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2乗可積分性をもつレヴィ過程に対するマリアヴァン解析とその応用

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1 Abstract

The representations of functionals of Brownian motions (or Lévy processes) by stochastic integrals are important theorems in Probability theory. In particular, the Clark-Ocone formula is an explicit stochastic integral representation for random variables in terms of Malliavin derivatives that turns to be central in the application to mathematical finance. On the other hand, a Stroock formula is an explicit representation for chaos expansion by using Malliavin derivative. In this paper, we introduce a Clark-Ocone type formula under change of measure for Lévy processes with L^2-Lévy measure ([7]). We also introduce a Stroock type formula for L^2-Lévy functionals ([5]). This paper is résumé of [5] and [7].

2 A history of Clark-Ocone formulae

- The Clark-Ocone formula is an explicit stochastic integral representation for random variables in terms of Malliavin derivatives: For $F \in \mathbb{D}^{1,2}(\mathbb{R})$,

$$F = \mathbb{E}[F] + \int_0^T \int_{\mathbf{R}} \mathbb{E}[D_{t,z}F|\mathcal{F}_{t-}]Q(dt, dz).$$

- One for Brownian functionals: Clark (1970, 1971, Stochastics **41**, **42**), Ocone (1984, Stochastics **12**) and Haussmann (1979 ,Stochastics **3**).
- One for pure jump Lévy functionals: Løkka (2004, Stochastic Anal. Appl. **22**)
- Clark-Ocone formula under change of measure for Brownian motions: Ocone-Karatzas (1991, Stochastics **34**).
- Clark-Ocone formula under change of measure for pure jump Lévy processes: Huehne (2005, Working Paper)
- White noise generalization of the Karatzas-Ocone formula: Okur (2010, Stochastic Anal. Appl. **28**)
- Di Nunno et al. (2009, Universitext) and Okur (2012, Stochastics **84**) introduced one for Lévy processes and their results are different from our result (different setting, different representation).

3 A history of Stroock type formulae

- Stroock formula is a useful tool to compute Wiener-Itô chaos expansions: If $F \in \cap_{k=1}^{\infty} \mathbb{D}^{k,2}(\mathbb{R})$, then,

$$F = \mathbb{E}[F] + \sum_{n=1}^{\infty} I_n(f_n),$$

where,

$$f_n = \frac{1}{n!}\mathbb{E}[D^n F].$$

We review a history of Stroock type formula:

- In 1987, D.W. Stroock proved the Stroock formula for Brownian motions.
- N. Privault (2002) got one on the Poisson space.
- Eddahbi et al. (2005) showed a Stroock formula for a certain class of Lévy processes.

4 Malliavin calculus for square integrable Lévy processes

Throughout this paper, we consider Malliavin calculus for Lévy processes, based on, [4] and [2].

For given an infinitely divisible distribution μ on $\mathbb{R}$, we can construct a Lévy process from Lévy-Ito decomposition. For details, see the book by Sato [6].

Given an infinitely divisible distribution μ on $\mathbb{R}$, we have Lévy-Khintchine representation: there exist unique $\sigma^2 \geq 0$, $\gamma \in \mathbb{R}$ and Lévy measure ν, that is

$$\nu(\{0\}) = 0 \text{ and } \int_{\mathbb{R}} \min(1, |z|^2)\nu(dz) < \infty,$$

such that its characteristic function has following form:

$$\int_{\mathbb{R}} e^{iuz}\mu(dz) = \exp\left(-\frac{\sigma^2}{2}u^2 + i\gamma u + \int_{\mathbb{R}_0} (e^{iuz} - 1 - iuz\mathbb{1}_{|z|<1})\nu(dz)\right).$$

where $\mathbb{R}_0$ means $\mathbb{R}\backslash\{0\}$. To construct centered square integral Lévy process, we assume that

$$\gamma = 0 \text{ and } \int_{\mathbb{R}_0} z^2\nu(dz) < \infty.$$

In fact, the second condition is equivalent to existence of second moment of μ.

