambda_\nu(y).$$

Here K_ν is the modified Bessel function of second kind of order ν. We write $f(t) \sim g(t)$ if $f(t)/g(t) \to 1$ in any process of taking limit like '$t \to \infty$'. From the known properties of $K_\nu(z)$ it follows that

$$\Lambda_\nu(0) = \frac{2\pi^{\nu+1}}{\Gamma(\nu)} \quad \text{for} \quad \nu > 0; \quad \Lambda_0(y) \sim \frac{\pi}{-\lg y} \quad \text{as} \quad y \downarrow 0; \text{ and}$$

$$\Lambda_\nu(y) = (2\pi)^{\nu+1/2} y^{-\nu+1/2}\, e^y (1 + O(1/y)) \quad \text{as} \quad y \to \infty.$$

1. Density of Hitting Time Distribution

The definition of $q^{(d)}(x,t)$ may be naturally extended to Bessel processes of order ν and the results concerning it given below may be applied to such extension if $\nu \geq 0$.

Theorem 1 *Uniformly for $x > a$, as $t \to \infty$,*

$$q^{(d)}(x,t) \sim a^{2\nu}\Lambda_\nu\Big(\frac{ax}{t}\Big)p_t^{(d)}(x)\Big[1-\Big(\frac{a}{x}\Big)^{2\nu}\Big] \qquad (d \neq 2) \tag{1}$$

and for $d = 2$,

$$q^{(2)}(x,t) = p_t^{(2)}(x) \times \begin{cases} \dfrac{4\pi \lg(x/a)}{(\lg t)^2}\Big(1+o(1)\Big) & (x \leq \sqrt{t}), \\ \Lambda_0\Big(\dfrac{ax}{t}\Big)\Big(1+o(1)\Big) & (x > \sqrt{t}). \end{cases} \tag{2}$$

Remark 1. A weaker version of the result above is given in [2] : the upper and lower bounds by some constants are obtained instead of the exact factor $(1+o(1))$ (although in some cases the results of [2] are very close to and even finer than ours). For each $x > a$ fixed the result also is given in [4] but with some coefficient being not explicit. The estimate (2) restricted to the parabolic region $x < \sqrt{t}$ is an immediate consequence of the results in [10]. For the random walks the results corresponding to (1) and (2) but restricted to within the parabolic region are given in [9].

Remark 2 (Scaling property). If $q^\circ(x,t)$ designates the density q when $a = 1$, then from the scaling property of Bessel processes it follows that $q(x,t) = a^{-2}q^\circ(x/a, t/a^2)$.

The estimation of $q(x,t)$ will be made in the following three cases

(i) $x < \sqrt{t}$; (ii) $\sqrt{t} < x \leq Mt$ (with M arbitrarily fixed) ; (iii) $x/t \to \infty$.

The methods employed in these cases are different from one another. Roughly speaking, for the case (i) the estimation is based on the well known formula for the Laplace transform of $q^{(d)}(x,\cdot)$, to which we apply the Laplace inversion formula. For the case (ii) we exploit the fact that any Bessel process of order $\nu > -1$ can be decomposed as a sum of two independent Bessel processes and apply the result of the case (i). The case (iii) follows from Lemma 4 [2] where a better estimate than required for Theorem 1 is given. The proof of it, mostly purely analytic, rests on the integral representation obtained in [1] and the derivation from it is somewhat involved. In the last section of this note there will be given a relatively more probabilistic proof for the case (iii) of Theorem 1.

2. Density of Hitting Site Distribution Conditioned on $\sigma = t$

2.1. The case $d = 2$. For convenience sake I use complex notation; in particular B_t is considered to be a complex Brownian motion, while the expression $x\mathbf{e}$ is retained. Let $\arg B_t \in \mathbf{R}$ be the argument of B_t $(\in \mathbf{C})$, which is a.s. uniquely determined by continuity under the convention $\arg B_0 \in (-\pi, \pi]$. The following limits can be shown to exist.

$$f_v(\theta) = \lim_{x/t \to v} \frac{P_{x\mathbf{e}}[\arg B_t \in d\theta | \sigma = t]}{d\theta} \qquad (-\infty < \theta < \infty, v > 0).$$

$$\Phi_v(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda\theta} f_v(\theta)d\theta = \lim_{x/t \to v} E_{x\mathbf{e}}[e^{i\lambda \arg B_\sigma} | \sigma = t] \qquad (\lambda \in \mathbf{R}).$$

