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A stochastic process with a Stochastic Hamiltonian equation phase and a Uniform Motion phase

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1 Description of the model

For any $\lambda \geq 1$, let $Q_t^\lambda, V_t^\lambda, P_t^\lambda \in \mathbf{R}^d$ denote the position, the velocity and the momentum of a particle, respectively. We assume that the velocity V_t^λ is given by $V_t^\lambda = \frac{P_t^\lambda}{\sqrt{1+|P_t^\lambda|^2}}$, *i.e.*, we assume that the system is relativistic, and for the sake of simplicity, we wrote the speed of light as 1. Moreover, we assume that the motion of the particle is given by the following stochastic differential equation:

$$\begin{cases} dQ_t^\lambda = \frac{P_t^\lambda}{\sqrt{1+|P_t^\lambda|^2}}dt \\ dP_t^\lambda = \sigma(Q_t^\lambda)dB_t - \gamma\frac{P_t^\lambda}{\sqrt{1+|P_t^\lambda|^2}}dt - \lambda\nabla U(Q_t^\lambda)dt, \\ (Q_0^\lambda, P_0^\lambda) = (q_0, p_0). \end{cases} \tag{1.1}$$

Here $\gamma > 0$ is a constant. We will take the limit $\lambda \to \infty$ later. Our system (1.1) can be considered as a decayed and randomized system with Hamiltonian

$$H(q,p) = \sqrt{1+p^2} + \lambda U(q).$$

We assume that $\sigma \in C^\infty(\mathbf{R}^d, \mathbf{R}^{d\times d})$ is bounded and ${}^t\sigma\sigma$ is uniformly elliptic, where ^{t} means the tranpose of a matrix.

As for the potential function U, we assume that $U \in C_0^\infty(\mathbf{R}^d;\mathbf{R})$ is spherical symmetric and satisfies the following conditions: There exist constants $r_2 > r_1 > 0$ such that $U(x) = 0$ if $|x| \geq r_2$, $U(x) > 0$ if $|x| < r_1$, and $U(x) < 0$ if $|x| \in (r_1, r_2)$. Let h be the real-valued function such that $U(x) = h(|x|)$. In addition, we assume that there exists a constant $\varepsilon_0 \in (0, r_1/2 \wedge (r_2 - r_1)/2)$ and a function $k \in C_0^\infty(\mathbf{R}^d;\mathbf{R})$ such that $\|k\|_\infty \leq 1$ and $|h'(|x|)| = h'(|x|)k(x)$ if $x \in A$, where $A := \{x \in \mathbf{R}^d \big| ||x| - r_1| \leq \varepsilon_0 \text{ or } |x| \geq r_2 - \varepsilon_0\}$.

We also assume that $U(q_0) = 0$, *i.e.*, we assume that the particle starts from a position that is far enough from the origin such that the initial potential is 0.

We are interested in the behavior of the particle described by (1.1) when $\lambda \to \infty$. [2] considered a similar question for the non-relative model, in the case where U gives a reflecting force, precisely, in the case where there exist constants $r, \varepsilon > 0$ such that $U(q) = 0$ when $|q| > r$ and $U(q) > 0$ when $|q| \in (r - \varepsilon, r)$, and got a limit process given as a diffusion process reflecting at $|Q_t| = r$. In contrast, in our model, U gives an absorbing force as soon as the particle enters $|Q_t| < r_2$, which means that when $\lambda \to \infty$, P_t becomes infinity in an instant. (This constitutes the main difficulty in the treatment of our model).

As in the relation between [2] and [3], this problem is also closely related to the problem of "mechanical models of Brownian motions" with absorbing resulting-interactions. Our limit $\lambda \to \infty$ in this study corresponds to the fact that the mass m of the environmental gas particles converges to 0 in that problem (see [4] for details).

2 Idea

Recall that we are interested in the limit behavior of the particle evolving according to (1.1) when $\lambda \to \infty$.