Second, We give a Lévy process from an infinitely divisible distribution. Let $\{W_t; t \in [0,T]\}$ be a standard Brownian motion and N be a Poisson random measure independent of W defined by

$$N(A,t) = \sum_{s\leq t} \mathbf{1}_A(\Delta X_s),\ A \in \mathcal{B}(\mathbb{R}_0),\ \Delta X_s := X_s - X_{s-},$$

We denote the compensated Poisson random measure by $\tilde{N}(dt,dz) = N(dt,dz) - dt\nu(dz)$, where $dt\nu(dz) = \lambda(dt)\nu(dz)$ is the compensator of N, $\nu(\cdot)$ the Lévy measure of μ. We give a centered square integrable Lévy process $X = \{X_t; t \in [0,T]\}$ on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P}; \{\mathcal{F}_t\}_{t\in[0,T]})$, as follows:

$$X_t = \sigma W_t + \int_0^t \int_{\mathbb{R}_0} z\tilde{N}(ds,dz)$$

where $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ is the augmented filtration generated by X.

To consider multiple integral, we consider the finite measure q defined on $[0,T]\times\mathbb{R}$ by

$$q(E) = \sigma^2 \int_{E(0)} dt\delta_0(dz) + \int_{E'} z^2 dt\nu(dz), \quad E \in \mathcal{B}([0,T]\times\mathbb{R}),$$

where $E(0) = \{(t,0) \in [0,T]\times\mathbb{R}; (t,0) \in E\}$ and $E' = E - E(0)$, and the random measure Q on $[0,T]\times\mathbb{R}$ by

$$Q(E) = \sigma \int_{E(0)} dW_t\delta_0(dz) + \int_{E'} z\tilde{N}(dt,dz), \quad E \in \mathcal{B}([0,T]\times\mathbb{R}).$$

Let $L^2_{T,q,n}(\mathbb{R})$ denote the set of product measurable, deterministic functions $h : ([0,T]\times\mathbb{R})^n \to \mathbb{R}$ satisfying

$$\|h\|^2_{L^2_{T,q,n}} := \int_{([0,T]\times\mathbb{R})^n} |h((t_1,z_1),\cdots,(t_n,z_n))|^2 q(dt_1,dz_1)\cdots q(t_n,z_n) < \infty.$$

For $n \in \mathbb{N}$ and $h_n \in L^2_{T,q,n}(\mathbb{R})$, we denote

$$I_n(h_n) := \int_{([0,T]\times\mathbb{R})^n} h((t_1,z_1),\cdots,(t_n,z_n))Q(dt_1,dz_1)\cdots Q(dt_n,dz_n).$$

It is easy to see that $\mathbb{E}[I_0(h_0)] = h_0$ and $\mathbb{E}[I_n(h_n)] = 0$, for $n \geq 1$. In this setting, we introduce the following chaos expansion (see Theorem 2 in [3], Section 2 of [4]).

Proposition 1 *Any $\mathcal{F}$-measurable square integrable random variable F has a unique representation*

$$F = \sum_{n=0}^{\infty} I_n(f_n), \mathbb{P}\text{−a.s.}$$

with functions $f_n \in L^2_{T,q,n}(\mathbb{R})$ that are symmetric in the n pairs $(t_i,z_i), 1 \leq i \leq n$ and we have the isometry

$$\mathbb{E}[F^2] = \sum_{n=0}^{\infty} n!\|f_n\|^2_{L^2_{T,q,n}}.$$

We next define the follows:

Definition 1 *Let $\mathbb{D}^{k,2}(\mathbb{R}), k \geq 1$ denote the set of $\mathcal{F}$ -measurable random variables $F \in L^2(\mathbb{P})$ with the representation $F = \sum_{n=0}^{\infty} I_n(h_n)$ satisfying*

$$\sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)n!\|h_n\|^2_{L^2_{T,q,n}} < \infty.$$

For $F \in \mathbb{D}^{k,2}(\mathbb{R}), k \geq 1$, we define the k-th Malliavin derivative as follows:

$$D^k_{t_1,z_1,\cdots,t_k,z_k}F = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)I_{n-k}(h_n((t_1,z_1),\cdots,(t_k,z_k),\cdot)),$$

$(t_k,z_k) \in [0,T]\times\mathbb{R}, k \geq 1$.

We next establish the following fundamental result.

Proposition 2 (The closability of operator D,[7]) *Let $F \in L^2(\mathbb{P})$ and $F_k \in \mathbb{D}^{1,2}(\mathbb{R}), k \in \mathbb{N}$ such that*

1. $\lim_{k\to\infty} F_k = F$ *in* $L^2(\mathbb{P})$,

2. $\{D_{t,z}F_k\}_{k=1}^{\infty}$ *converges in* $L^2(q \times \mathbb{P})$.