Proposition 2 $\Phi_v(\lambda) = \dfrac{K_0(av)}{K_\lambda(av)} \quad (v > 0).$

From Proposition 2 it follows that

$$\Phi_{0+}(\lambda) = 0 \ (\lambda \neq 0) \quad \text{and} \quad \Phi_{+\infty}(\lambda) = 1,$$

which show that $f_v(\theta)d\theta$ concentrates in the limit at infinity as $v \downarrow 0$ and at zero as $v \to \infty$, respectively. The latter result is made precise in Theorem 4 below. Since $\lg \Phi_v(\lambda) \sim -\lambda \lg \lambda$ $(\lambda \to \pm\infty)$, f_v can be extended to an entire function, in particular its support (as a function on $\mathbf{R}$) is the whole real line. $K_{i\eta}(av)$ is an entire function of η and has zeros on and only on the real axis. If η_0 is the smallest positive zero, then

$$\textstyle\int_0^\infty f_v(\theta)e^{\eta\theta}d\theta \text{ is finite or infinity according as } \eta < \eta_0 \text{ or } \eta \geq \eta_0;$$

it can be shown that $0 < \eta_0 - av \leq Cv^{1/3}$.

Proposition 3 *For $v > 0$*

$$f_v(\theta) \geq \pi^{-1} av K_0(av)\, e^{av\cos\theta}\cos\theta \qquad (|\theta| \leq \tfrac{1}{2}\pi). \tag{3}$$

Proposition 3 and Corollary 11 of the next section together show that the probability $f_v(\theta)d\theta$ divided by $\pi^{-1}2avK_0(av)\,e^{av\cos\theta}$ weakly converges to the probability $\frac{1}{2}\mathbf{1}(|\theta| \leq \frac{1}{2}\pi)\cos\theta\, d\theta$ as $v \to \infty$. In fact, the following stronger result holds true.

Theorem 4 *Write v for the ratio x/t. Then, as $x/t \to \infty$ and $t \to \infty$*

$$\frac{1}{2}\sqrt{\frac{2\pi}{av}}e^{av(1-\cos\theta)}P_{x\mathbf{e}}[\arg B(\sigma) \in d\theta \,|\, \sigma = t] \implies \frac{1}{2}\mathbf{1}(|\theta| < \pi/2)\cos\theta\, d\theta,$$

where '$\Rightarrow$' designates the weak convergence of finite measures and $\mathbf{1}(A)$ the indicator function of a statement A.

Proposition 2 follows from Theorem 1 and the next lemma, the latter of which in turn is derived by the skew product representation of multi-dimensional Brownian motion combined with Lemma 15 given in Section 6.

Lemma 5

$$E_{x\mathbf{e}}[e^{i\lambda \arg B_\sigma} | \sigma = t] = \frac{q^{(2|\lambda|+2)}(x,t)}{q^{(2)}(x,t)} x^{|\lambda|}.$$

2.2. THE GENERAL CASE $d \geq 2$. Let $\Theta \in [0, \pi)$ denote the colatitude of B_σ when $(a, 0, \dots, 0)$ is chosen to be the north pole so that

$$a\cos\Theta = \mathbf{e}\cdot B_\sigma.$$

Results similar to those in the case $d = 2$ hold but here we give only the following analogue of Theorem 4.

Theorem 6 *As $v := x/t \to \infty$ and $t \to \infty$*

$$\frac{1}{c_{d-1}}\left(\frac{2\pi}{av}\right)^{(d-1)/2} e^{av(1-\cos\theta)}P_{x\mathbf{e}}[\Theta \in d\theta \,|\, \sigma = t] \implies (d-1)\mathbf{1}(0 \leq \theta < \tfrac{1}{2}\pi)\sin^{d-2}\theta\, \cos\theta\, d\theta,$$

where $c_n = \pi^{n/2}/n\Gamma(\frac{1}{2}n)$, the volume of n-dimensional ball of radius 1.