First notice that although P_t, instead of V_t, is the one that seems to be more natural to be considered, it is hopeless to have P_t^λ to converge as $\lambda \to \infty$ or to track the behavior of it directly: when $\lambda \to \infty$, P_t^λ, although keeps finite in the domain $U(Q_t) = 0$, actually diverges to ∞ in the domain $U(Q_t) \neq 0$. However, although P_t^λ might diverge to ∞ as $\lambda \to \infty$, we have that V_t^λ is always bounded by 1, and whenever $|P_t^\lambda| < \infty$, we always have that $P_t^\lambda = \frac{V_t^\lambda}{\sqrt{1-|V_t^\lambda|^2}}$. Also, it is V_t instead of P_t that gives us the velocity of the particle. Therefore, we use (Q_t, V_t) to describe the behavior of a particle.

We can prove that $\left\{\text{the distribution of } \{(Q_t^\lambda, V_t^\lambda)\}_t; \lambda \geq 1\right\}$ is tight, which makes it prospective that the mentioned distribution converges as $\lambda \to \infty$. But how to describe the limit process?

In order to explain the main difficulty of this problem, let us first make some observation about the behavior of the particle when $\lambda \to \infty$, under the assumption of the desired convergence. First of all, notice that in the limit $\lambda \to \infty$, by looking at the total energy of the particle, we find easily that the particle keeps in the domain $|Q_t| \geq r_1$. Also, the behavior of the particle in the area $|Q_t| > r_2$ is trivial: in this domain, we have $U(Q_t) = 0$, so the particle evolves according to the diffusion process without the term $-\lambda \nabla U(Q_t)dt$, so after taking $\lambda \to \infty$, we still have the same diffusion. This gives us a "diffusion phase". When $|Q_t| \in (r_1, r_2)$, the term $-\lambda \nabla U(Q_t)dt$ gives us a very strong "absorbing" force when $\lambda \to \infty$, which is parallel to Q_t, hence P_t becomes very large (and parallel to Q_t) in a very short time, therefore, heuristically V_t should be $\pm \frac{Q_t}{|Q_t|}$ in the area $|Q_t| \in (r_1, r_2)$. This gives us our second phase: the "uniform motion phase". Therefore, it is not difficult to see heuristically that our limit process should be a combination of two phases: a diffusion phase (for $|Q_t| > r_2$) and a uniform motion phase (for $|Q_t| \in (r_1, r_2)$).

The problem is, how does the particle evolve when it reaches the boundary $|Q_t| = r_2$? Precisely, notice that in the limit, when the particle crosses $|Q_t| = r_2$, since the value of $|P_t|$ jumps between ∞ and a finite value as we just mentioned, we have that $|V_t|$ also jumps between 1 and a number that is strictly less than 1. So V_t is not continuous either.

There is no problem when the particle reaches the boundary from the diffusion phase: Since Q_t is continuous with respect to t, the particle simply enters the uniform motion phase with initial condition $V_t = -\frac{Q_t}{|Q_t|}$. However, the answer is not so clear when the particle reaches the boundary from the uniform motion phase: we have to determine whether it stays in the uniform motion phase by taking $V_t = -\frac{Q_t}{|Q_t|}$ or re-enters the diffusion phase; and in the latter case, what is the new initial velocity V_t of the particle? So we have to answer the question "what is the value of V_t (or P_t) at this moment"? Notice that as just mentioned, when $\lambda \to \infty$, $|P_t|$ becomes ∞ in the domain $|Q_t| \in (r_1, r_2)$, so it is hopeless to track P_t (or V_t) directly.