Then, $F \in \mathbb{D}^{1,2}$ *and* $\lim_{k\to\infty} D_{t,z}F_k = D_{t,z}F$ *in* $L^2(q \times \mathbb{P})$.

We also introduce a Clark-Ocone type formula for Lévy functionals.

Proposition 3 (Clark-Ocone type formula for Lévy functionals) *Let* $F \in \mathbb{D}^{1,2}(\mathbb{R})$. *Then,*

$$\begin{aligned} F &= \mathbb{E}[F] + \int_{[0,T]\times\mathbb{R}} \mathbb{E}[D_{t,z}F|\mathcal{F}_{t-}]Q(dt,dz) \\ &= \mathbb{E}[F] + \sigma\int_0^T \mathbb{E}[D_{t,0}F|\mathcal{F}_{t-}]dW(t) + \int_0^T\int_{\mathbb{R}_0} \mathbb{E}[D_{t,z}F|\mathcal{F}_{t-}]z\tilde{N}(dt,dz). \end{aligned}$$

Proof *The proof is same to the one for the Brownian motion case (see, Theorem 4.1 in Di Nunno et al (2009)) and pure jump Lévy case (see, Theorem 12.16 in Di Nunno et al (2009)).* □

We also introduce the follows.

Lemma 1 *Let* $F \in \mathbb{D}^{1,2}(\mathbb{R})$. *Then, for* $0 \le t \le T$, $\mathbb{E}[F|\mathcal{F}_t] \in \mathbb{D}^{1,2}(\mathbb{R})$ *and*

$$D_{s,x}\mathbb{E}[F|\mathcal{F}_t] = \mathbb{E}[D_{s,x}F|\mathcal{F}_t]\mathbf{1}_{\{s\le t\}}, \text{ for } q\text{-a.e. } (s,x) \in [0,T]\times\mathbb{R},\ \mathbb{P}\text{-a.s.}$$

Proof *We can show the same step as Lemma 3.1 of [1].* □

Next we introduce a chain rule. First we define the following.

Definition 2 1. *Let* $C_0^\infty(\mathbb{R}^n)$ *denote the space of smooth functions* $f : \mathbb{R}^n \to \mathbb{R}$ *with compact support.*

2. *A random variable of the form* $F = f(X_{t_1},\cdots,X_{t_n})$, *where* $f \in C_0^\infty(\mathbb{R}^n)$, $n \in \mathbb{N}$, *and* $t_1,\cdots,t_n \ge 0$, *is said to be a smooth random variable. The set of all smooth random variables is denoted by* $\mathcal{S}$.

3. *For* $F \in \mathcal{S}$, *we define the Malliavin derivative operator* $\mathcal{D}$ *as a map from* $\mathcal{S}$ *into* $L^2(q \times \mathbb{P})$

$$\begin{aligned} \mathcal{D}_{t,z}F &:= \sum_{i=1}^n \frac{\partial f}{\partial x_i}(X_{t_1},\cdots,X_{t_n})\mathbf{1}_{[0,t_i]\times\{0\}}(t,z) \\ &\quad + \frac{f(X_{t_1}+z\mathbf{1}_{[0,t_1]}(t),\cdots,X_{t_n}+z\mathbf{1}_{[0,t_n]}(t)) - f(X_{t_1},\cdots,X_{t_n})}{z}\mathbf{1}_{\mathbb{R}_0}(z) \end{aligned}$$

for $(t,z) \in [0,T]\times\mathbb{R}$.

By Lemma 3.1 and Theorem 4.1 in [2], we can see that the closure of the domain of $\mathcal{D}$ with respect to the norm

$$||F||_{\mathcal{D}} := \{\mathbb{E}[|F|^2] + \mathbb{E}[||\mathcal{D}F||^2_{L^2_q}]\}^{1/2}$$