3. Expected Volume of Wiener Sausage for Brownian Bridge

Let S_t be a Wiener sausage of length t, namely, the region swept by the ball of radius $a > 0$ attached to B_s at its center as s runs from 0 to t:

$$S_t = \{\mathbf{x} \in \mathbf{R}^d : |B_s - \mathbf{x}| \le a \ \text{ for some } \ s \in [0,t]\}.$$

The d-dimensional volume of a set $A \subset \mathbf{R}^d$ is denoted by $\mathrm{Vol}_d(A)$.

Theorem 7 *Let $d \ge 3$.*

$$E_0\left[\mathrm{Vol}_d(S_t) \mid B_t = \mathbf{x}\right] \sim a^{d-2} t \Lambda_\nu(0) \quad \text{as } t \to \infty, \ x/t \to 0. \tag{4}$$

Theorem 8 *Let $d = 2$. The asymptotic formula (4) (with $d = 2$) holds if restricted to the region $|\mathbf{x}| > \sqrt{t}$ and $\Lambda_\nu(0)$ is replaced by $\Lambda_0(ax/t)$), which is asymptotic to $\pi/\lg(t/x)$.*

Remark 3. In [11], it is shown that if $d = 2$, for each $M > 1$, uniformly for $|\mathbf{x}| \le M\sqrt{t}$,

$$E_0\left[\mathrm{Vol}_2(S_t) \mid B_t = \mathbf{x}\right] = 2\pi t N(\kappa t/a^2) + \frac{\pi x^2}{(\lg t)^2}\left[\lg \frac{t}{x^2 \vee 1} + O(1)\right] + O(1)$$

as $t \to \infty$, where $\kappa = 2e^{-2\gamma}$ and $N(\lambda), \lambda \ge 0$ is given by

$$N(\lambda) = \int_0^\infty \frac{e^{-\lambda u}}{(\lg u)^2 + \pi^2} \frac{du}{u} = \frac{1}{\lg \lambda} - \frac{\gamma}{(\lg \lambda)^2} + \frac{\gamma^2 - \frac{1}{6}\pi^2}{(\lg t)^3} + \cdots \quad (\text{as } \lambda \to \infty)$$

(asymptotic expansion). Here $\gamma = -\int_0^\infty e^{-u} \lg u du$ (Euler's constant). The special function $N(\lambda)$ is called Ramanujan's function by some authors.

Theorems 7 and 8 follow from the next proposition and Theorem 1.

Proposition 9 *As $x^2/t \to \infty, t \to \infty$*

$$\begin{aligned} E_0\left[\mathrm{Vol}_d(S_t) \mid B_t = \mathbf{x}\right] &\sim \frac{t}{p_t^{(d)}(x)} \cdot \frac{E_{\mathbf{x}}[e^{-B_\sigma \cdot \mathbf{x}/t}; \sigma \in dt]}{dt} \\ &= \frac{t q^{(d)}(x,t)}{p_t^{(d)}(x)} \int_{\partial U} e^{-\xi \cdot \mathbf{x}/t} P_{\mathbf{x}}[B_\sigma \in d\xi \mid \sigma = t]. \end{aligned}$$

Proposition 10 *As $x/t \to \infty$, $E_0\left[\mathrm{Vol}_d(S_t) \mid B_t = \mathbf{x}\right] \sim c_{d-1} a^{d-1} x$.*

Combining these two propositions yields

Corollary 11 *For $d = 2$, as $v := x/t \to \infty$,*

$$E_{x\mathbf{e}}\left[e^{av(1-\cos(\Theta))} \mid \sigma = t\right] \sim \sqrt{\frac{av}{2\pi}}.$$

Proposition 12 *As x/t tends to a constant $v > 0$, the density $P_{x\mathbf{e}}[\Theta \in d\theta | \sigma = t]/d\theta$ weakly converges to a probability density $g_v^{(d)}(\theta)$ on $0 < \theta < \pi$ and for $d = 2, 3, \ldots$,*

$$E_0\left[\mathrm{Vol}_d(S_t) \mid B_t = \mathbf{x}\right] \sim a^{d-2} t \Lambda_\nu(av) \int_0^\pi e^{-av\cos\theta} g_v^{(d)}(\theta) d\theta. \tag{5}$$

(Note that for $d = 2$, $g_v^{(2)}(\theta) = 2\sum_{n=0}^\infty f_v(\theta + 2n\pi)$.)