Let us explain the idea of our solution to this problem in the rest of this section. First notice that the norm $|V_t|$ of the velocity for $|Q_t| > r_2$ could be determined in the following way: Let

$$H_t^\lambda := \sqrt{1 + |P_t^\lambda|^2} + \lambda U(Q_t^\lambda) = \frac{1}{\sqrt{1 - |V_t^\lambda|^2}} + \lambda U(Q_t^\lambda).$$

Notice that when $|Q_t^\lambda| > r_2$, we have that $U(Q_t^\lambda) = 0$, hence $H_t^\lambda = \frac{1}{\sqrt{1-|V_t^\lambda|^2}}$, equivalently, $|V_t^\lambda| = \sqrt{1 - \frac{1}{|H_t^\lambda|^2}}$. Therefore, in order to determine the value of $|V_t^\lambda|$, it suffices to find the value of H_t^λ. On the other hand, by Ito's formula and (1.1), we get that H_t^λ satisfies a stochastic differential equation which does not include λ explicitly, precisely, it satisfies

$$dH_t^\lambda = A_1^h(Q_t^\lambda, V_t^\lambda)dB_t + A_2^h(Q_t^\lambda, V_t^\lambda)dt$$

with some proper $A_1^h(q, v)$ and $A_2^h(q, v)$ (see Section 3 for their precise expressions). Therefore, after taking the limit $\lambda \to \infty$, we still have that H_t satisfies the same equation, *i.e.*, H_t is continuous and tractable.

Now, with $|V_t|$ known, we need to determine the direction of V_t. This problem is solved based on the following observation. Notice that our main difficulty that P_t is possibly infinity, caused by the term $-\lambda\nabla U(Q_t^\lambda)dt$ of (1.1), occurs only in the Q_t-direction. Therefore, the component of P_t that is perpendicular to Q_t should also be tractable (*i.e.*, finite, continuous and could be expressed by a stochastic differential equation). Precisely, for any $a, b \in \mathbf{R}^d$ with $a \neq 0$, let $\pi_a b$ and $\pi_a^\perp b$ denote the components of b that are parallel to a and perpendicular to a, respectively, *i.e.*,

$$\pi_a b = \frac{b\cdot a}{|a|^2}a, \qquad \pi_a^\perp b = b - \frac{b\cdot a}{|a|^2}a.$$

We define

$$R_t^\lambda := \pi_{Q_t^\lambda}^\perp P_t^\lambda.$$

Then R_t is tractable (see Section 3 for details).

(Q_t, H_t, R_t) determines the behavior of the particle when it arrives the boundary $|Q_t| = r_2$ in the following way: in the diffusion phase, we have that $|\pi_{Q_t^\lambda}P_t^\lambda|^2 = |P_t^\lambda|^2 - |R_t^\lambda|^2 = |H_t^\lambda|^2 - 1 - |R_t^\lambda|^2$, *i.e.*, $\pi_{Q_t^\lambda}P_t^\lambda$ is determined by $(Q_t^\lambda, H_t^\lambda, R_t^\lambda)$ up to ± 1. Especially, in the moment that the particle re-enters the diffusion domain $|Q_t^\lambda| > r_2$ from the uniform motion domain $|Q_t^\lambda| \in (r_1, r_2)$, we must have that $\pi_{Q_t^\lambda}P_t^\lambda$ has the same direction as Q_t^λ, so P_t^λ and V_t^λ are uniquely determined by $(Q_t^\lambda, H_t^\lambda, R_t^\lambda)$. See Section 3 for the precise expression. This fact keeps true when $\lambda \to \infty$. Moreover, we can prove that the sufficient and necessary condition for the particle to re-enter the diffusion phase is that this precise expression is well-defined. Since as mentioned, H_t and R_t are continuous and tractable even after $\lambda \to \infty$, this enables us to determine V_t for $|Q_t| = r_2$ after taking limit $\lambda \to \infty$.

3 Result

Let us present our result in this section.