is the space $\mathbb{D}^{1,2}(\mathbb{R})$ and $D_{t,z}F = \mathcal{D}_{t,z}F$ for all $F \in \mathcal{S} \subset \mathbb{D}^{1,2}(\mathbb{R})$. Moreover, by Corollary 4.1 in [2], the set $\mathcal{S}$ of smooth random variables is dense in $L^2(\mathbb{P})$, $\mathbb{D}^{1,2}(\mathbb{R})$. Hence, we can see the following:
$F \in \mathbb{D}^{1,2}(\mathbb{R})$ if and only if there exists a sequence $\{F_k\}_{k=1}^\infty$, $F_k \in \mathcal{S}$ with $F_k \to F$ in $L^2(\mathbb{P})$ and $D_{t,z}F_k \to D_{t,z}F$ in $L^2(q \times \mathbb{P})$.
Similarly, for $F \in \mathcal{S}$ and $k \in \mathbb{N}$, we can introduce a k-th Malliavin derivative operator $\mathcal{D}^k$ as a map from $\mathcal{S}$ into $L^2(q^k \times \mathbb{P})$

$$\mathcal{D}^k_{t_1,z_1,\cdots,t_k,z_k}F = \mathcal{D}_{t_1,z_1}\cdots\mathcal{D}_{t_k,z_k}F.$$

By induction we can show that $\mathcal{D}^k$ is closable and the closure of the domain of definition of $\mathcal{D}^k$ with respect to the norm

$$||F||_{\mathcal{D}^k} := \{\mathbb{E}[|F|^2] + \sum_{i=1}^k \mathbb{E}[||\mathcal{D}^iF||^2_{L^2_{q^i}}]\}^{1/2}$$

is the space $\mathbb{D}^{k,2}(\mathbb{R})$ and $D_{t_1,z_1,\cdots,t_n,z_n}F = \mathcal{D}_{t_1,z_1,\cdots,t_n,z_n}F$ for all $F \in \mathcal{S} \subset \mathbb{D}^{k,2}(\mathbb{R})$. Hence, we also see that the set $\mathcal{S}$ of smooth random variables is dense in $L^2(\mathbb{P})$, $\mathbb{D}^{k,2}(\mathbb{R})$ and that $F \in \mathbb{D}^{k,2}(\mathbb{R}), k \geq 1$ if and only if there exists a sequence $\{F_n\}_{n=1}^{\infty}$, $F_n \in \mathcal{S}$ with $F_n \to F$ in $L^2(\mathbb{P})$ and $D^k_{t_1,z_1,\cdots,t_k,z_k}F_n \to D^k_{t_1,z_1,\cdots,t_k,z_k}F$ in $L^2(q^k \times \mathbb{P})$ for $k \geq 1$.
Now we introduce a chain rule.

Proposition 4 (Chain rule) *Let $\varphi \in C^1(\mathbb{R}^n;\mathbb{R})$ and $F = (F_1,\cdots,F_n)$, where, $F_1,\cdots,F_n \in \mathbb{D}^{1,2}(\mathbb{R})$. Suppose that $\varphi(F) \in L^2(\mathbb{P}), \sum_{k=1}^n \frac{\partial}{\partial x_k}\varphi(F)D_{t,0}F_k \in L^2(\lambda \times \mathbb{P})$ and $\frac{\varphi(F_1+zD_{t,z}F_1,\cdots,F_k+zD_{t,z}F_k)-\varphi(F_1,\cdots,F_k)}{z} \in L^2(z^2\nu(dz)dtd\mathbb{P})$. Then, $\varphi(F) \in \mathbb{D}^{1,2}(\mathbb{R})$,*

$$D_{t,0}\varphi(F) = \sum_{k=1}^{n} \frac{\partial}{\partial x_k}\varphi(F)D_{t,0}F_k$$

and

$$D_{t,z}\varphi(F) = \frac{\varphi(F_1 + zD_{t,z}F_1,\cdots,F_k + zD_{t,z}F_k) - \varphi(F_1,\cdots,F_k)}{z}, z \neq 0.$$

5 Commutation of integration and the Malliavin differentiability

In this section, we consider about commutations of integration and the Malliavin differentiability (see [7]).