4. Brownian Motion with A Constant Drift $-v\mathbf{e}$

The Brownian bridge $P_{\mathbf{0}}[\cdot|B_t = x\mathbf{e}]$ with $v := x/t$ kept away from zero may be comparable or similar to the process $B_t - v\mathbf{e}\,t$ in significant respects and here are given the results for the latter that are readily derived from those given above for the bridge. We label the objects defined with $B_t - v\mathbf{e}\,t$ in place of B_t by the superscript $^{(v)}$ like $\sigma^{(v)}, \Theta^{(v)}$, etc. The translation is made by using the transformation of drift, which in particular shows that for every positive v, x, t,

$$P_{x\mathbf{e}}[\Theta^{(v)} \in d\theta, \sigma^{(v)} \in dt] = e^{-av\cos\theta + vx - \frac{1}{2}v^2 t} P_{x\mathbf{e}}[\Theta \in d\theta, \sigma \in dt].$$

In the following three statements, the limit is taken as $x/t \to v$ (with $v > 0$ fixed) and $t \to \infty$:

(i) $P_{x\mathbf{e}}[\sigma^{(v)} \in dt]/dt \sim \Lambda_\nu(av)/2\pi t$;

(ii) $P_{x\mathbf{e}}[\Theta^{(v)} \in d\theta \,|\, \sigma^{(v)} = t] \sim e^{-av\cos\theta} g_v^{(d)}(\theta)/\Xi, \qquad \Xi := \int_{-\infty}^{\infty} e^{-av\cos\theta} g_v^{(d)}(\theta) d\theta$;

(iii)
$$\begin{aligned} P_{\mathbf{0}}[\mathrm{Vol}_d(S_t^{(v)})] &= \int_{\mathbf{R}^d} P_{\mathbf{0}}[\mathrm{Vol}_d(S_t) \,|\, B_t = \mathbf{x}] p_t^{(d)}(|\mathbf{x} - v\mathbf{e}\,t|) d\mathbf{x} \\ &\sim a^{d-2}\, t\Lambda_\nu(av) \int_0^\pi e^{-av\cos\theta} g_v^{(d)}(\theta) d\theta, \end{aligned}$$

where the equality in (iii) may follow from the fact that the law of the Brownian bridge does not depends on the strength of drift. Similarly, Theorem 6 may translate into the statement that as $v := x/t \to \infty$ and $t \to \infty$

$$P_{x\mathbf{e}}[\Theta^{(v)} \in d\theta \,|\, \sigma^{(v)} = t] \Longrightarrow (d-1)\mathbf{1}(0 \le \theta < \tfrac{1}{2}\pi) \sin^{d-2}\theta \,\cos\theta \, d\theta,$$

which may be intuitively understandable if one notices that the right-hand side is the law of the colatitude of a random variable taking values in the right half of the sphere $|\mathbf{x}| = 1$ whose projection on the $d-1$ dimensional plane perpendicular to the first coordinate axis is uniformly distributed on the "hyper unit disc" on the plane.

5. Range of Pinned Random Walk.

Let $S_n = X_1 + \cdots + X_n$ be a random walk on $\mathbf{Z}^2$ that is irreducible and of mean zero. For $\lambda \in \mathbf{R}^2$, put $\phi(\lambda) = \lg E[e^{\lambda \cdot X_1}]$ and $\Xi = \{\lambda : E[\,|X_1|e^{\lambda\cdot X_1}] < \infty\}$, and for $\mu \in \mathbf{R}^2$ let $c(\mu)$ be the value of λ determined by

$$\nabla\phi(\lambda)\Big|_{\lambda = c(\mu)} = \mu. \tag{6}$$

$c(\mu)$ is well defined if μ is in the image set of an interior of Ξ under $\nabla\phi$. Let Q be the covariance matrix of X_1 and $f_0(n)$ the probability that the walk returns to the origin for the first time at the n-th step ($n \ge 1$). Put

$$H(\mu) = \sum_{k=1}^{\infty} f_0(k)\Big(1 - e^{-k\phi(c(\mu))}\Big), \qquad Z_n = \sharp\{S_1, S_2, \dots, S_n\}.$$