Let $\widetilde{W}^d := C([0,\infty);\mathbf{R}^d) \times D([0,\infty);\mathbf{R}^d) \times C([0,\infty);\mathbf{R}) \times C([0,\infty);\mathbf{R}^d)$, where $D([0,\infty);\mathbf{R}^d)$ denotes the set of $\mathbf{R}^d$-valued functions defined on $[0,\infty)$ that are right-continuous with left limit which exists at every point. We define the metric function $dist(\cdot,\cdot)$ on $\widetilde{W}^d$ given by

$$\begin{aligned} dist(w_1, w_2) \quad := \quad & \sum_{n=1}^{\infty} 2^{-n}\Big(1 \wedge \Big[\max_{t\in[0,n]} |q_1(t) - q_2(t)| + \Big(\int_0^n |v_1(t) - v_2(t)|^n\Big)^{1/n} \\ & + \max_{t\in[0,n]} |h_1(t) - h_2(t)| + \max_{t\in[0,n]} |r_1(t) - r_2(t)|\Big]\Big) \end{aligned}$$

for any $w_i(\cdot) = (q_i(\cdot), v_i(\cdot), h_i(\cdot), r_i(\cdot)) \in \widetilde{W}^d$, $i = 1, 2$. In other words, we are considering $\|\cdot\|_\infty$-norm for q, h, r and L^p-norm for v until any finite time.

Let μ_λ denote the distribution of $\{(Q_t^\lambda, V_t^\lambda, H_t^\lambda, R_t^\lambda); t \in [0,\infty)\}$. We prove that when $\lambda \to \infty$, μ_λ converge weakly as probabilities on $\widetilde{W}^d$.

In order to present our limit process, let us first prepare some notations. For any $q, v \in \mathbf{R}^d$, let

$$\begin{aligned}
A_1^h(q,v) &= {}^t v\sigma(q), \\
A_2^h(q,v) &= -\gamma|v|^2 + \frac{1}{2}\sqrt{1-|v|^2}\sum_{i,j=1}^d \sigma_{ij}^2(q) - \frac{1}{2}\sqrt{1-|v|^2}\left|{}^t\sigma(q)v\right|^2, \\
A_1^r(q,v) &= \sigma(q) - \frac{1}{|q|^2} q\, {}^t q\sigma(q), \\
A_2^r(q,v,r) &= -\gamma\sqrt{1-|v|^2}r - \sqrt{1-|v|^2}|r|^2\frac{q}{|q|^2} - \frac{(q,v)}{|q|^2}r, \\
A_1^v(q,v) &= \sqrt{1-|v|^2}\Big(\sigma(q) - v\,{}^t v\sigma(q)\Big), \\
A_2^v(q,v) &= -\gamma(1-|v|^2)^{3/2}v - \frac{1}{2}(1-|v|^2)\Big(\sum_{i,j=1}^d \sigma_{ij}^2(q)\Big)v \\
&\quad + \frac{3}{2}(1-|v|^2)\left|{}^t\sigma(q)v\right|^2 v - (1-|v|^2)\sigma(q)\,{}^t\sigma(q)v,
\end{aligned}$$

let $K_1(q,v)$ be the $(3d+1)\times d$-matrix and let $K_2(q,v,r)$ be the $(3d+1)\times 1$-matrix given by the following, respectively:

$$K_1(q,v) = \begin{pmatrix} 0 \\ 1_{\{|q|>r_2\}}A_1^v(q,v) \\ A_1^h(q,v) \\ A_1^r(q,v) \end{pmatrix}, \qquad K_2(q,v,r) = \begin{pmatrix} v \\ 1_{\{|q|>r_2\}}A_2^v(q,v) \\ A_2^h(q,v) \\ A_2^r(q,v,r) \end{pmatrix}.$$

Finally, let L be the generator

$$Lf(q,v,h,r) = \frac{1}{2}\sum_{i,j=d+1}^{3d+1}\Big(K_1(q,v)\,{}^tK_1(q,v)\Big)_{ij}\nabla_i\nabla_j f(q,v,h,r) + K_2(q,v,r)\cdot\nabla f(q,v,h,r).$$

Here $(*)_{ij}$ stands for the (i,j)-element of the matrix $*$, $\nabla = {}^t(\nabla_1, \cdots, \nabla_{3d+1})$, and

$$\nabla_i = \begin{cases} \nabla_{q_i}, & i = 1,\cdots,d, \\ \nabla_{v_{i-d}}, & i = d+1,\cdots,2d, \\ \nabla_h, & i = 2d+1, \\ \nabla_{r_{i-2d-1}}, & i = 2d+2,\cdots,3d+1. \end{cases}$$

Our main result is the following.