Definition 3

1. *Let $\mathbb{L}^{1,2}(\mathbb{R})$ denote the space of product measurable and $\mathbb{F}$-adapted processes $G : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ satisfying*

$$\mathbb{E}\left[\int_{[0,T]\times\mathbb{R}} |G(s,x)|^2 q(ds,dx)\right] < \infty,$$

$G(s,x) \in \mathbb{D}^{1,2}(\mathbb{R}), q$–a.e. $(s,x) \in [0,T] \times \mathbb{R}$ *and*

$$\mathbb{E}\left[\int_{([0,T]\times\mathbb{R})^2} |D_{t,z}G(s,x)|^2 q(ds,dx)q(dt,dz)\right] < \infty.$$

2. *Let $\mathbb{L}_0^{1,2}(\mathbb{R})$ denote the space of measurable and $\mathbb{F}$-adapted processes $G : \Omega \times [0,T] \to \mathbb{R}$ satisfying*

$$\mathbb{E}\left[\int_{[0,T]} |G(s)|^2 ds\right] < \infty,$$

$G(s) \in \mathbb{D}^{1,2}(\mathbb{R}),\ s \in [0,T]$, a.e. *and*

$$\mathbb{E}\left[\int_{[0,T]\times\mathbb{R}}\int_{[0,T]} |D_{t,z}G(s)|^2 ds q(dt,dz)\right] < \infty.$$

3. *Let $\tilde{\mathbb{L}}_1^{1,2}(\mathbb{R})$ denote the space of product measurable and $\mathbb{F}$-adapted processes $G : \Omega \times [0,T] \times \mathbb{R}_0 \to \mathbb{R}$ satisfying*

$$\mathbb{E}\left[\int_{[0,T]\times\mathbb{R}_0} |G(s,x)|^2 \nu(dx)ds\right] < \infty,$$

$$\mathbb{E}\left[\left(\int_{[0,T]\times\mathbb{R}_0} |G(s,x)|\nu(dx)ds\right)^2\right] < \infty,$$

$G(s,x) \in \mathbb{D}^{1,2}(\mathbb{R}), (s,x) \in [0,T] \times \mathbb{R}_0$, a.e.,

$$\mathbb{E}\left[\int_{[0,T]\times\mathbb{R}} \left(\int_{[0,T]\times\mathbb{R}_0} |D_{t,z}G(s,x)|\nu(dx)ds\right)^2 q(dt,dz)\right] < \infty$$

and

$$\mathbb{E}\left[\int_{[0,T]\times\mathbb{R}}\int_{[0,T]\times\mathbb{R}_0}|D_{t,z}G(s,x)|^2\nu(dx)dsq(dt,dz)\right]<\infty.$$

We next discuss the commutation relation of the stochastic integral with the Malliavin derivative. A canonical space version of it was shown by [1].

Proposition 5 *Let $G:\Omega\times[0,T]\times\mathbb{R}\to\mathbb{R}$ be a predictable process with $\mathbb{E}\left[\int_{[0,T]\times\mathbb{R}}|G(s,x)|^2q(ds,dx)\right]<\infty$. Then*

$$G\in\mathbb{L}^{1,2}(\mathbb{R})\ \text{ if and only if }\ \int_{[0,T]\times\mathbb{R}}G(s,x)Q(ds,dx)\in\mathbb{D}^{1,2}(\mathbb{R}). \tag{1}$$

Furthermore, if $\int_{[0,T]\times\mathbb{R}}G(s,x)Q(ds,dx)\in\mathbb{D}^{1,2}(\mathbb{R})$, then, for q -a.e. $(t,z)\in[0,T]\times\mathbb{R}$, we have

$$D_{t,z}\int_{[0,T]\times\mathbb{R}}G(s,x)Q(ds,dx)=G(t,z)+\int_{[0,T]\times\mathbb{R}}D_{t,z}G(s,x)Q(ds,dx),\quad \mathbb{P}\text{-a.s.}, \tag{2}$$

and $\int_{[0,T]\times\mathbb{R}}D_{t,z}G(s,x)Q(ds,dx)$ is a stochastic integral in Itô sense.

Next proposition provides a commutation of the Lebesgue integration and the Malliavin differentiability. Delong and Imkeller ([1]) also derived a canonical space version of it.