Theorem 13 *Suppose that $\phi(\lambda) < \infty$ in a neighborhood of the origin and let K be a compact set contained in the interior of Ξ. Then,*

$$H(\mu) = \frac{2\pi |Q|^{1/2}}{-\lg[\frac{1}{8}\mu \cdot Q^{-1}\mu]} + O\Big(\frac{1}{(\lg|\mu|)^2}\Big) \qquad as \quad |\mu| \to 0$$

and, uniformly for $\mathbf{x} \in \mathbf{Z}^2$ satisfying $\mathbf{x}/n \in K$ and $|\mathbf{x}| \geq \sqrt{n}$,

$$E_0\Big[Z_n \,\Big|\, S_n = \mathbf{x}\Big] = nH(\mathbf{x}/n) + O\Big(\frac{n}{(\lg n) \vee (\lg|\mathbf{x}/n|)^2}\Big) \qquad as \quad n \to \infty. \tag{7}$$

The case $|\mathbf{x}| < \sqrt{n}$ is studied in [8], where one finds the asymptotic form quite similar to that for Brownian motion presented in Remark 3.

6. Proof of Theorem 1 in the Case $x/t \to \infty$.

Here we give a proof of the following result which slightly refines the estimate in the case $x/t \to \infty$ of Theorem 1.

Proposition 14 *For each $\nu \geq 0$, uniformly for $x > 1$ and $t > 0$,*

$$q^{(d)}(x,t) = \frac{x-a}{\sqrt{2\pi t^3}} \exp\Big(-\frac{(x-a)^2}{2t}\Big)\Big(\frac{a}{x}\Big)^{(d-1)/2}\Big[1 + O\Big(\frac{t}{x}\Big)\Big]. \tag{8}$$

The estimate of (8) determine the exact asymptotic form of $q^{(d)}$ only when either t is small or x/t is large. In [2] a more precise estimate is obtained, where the error term expressed by O in (8) is identified with $\beta t/x$ apart from the smaller error of order $O(\frac{t}{x}[\sqrt{t} \wedge \frac{t}{x-a}])$. Here and in below

$$\beta = (1 - 4\nu^2)/8 = (d-1)(3-d)/8. \tag{9}$$

6.1. Let P_x^{BM} designate the probability measure of a linear Brownian motion B_t started at x, E_x^{BM} the expectation w.r.t. and σ_a the first passage time of a for B_t.

Lemma 15

$$q^{(d)}(x,t) = \frac{x-a}{\sqrt{2\pi t^3}} e^{-(x-a)^2/2t}\Big(\frac{a}{x}\Big)^{(d-1)/2} E_x^{BM}\Big[\exp\Big\{\beta\int_0^t \frac{ds}{B_s^2}\Big\}\,\Big|\,\sigma_a = t\Big]. \tag{10}$$

Proof. Put $\gamma(x) = (d-1)/2x$ and $Z(t) = e^{\int_0^t \gamma(B_s)dB_s - \frac{1}{2}\int_0^t \gamma^2(B_s)ds}$ where B_t is the linear Brownian motion. Then by the Cameron-Martin-Girsanov formula

$$\int_{t-h}^t q^{(d)}(x,s)ds = P_x[t-h \leq \sigma < t] = E_x^{BM}[Z(\sigma_a); t-h \leq \sigma_a < t] \tag{11}$$

for $0 < h < t$. By Ito's formula we have $\int_0^t dB_s/B_s = \lg(B_t/B_0) + \frac{1}{2}\int_0^t ds/B_s^2$ $(t < \sigma_0)$. Hence

$$Z(\sigma_a) = \Big(\frac{a}{B_0}\Big)^{(d-1)/2} \exp\Big[\frac{(d-1)(3-d)}{8}\int_0^{\sigma_a} \frac{ds}{B_s^2}\Big],$$

which together with (11) leads to the identity (10).

