数理解析研究所講究録
第 1855 巻 2013 年 80-81

Application of the lace expansion to the φ^4 model

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The φ^4 model is a standard model in scaler field theory. It is defined as the Gaussian free field combined with quartic self-interaction. Let $\Lambda\subset\mathbb{Z}^d$ and define the Hamiltonian for the spin configuration $\varphi=\{\varphi_x\}_{x\in\Lambda}$ as

$$\mathscr{H}_\Lambda(\varphi)=-\sum_{\{u,v\}\subset\Lambda}\mathscr{J}_{u,v}\varphi_u\varphi_v+\sum_{v\in\Lambda}\Big(\frac{\mu}{2}\varphi_v^2+\frac{\lambda}{4!}\varphi_v^4\Big),$$

where $\mu\in\mathbb{R}$ plays the role of the temperature, while the intensity λ of self-interaction is fixed nonnegative. We assume that the spin-spin coupling $\mathscr{J}_{u,v}$ is ferromagnetic (i.e., $\mathscr{J}_{u,v}\geq 0$), translation-invariant (i.e., $\mathscr{J}_{u,v}=\mathscr{J}_{o,v-u}$), $\mathbb{Z}^d$-symmetric and finite-range. For example, the nearest-neighbor coupling $\mathscr{J}_{o,x}=\delta_{|x|,1}$ satisfies all those properties. Let

$$\langle\varphi_o\varphi_x\rangle_\mu=\lim_{\Lambda\uparrow\mathbb{Z}^d}\frac{\int_{\mathbb{R}^\Lambda}\varphi_o\varphi_x\,e^{-\mathscr{H}_\Lambda(\varphi)}\,\mathrm{d}^\Lambda\varphi}{\int_{\mathbb{R}^\Lambda}e^{-\mathscr{H}_\Lambda(\varphi)}\,\mathrm{d}^\Lambda\varphi}.$$

It is known to exhibit a phase transition and critical behavior: there is a $\mu_c=\mu_c(d,\mathscr{J},\lambda)$, which is not larger than the critical point $\hat{\mathscr{J}}\equiv\sum_{x\in\mathbb{Z}^d}\mathscr{J}_{o,x}$ for the Gaussian free field, such that $\chi_\mu\equiv\sum_{x\in\mathbb{Z}^d}\langle\varphi_o\varphi_x\rangle_\mu$ is finite if and only if $\mu>\mu_c$ and diverges as $\mu\downarrow\mu_c$ [7]. There were intensive researches in the 1980's when Aizenman [1] and Fröhlich [2] succeeded in showing mean-field behavior (e.g., χ_μ is bounded above and below by a positive multiple of $(\mu-\mu_c)^{-1}$ as $\mu\downarrow\mu_c$) above 4 dimensions under the assumption of reflection-positivity. The nearest-neighbor model satisfies this assumption. In 4 dimensions, Gawędzki and Kupiainen [4] and Hara and Tasaki [5, 6] succeeded in showing the mean-field behavior (with log corrections) for the weakly coupled nearest-neighbor model using a rigorous renormalization-group method.

The sufficient condition for the mean-field behavior that Aizenman suggested in [1] is the bubble condition

$$\sum_{x\in\mathbb{Z}^d}\langle\varphi_o\varphi_x\rangle_{\mu_c}^2<\infty.$$

For reflection-positive models, the Fourier transform of $\langle\varphi_o\varphi_x\rangle_\mu$ is known to obey the Gaussian infrared bound [3]

$$0\leq\sum_{x\in\mathbb{Z}^d}e^{ik\cdot x}\langle\varphi_o\varphi_x\rangle_\mu\leq O(|k|^{-2})\quad\text{uniformly in }\mu>\mu_c,$$

which implies that the bubble condition holds for $d>4$, hence the mean-field behavior. Although the result is satisfactory, it is often hard to verify the assumption of reflection-positivity.

The goal of my research is to investigate asymptotic behavior of the critical two-point function $\langle\varphi_o\varphi_x\rangle_{\mu_c}$ above the upper-critical dimension, without the assumption of reflection-positivity. In [9], we prove the following:

[1]`http://www.math.sci.hokudai.ac.jp/~sakai/`

Theorem 1. *Let $\rho = 2(d-4) > 0$ and $0 < \lambda \ll 1$ (depending on d and $\mathscr{J}$). Then there is a $\Phi_\mu(x) = \langle\varphi_o^2\rangle_\mu\delta_{o,x} + O(\lambda)\,(|x| \vee 1)^{-(d+2+\rho)}$, where $O(\lambda)$ is uniform in $\mu \geq \mu_c$, such that*

$$\mu_c = \hat{\mathscr{J}} - \frac{\lambda}{2}\hat{\Phi}_{\mu_c}, \qquad \langle\varphi_o\varphi_x\rangle_{\mu_c} \underset{|x|\uparrow\infty}{\sim} \frac{\frac{d}{2}\Gamma(\frac{d-2}{2})\pi^{-d/2}}{\sum_{y\in\mathbb{Z}^d}|y|^2\big(\mathscr{J}_{o,y} - \frac{\lambda}{2}\Phi_{\mu_c}(y)\big)}|x|^{2-d}.$$

The key elements for the proof of the above theorem are the following:

1. The Griffiths-Simon construction [10] to approximate the φ^4 model on Λ to some Ising model on $\Lambda \times \{1, 2, \ldots, N\}$.
2. The lace expansion for the Ising two-point function [8].
3. Detail estimates on the expansion coefficients in terms of N [9].

These steps yield a linearized version of the Schwinger-Dyson equation, which further yields the aforementioned asymptotic expression for $\langle\varphi_o\varphi_x\rangle_{\mu_c}$.

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