Proposition 6 *Assume that $G:\Omega\times[0,T]\times\mathbb{R}\to\mathbb{R}$ is a product measurable and $\mathbb{F}$ -adapted process, η on $[0,T]\times\mathbb{R}$ a finite measure, so that conditions*

$$\begin{aligned}&\mathbb{E}\left[\int_{[0,T]\times\mathbb{R}}|G(s,x)|^2\eta(ds,dx)\right]<\infty,\\ &G(s,x)\in\mathbb{D}^{1,2}(\mathbb{R}),\quad \text{for } \eta\text{-a.e. } (s,x)\in[0,T]\times\mathbb{R},\\ &\mathbb{E}\left[\int_{([0,T]\times\mathbb{R})^2}|D_{t,z}G(s,x)|^2\eta(ds,dx)q(dt,dz)\right]<\infty\end{aligned}$$

are satisfied. Then we have

$$\int_{[0,T]\times\mathbb{R}}G(s,x)\eta(ds,dx)\in\mathbb{D}^{1,2}(\mathbb{R})$$

and the differentiation rule

$$D_{t,z}\int_{[0,T]\times\mathbb{R}}G(s,x)\eta(ds,dx)=\int_{[0,T]\times\mathbb{R}}D_{t,z}G(s,x)\eta(ds,dx)$$

holds for q -a.e. $(t,z)\in[0,T]\times\mathbb{R},\mathbb{P}$ -a.s.

By using σ-finiteness of ν and Proposition 6, we can show the following proposition.

Proposition 7 *Let $G\in\tilde{\mathbb{L}}_1^{1,2}(\mathbb{R})$. Then,*

$$\int_{[0,T]\times\mathbb{R}_0}G(s,x)\nu(dx)ds\in\mathbb{D}^{1,2}(\mathbb{R})$$

and the differentiation rule

$$D_{t,z}\int_{[0,T]\times\mathbb{R}_0}G(s,x)\nu(dx)ds=\int_{[0,T]\times\mathbb{R}_0}D_{t,z}G(s,x)\nu(dx)ds$$

holds for q -a.e. $(t,z)\in[0,T]\times\mathbb{R},\mathbb{P}$ -a.s.

6 A Clark-Ocone type formula under change of measure for Lévy processes

In this section, we introduce a Clark-Ocone type formula under change of measure for Lévy processes ([7]). Now, we assume the following.

Assumption 1 *Let $\theta(s,x) < 1, s \in [0,T], x \in \mathbb{R}_0$ and $u(s), s \in [0,T]$, be predictable processes such that*

$$\int_0^T \int_{\mathbb{R}_0} \{|\log(1-\theta(s,x))| + \theta^2(s,x)\}\nu(dx)ds < \infty, \text{ a.s.},$$
$$\int_0^T u^2(s)ds < \infty, \text{ a.s.}$$

Moreover we denote

$$\begin{aligned} Z(t) &:= \exp\Big(-\int_0^t u(s)dW(s) - \frac{1}{2}\int_0^t u(s)^2 ds + \int_0^t \int_{\mathbb{R}_0} \log(1-\theta(s,x))\tilde{N}(ds,dx) \\ &\quad + \int_0^t \int_{\mathbb{R}_0} (\log(1-\theta(s,x)) + \theta(s,x))\nu(dx)ds\Big), t \in [0,T]. \end{aligned}$$

Define a measure $\mathbf{Q}$ on $\mathcal{F}_T$ by

$$d\mathbf{Q}(\omega) = Z(\omega,T)d\mathbb{P}(\omega),$$

and we assume that $Z(T)$ satisfies the Novikov condition, that is,

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T u^2(s)ds + \int_0^T \int_{\mathbb{R}_0} \{(1-\theta(s,x))\log(1-\theta(s,x)) + \theta(s,x)\}\nu(dx)ds\right)\right] < \infty.$$

Furthermore we denote

$$\tilde{N}_{\mathbf{Q}}(dt,dx) := \theta(t,x)\nu(dx)dt + \tilde{N}(dt,dx)$$

and

$$dW_{\mathbf{Q}}(t) := u(t)dt + dW(t).$$

Second, we assume the following.

Assumption 2 *We denote*

$$\begin{aligned} \tilde{H}(t,z) &:= \exp\Big(-\int_0^T zD_{t,z}u(s)dW_{\mathbf{Q}}(s) - \frac{1}{2}\int_0^T (zD_{t,z}u(s))^2 ds \\ &\quad + \int_0^T \int_{\mathbb{R}_0} \left[zD_{t,z}\theta(s,x) + \log\left(1 - z\frac{D_{t,z}\theta(s,x)}{1-\theta(s,x)}\right)(1-\theta(s,x))\right]\nu(dx)ds \\ &\quad + \int_0^T \int_{\mathbb{R}_0} \log\left(1 - z\frac{D_{t,z}\theta(s,x)}{1-\theta(s,x)}\right)\tilde{N}_{\mathbf{Q}}(ds,dx)\Big), \end{aligned}$$

and

$$K(t) := \int_0^T D_{t,0}u(s)dW_{\mathbf{Q}}(s) + \int_0^T \int_{\mathbb{R}_0} \frac{D_{t,0}\theta(s,x)}{1-\theta(s,x)}\tilde{N}_{\mathbf{Q}}(ds,dx)$$

and assume that $\sigma \neq 0$. Furthermore, we assume the following:

1. *$F, Z(T) \in \mathbb{D}^{1,2}(\mathbb{R})$, with $FZ(T) \in L^2(\mathbb{P})$,*

$$Z(T)D_{t,z}F + FD_{t,z}Z(T) + zD_{t,z}F \cdot D_{t,z}Z(T) \in L^2(q \times \mathbb{P}),$$

2. $Z(T)D_{t,0}\log Z(T) \in L^2(\lambda \times \mathbb{P}), Z(T)(e^{zD_{t,z}\log Z(T)} - 1) \in L^2(\nu(dz)dtd\mathbb{P})$,
3. $u(s)D_{t,0}u(s) \in L^2(\lambda \times \mathbb{P}), 2u(s)D_{t,z}u(s) + z(D_{t,z}u(s))^2 \in L^2(z^2\nu(dz)dtd\mathbb{P})$, *s-a.e.*
4. $\log\left(1 - z\frac{D_{t,z}\theta(s,x)}{1-\theta(s,x)}\right) \in L^2(\nu(dz)dtd\mathbb{P}), \frac{D_{t,0}\theta(s,x)}{1-\theta(s,x)} \in L^2(\lambda \times \mathbb{P}), (s,x)$ *-a.e.*
5. $\sigma^{-1}u, x^{-1}\log(1-\theta(s,x)) \in \mathbb{L}^{1,2}(\mathbb{R})$,
6. $u(s)^2 \in \mathbb{L}_0^{1,2}$ *and* $\theta(s,x), \log(1-\theta(s,x)) \in \tilde{\mathbb{L}}_1^{1,2}(\mathbb{R})$,
7. *and* $F\tilde{H}(t,z), \tilde{H}(t,z)D_{t,z}F \in L^1(\mathbb{Q}), (t,z)$ *-a.e.*

We next introduce a Clark-Ocone type formula under change of measure for Lévy processes.

Theorem 1 *Under Assumption 1 and Assumption 2, we have*

$$\begin{aligned} F &= \mathbb{E}_{\mathbb{Q}}[F] + \sigma\int_0^T \mathbb{E}_{\mathbb{Q}}\left[D_{t,0}F - FK(t)\middle|\mathcal{F}_{t-}\right] dW_{\mathbb{Q}}(t) \\ &\quad + \int_0^T \int_{\mathbb{R}_0} \mathbb{E}_{\mathbb{Q}}[F(\tilde{H}(t,z)-1) + z\tilde{H}(t,z)D_{t,z}F|\mathcal{F}_{t-}]\tilde{N}_{\mathbb{Q}}(dt,dz). \end{aligned}$$

Corollary 1 *Assume in addition to all assumptions of Theorem 1, u and θ are deterministic functions, then we have*

$$F = \mathbb{E}_{\mathbb{Q}}[F] + \sigma\int_0^T \mathbb{E}_{\mathbb{Q}}[D_{t,0}F|\mathcal{F}_{t-}]dW_{\mathbb{Q}}(t) + \int_0^T\int_{\mathbb{R}_0} \mathbb{E}_{\mathbb{Q}}[D_{t,z}F|\mathcal{F}_{t-}]z\tilde{N}_{\mathbb{Q}}(dt,dz).$$

7 Stroock type formula for L^2-Levy functionals

Finally, we introduce a Stroock type formula for L^2-Levy functionals ([5]).

Theorem 2 *Let $F \in \cap_{k=1}^{\infty}\mathbb{D}^{k,2}(\mathbb{R})$. Then, we have*

$$F = \mathbb{E}[F] + \sum_{n=1}^{\infty} I_n(f_n),$$

where,

$$f_k((t_1,z_1),\cdots,(t_k,z_k)) = \frac{\mathbb{E}[D^k_{t_1,z_1,\cdots,t_k,z_k}F]}{k!}$$

for all $k \geq 1$.