6.2. THE CASE $\nu \geq 1/2$. In view of (10) the case $\nu = 0$ is essential, but we at first deal with the easier case $\nu > 1/2$. Let $\nu > 1/2$ so that $\beta < 0$. Owing to Lemma 15 it suffices to show

$$E_x^{BM}\left[e^{\beta \int_0^t B_s^{-2} ds} \middle| \sigma_a = t\right] = 1 + O(t/x) \tag{12}$$

as $x/t \to \infty$. Put

$$v = x/t \quad \text{and} \quad J_t(x) = E_x^{BM}\left[e^{\beta \int_0^t B_s^{-2} ds} \middle| \sigma_a = t\right].$$

Then, by the strong Markov property of Brownian motion

$$J_t(x) = \int_a^\infty J_{1/v}(y) E_x^{BM}\left[e^{\beta \int_0^{t-1/v} B_s^{-2} ds}; \sigma_a > t - \frac{1}{v}, B_{t-1/v} \in dy\right] \frac{q^{(1)}(y-a, 1/v)}{q^{(1)}(x-a, t)}.$$

By bringing in the measure

$$\mu(dy) = \mu_{t,v}(dy) = \frac{q^{(1)}(y-a, 1/v)}{q^{(1)}(x-a, t)} P_x^{BM}[B_{t-1/v} \in dy],$$

this may be written as

$$J_t(x) = \int_a^\infty J_{1/v}(y) E_y^{BM}\left[e^{\beta \int_0^{t-1/v} B_s^{-2} ds}; \sigma_a > t - \frac{1}{v} \,\middle|\, B_{t-1/v} = x\right] \mu(dy). \tag{13}$$

An elementary (but careful) computation shows that

$$\mu(dy) = \frac{y-a}{\sqrt{2\pi/v}} \exp\left(-\frac{1}{2}\left[y - (a+1)\right]^2 v - \frac{y^2}{2t}(1 + o(1))\right)(1 + o(1)) dy \tag{14}$$

with $o(1) \to 0$ as $v \to \infty$ uniformly in $y > a$, entailing that μ converges to the unit measure concentrated at $y = a+1$ in the limit as $v \to \infty$.

For each non-random $t_0 > 0$ the conditional law $P_y^{BM}[\,\cdot\,|B_{t_0} = x]$ is the same as the law under P_y^{BM} of

$$\Big(s(x - B_{t_0})/t_0 + B_s\Big)_{0 \leq s \leq t_0}. \tag{15}$$

Combining this with the well known fact that may read

$$P_{a+1}^{BM}[B_t + vs < a + 2^{-1} vs \ \text{ for some } \ s > 0] = e^{-v},$$

we infer that

$$E_{a+1}^{BM}\left[B_s + vs > a + 2^{-1}vs \ \text{ for } \ 0 < s < t - \frac{1}{v}; \sigma_a > t - \frac{1}{v} \,\middle|\, B_{t-1/v} = x\right] = 1 + O(1/v).$$

If the event in this conditional probability occurs, we have $\int_0^{t-1/v} B_s^{-2} ds \leq 2/av$. Hence

$$E_{a+1}^{BM}\left[e^{\beta \int_0^{t-1/v} B_s^{-2} ds}; \sigma_a > t - \frac{1}{v} \,\middle|\, B_{t-1/v} = x\right] = 1 + O(1/v).$$

This remains true if the initial position $a+1$ is replaced by any $y > 1 + a/2$. Therefore, with the help of (14) and the trivial relation $J_{1/v}(y) = 1 + O(1/v)$ valid uniformly in $y > a$ we conclude (12) from (13).

6.3. THE CASE $0 \le \nu < 1/2$. Here we have $\beta > 0$ and we must evaluate the conditional expectation appearing in (12) from above, the lower bound being trivial. To this end we apply the Kac formula and resort to the exact solution of a certain differential equation. In view of (13) and the result mentioned right after (14) it suffices to show that for all $y > a$

$$W := E_y^{BM}\left[\exp\left\{\beta\int_0^{t-1/v}\frac{ds}{B_s^2}\right\};\ \sigma_a > t-\frac{1}{v}\,\middle|\, B_{t-1/v}=x\right] \le 1+\frac{C_0}{v}$$

provided v is large enough, where $v = x/t$ as in **5.2**.