THEOREM 3.1 *1. There exists a unique probability measure μ on $\widetilde{W}^d$ that satisfies the following:*

($\mu 1$) $\mu\Big(Q_0 = q_0, V_0 = \frac{p_0}{\sqrt{1+|p_0|^2}}, H_0 = \sqrt{1+|p_0|^2}, R_0 = \pi_{q_0}^{\perp} p_0\Big) = 1.$

($\mu 2$) $\mu(|Q(t)| \geq r_1, |V(t)| \leq 1, t \in [0,\infty)) = 1.$

($\mu 3$) *For any* $f \in C_0^\infty(\mathbf{R}^{3d+1})$ *with supp*$f \subset \Big\{\big(B(r_2)\backslash\overline{B(r_1)}\big)\cup(\overline{B(r_2)})^C\Big\}\times\mathbf{R}^d\times \mathbf{R}\times\mathbf{R}^d$, *we have that* $\Big\{f(Q_t,V_t,H_t,R_t) - \int_0^t Lf(Q_s,V_s,H_s,R_s)ds; t\geq 0\Big\}$ *is a continuous martingale under* μ.

($\mu 4$) *We have* μ*-almost surely the following: For any* $t \in [0,\infty)$, $|Q_t| \in (r_1, r_2)$ *implies that* $V_t = \pm\frac{Q_t}{|Q_t|}$ *and that* $V_t = V_{t-}$, *also,* $|Q_t| = r_1$ *implies that* $V_t = \frac{Q_t}{|Q_t|}$.

($\mu 5$) *We have* μ*-almost surely that for* $t \in [0,\infty)$ *with* $|Q_t| = r_2$,

(1) *if* $Q_t \cdot V_{t-} < 0$, *then* $V_t = -\frac{Q_t}{|Q_t|}$;

(2) *if* $Q_t \cdot V_{t-} > 0$ *and* $H_t < \sqrt{1+|R_t|^2}$, *then* $V_t = -\frac{Q_t}{|Q_t|}$;

(3) *if* $Q_t \cdot V_{t-} > 0$ *and* $H_t > \sqrt{1+|R_t|^2}$, *then* $V_t = \frac{\sqrt{H_t^2-1-|R_t|^2}Q_t/|Q_t|+R_t}{H_t}$.

2\. *In addition, we assume that* $h'(r_1) < 0$ *and* $\lim_{a\to r_2-0}\frac{h'(a)}{h(a)} = -\infty$. *Then when* $\lambda\to\infty$, $\mu_\lambda \to \mu$ *as probability measures on* $(\widetilde{W}^d, dist)$.

We remark that under μ, we have that Q_t, H_t and R_t are continuous, and V_t is right-continuous with left limit at each t.

Remark 1 *The elements* $1_{\{|Q_t|>r_2\}}A_i^v(Q_t,V_t)(i=1,2)$ *of* $K_1(Q_t,V_t)$ *and* $K_2(Q_t,V_t,R_t)$ *are not* 0 *only if* $|Q_t| > r_2$, *and in this domain, we get by a simple calculation that under* μ, *the following holds: (1)* $|V_t| < 1$, *(2) the distribution of* $(Q_t, \frac{V_t}{\sqrt{1-|V_t|^2}})$ *is a solution of the martingale problem corresponding to* $dQ_t = V_t dt, d(\frac{V_t}{\sqrt{1-|V_t|^2}}) = \sigma(Q_t)dB_t - \gamma V_t dt$, *equivalently,* $(Q_t, \frac{V_t}{\sqrt{1-|V_t|^2}})$ *satisfies (1.1) with* $\lambda = 0$, *(3)* (H_t, R_t) *is actually completely determined by* Q_t *and* V_t: $H_t = \frac{1}{\sqrt{1-|V_t|^2}}$ *and* $R_t = \frac{1}{\sqrt{1-|V_t|^2}}\pi_{Q_t}^{\perp}V_t$. *Also, when* $|Q_t| \in (r_1, r_2)$, *we have by (*$\mu 4$*) that* $|V_t| = 1$, *hence* $A_2^h(Q_t,V_t) = -\gamma$ *and* $A_2^r(Q_t,V_t,R_t) = -\frac{(Q_t,V_t)}{|Q_t|^2}R_t$. *Moreover, in this domain,* Q_t *and* V_t *are deterministic.*