Example 1 *Let $F = \int_{[0,T]\times\mathbb{R}} h(s,x)Q(ds,dx)$, where, h is a bounded function and we assume $\int_{\mathbb{R}_0} z^4\nu(dz) < \infty$. Now, we denote $G = F^2$. Then,*

$$G = \mathbb{E}[G] + \sum_{n=1}^{\infty} I_n(f_n),$$

where, $\mathbb{E}[G] = \int_{[0,T]\times\mathbb{R}} h(s,x)^2q(ds,dx)$, $f_1(t_1,z_1) = z_1h(t_1,z_1)^2$, $f_2(t_1,z_1,t_2,z_2) = h(t_1,z_1)h(t_2,z_2)$, and $f_n(t_1,z_1,\cdots,t_n,z_n) = 0, n \geq 3$. Moreover, we have

$$\begin{aligned} \mathbb{E}[G^2] &= \left(\int_{[0,T]\times\mathbb{R}} h(s,x)^2q(ds,dx)\right)^2 + \int_{[0,T]\times\mathbb{R}} z_1^2h(t_1,z_1)^4q(dt_1,dz_1) \\ &\quad +2\int_{([0,T]\times\mathbb{R})^2} h(t_1,z_1)^2h(t_2,z_2)^2q(t_1,z_1)q(dt_2dz_2). \end{aligned}$$

Example 2 *Let* $F = e^{X_T}$*, where,* $X_T = \sigma W_T + \int_0^T \int_{\mathbb{R}_0} z\tilde{N}(dt, dz) \in L^p(\mathbb{P})$ *for all* $p \geq 1$ *and we assume* $\int_{\mathbb{R}_0}(e^z - 1)^2 \nu(dz) < \infty$*. Then,*

$$F = \mathbb{E}[F] + \sum_{n=1}^{\infty} I_n(h_n),$$

where,

$$h_n(t_1, z_1, \cdots, t_n, z_n) = \frac{1}{n!}\mathbb{E}[F] \prod_{i=1}^{n} [\mathbf{1}_{[0,T]\times\{0\}}(t_i, z_i) + \mathbf{1}_{[0,T]\times\mathbb{R}_0}(t_i, z_i) z_i^{-1}(e^{z_i} - 1)]$$

and

$$\mathbb{E}[F] = \exp\left[\frac{1}{2}\sigma^2 T + T\int_{\mathbb{R}_0}(e^z - 1 - z)\nu(dz)\right].$$

Example 3 *Let* $L(x, T) := \int_0^T \delta(X(s) - x)ds \in \cap_{k=1}^{\infty} \mathbb{D}^{k,2}, x \in \mathbb{R}$*, where,* δ *is Dirac's delta function and* $T < \infty$*. Then,* $L(x, T) = \mathbb{E}[L(x, T)] + \sum_{n=1}^{\infty} I_n(f_n)$*, where,* $\mathbb{E}[L(x, T)] = \frac{1}{2\pi}\int_0^T \int_{\mathbb{R}} \mathbb{E}[e^{\sqrt{-1}\ell(X(s)-x)}]d\ell ds$,

$$\begin{aligned}&\mathbb{E}[e^{\sqrt{-1}\ell(X(s)-x)}]\\&= e^{-\sqrt{-1}\ell x}\exp\left(-\frac{\sigma^2 s^2 \ell^2}{2} + s\int_{\mathbb{R}_0}(e^{\sqrt{-1}\ell u} - 1 - \sqrt{-1}\ell \mathbf{1}_{|u|<1})\nu(du)\right),\end{aligned}$$

and

$$\begin{aligned}&f_k(t_1, z_1, \cdots, t_k, z_k)\\&= \frac{1}{2\pi \cdot n!}\int_0^T \int_{\mathbb{R}} \mathbb{E}[e^{\sqrt{-1}\ell(X(s)-x)}]\\&\times \prod_{i=1}^{k}\left(\sqrt{-1}\ell \mathbf{1}_{[0,s]\times\{0\}}(t_i, z_i) + \frac{e^{\sqrt{-1}\ell z_i} - 1}{z_i}\mathbf{1}_{[0,s]\times\mathbb{R}_0}(t_i, z_i)\right) d\ell ds.\end{aligned}$$

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