In order to obtain a tractable upper bound of W we discard the condition for non-absorption up to time $t-1/v$ and at the same time replace B_s by $(B_s \vee a)$ in the integral in the exponent: also, we express the conditional expectation by means of the unconditional realization of Brownian bridge given in (15) and thereafter restrict the range of the expectation to the event

$$|B_{t-1/v}| \le \sqrt{v}\,t,$$

which occurs with probability greater than $1-e^{-vt/2}$. Then, using the monotonicity of the function $x \vee a$ we obtain

$$W \le E_y^{BM}\left[\exp\left\{\beta\int_0^t\frac{ds}{[(B_s+(v-\sqrt{v}\,)s)\vee a]^2}\right\}\right] + e^{(\beta/a^2)t-vt/2}$$

for all sufficiently large v. For a positive number v_* put

$$U(y,t;v_*) = E_y^{BM}\left[\exp\left\{\beta\int_0^t\frac{ds}{[(B_s+v_*s)\vee a]^2}\right\}\right] \qquad (t \ge 0, y \in \mathbf{R}).$$

Then $W \le U(y,t;v_*) + e^{-x/4}$ if $v* \le v-\sqrt{v}$ and we have only to show that $U(y,t;v_*) = 1+O(1/v_*)$ uniformly for $y > a$.

For each v_* fixed, the function $U(y,t) = U(y,t;v_*)$ is a unique solution of the parabolic equation

$$\frac{\partial}{\partial t}U = \frac{1}{2}\frac{\partial^2}{\partial y^2}U + v_*\frac{\partial}{\partial y}U + \frac{\beta}{(y\vee a)^2}U \qquad (t > 0, y \in \mathbf{R})$$

that is uniformly bounded on each finite t-interval and satisfying the initial condition $U(y,+0) = 1$. It also satisfies the boundary condition $U(+\infty,t) = 1$. In view of this, we consider a stationary solution $S(y) = S(y;v_*)$ that satisfies $S(+\infty) = 1$. For $y \ge a$, on the one hand, it is given by

$$S(y;v_*) = \sqrt{2\pi v_* y}\,e^{-v_*y}\Big[I_\nu(v_*y) + \theta K_\nu(v_*y)\Big],$$

for some constant $\theta \in \mathbf{R}$. On the other hand we have two independent solutions $e^{\alpha_+(y-a)}$ and $e^{\alpha_-(y-a)}$ on $(-\infty,a]$ (for $v_* > \sqrt{2\beta}/a$), where $\alpha_\pm = -v_* \pm \sqrt{v_*^2 - 2\beta/a^2}$, so that for some constants A_+ and A_-,

$$S(y;v_*) = A_+e^{\alpha_+(y-a)} + A_-e^{\alpha_-(y-a)}.$$

For the present purpose we have only to consider, as it turns out shortly, a solution with $\theta = 0$, for which the continuity of $S(y)$ and $S'(y)$ at the joint a enforces

$$A_+ = \frac{\alpha_-S(a+)-S'(a+)}{\alpha_- - \alpha_+} \quad \text{and} \quad A_- = \frac{-\alpha_+S(a+)+S'(a+)}{\alpha_- - \alpha_+}.$$

We have the asymptotic formula

$$\sqrt{2\pi z}\, e^{-z} I_\nu(z) = 1 + \beta z^{-1} + 2^{-1}(1+\beta)\beta z^{-2} + O(z^{-3}) \qquad (z \to +\infty);$$

we need to have the asymptotic form of the derivative, for which however we may simply differentiate term-wise ([3], p.21). Hence, uniformly for $y \geq a$, as $v_* \to \infty$

$$S(y) = 1 + \frac{\beta}{yv_*} + O((yv_*)^{-2}) \quad \text{and} \quad S'(y) = -\frac{\beta}{y^2 v_*} - \frac{(1+\beta)\beta}{y^3 v_*^2} + O(v_*^{-3}).$$

From this as well as $\alpha_+ = -\beta/a^2 v^* + O(1/v_*^3)$ we can readily infer that both of $A_\pm$ are positive and in particular $S(y; v_*) > 1$ for all y, provided that v_* is large enough. By a simple comparison argument we conclude that $U(y,t) \leq S(y)$; in particular, $U(y,t) = 1 + O(1/v)$ as desired. The proof of Proposition 14 is finished.

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