The opposite is also true: if a probability satisfies ($\mu 5$*) and all of the conditions stated above, it also satisfies (*$\mu 1$*)* $\sim$ *(*$\mu 5$*).*

Therefore, we can "divide" our limit process as follows. Let

$$\begin{aligned}L_0 f(q,p) &= \sum_{i=1}^d \frac{p^i}{\sqrt{1+|p|^2}}\frac{\partial}{\partial q_i}f(q,p) + \frac{1}{2}\sum_{i,j=1}^d\Big(\sum_{k=1}^d \sigma_{ik}(q)\sigma_{jk}(q)\Big)\frac{\partial^2}{\partial p_i \partial p_j}f(q,p)\\ &\quad -\gamma\sum_{i=1}^d \frac{p^i}{\sqrt{1+|p|^2}}\frac{\partial}{\partial p_i}f(q,p),\end{aligned}$$

and

$$L_u f(q,v,h,r) = \sum_{i=1}^d v_i \frac{\partial}{\partial q_i} f - \gamma \frac{\partial}{\partial h} f(q,v,h,r) - \sum_{i=1}^d \frac{(q,v)}{|q|^2} r_i \frac{\partial}{\partial r_i} f(q,v,h,r)$$
$$+\frac{1}{2}\left|{}^t\sigma(q)v\right|^2 \nabla_h^2 f(q,v,h,r) + \frac{1}{2}\sum_{i,j=1}^d \left[\left(\sigma(q) - \frac{{}^t qq\sigma(q)}{|q|^2}\right)^2\right]_{i,j} \frac{\partial^2}{\partial r_i \partial r_j} f(q,v,h,r)$$
$$+\sum_{i=1}^d \Big(\sum_{j,k=1}^d v_j \sigma_{jk}(q)\Big(\sigma_{ik}(q) - \frac{1}{|q|^2} q_i \sum_{l=1}^d q_l \sigma_{lk}(q)\Big)\Big) \frac{\partial^2}{\partial h \partial r_i} f(q,v,h,r).$$

Then our limit process can also be described by L_0 and L_u in the following way. Our limit process consists of two phases, a diffusion phase and a uniform motion phase. Precisely, it satisfies the following:

1. *the particle keeps in the area $|Q_t| \geq r_1$;*
2. *when $|Q_t| > r_2$, $(Q_t, \frac{V_t}{\sqrt{1-|V_t|^2}})$ evolves according to the diffusion with generator L_0, and (H_t, R_t) are given by $H_t = \frac{1}{\sqrt{1-|V_t|^2}}$ and $R_t = \frac{1}{\sqrt{1-|V_t|^2}} \pi_{Q_t}^{\perp} V_t$;*
3. *the particle takes uniform motion in the area $|Q_t| \in (r_1, r_2)$ with $V_t = V_{t-} = \pm\frac{Q_t}{|Q_t|}$ and it reflects at $|Q_t| = r_1$ (hence (Q_t, V_t), the "visible" motion of the particle, is completely deterministic in this domain), and (Q_t, V_t, H_t, R_t) is a diffusion with generator L_u,*
4. *finally, its behavior at the boundary $|Q_t| = r_2$ of these two phases is determined as follows: when the particle arrives $|Q_t| = r_2$ from the diffusion phase, it simply enters the uniform motion phase by taking $V_t = -\frac{Q_t}{|Q_t|}$; when the particle arrives at $|Q_t| = r_2$ from the uniform motion phase, it either keeps in the uniform motion phase by reflecting or re-enters the diffusion phase, depending on the value of H_t and R_t at that moment, according to ($\mu 5$).*

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