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On stochastic differential equation for SLE on multiply connected planar domains

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1 Introduction

In 2000, Oded Schramm [S] formulated the *stochastic Loewner evolution* (SLE) on the upper half plane $\mathbb{H}$ with a finding that the possible candidates of the driving processes are $\xi(t)=\sqrt{\kappa}B_t$, where B_t is the standard Brownian motion on $\partial\mathbb{H}$ and κ is a positive constant. The SLE_κ was then produced as the solution of the chordal Loewner equation associated with this driving process.

We aim at extending the SLE to multiply connected domains. Based on recent results in [CFR] on the chordal Komatu-Loewner equation and following lines briefly laid by [BF2], we show that, for a corresponding evolution for a standard slit domain $D=\mathbb{H}\setminus\bigcup_{k=1}^{N}C_k$, the possible candidates of the driving processes are given by the solution $(\xi(t), \mathbf{s}(t))$ of a special Markov type stochastic differential equation, where $\xi(t)$ is a motion on $\partial\mathbb{H}$ and $\mathbf{s}(t)$ is a motion of slits C_k, $1\leq k\leq N$. When no slit is present, it reduces to $\sqrt{\kappa}B_t$ as above. The solution of the SDE is then substituted into the KL equation to produce *stochastic Komatu-Loewner evolution.*

A domain of the form $D=\mathbb{H}\setminus\bigcup_{k=1}^{N}C_k$ is called a *standard slit domain* where $\{C_k\}$ are mutually disjoint line segments parallel to x-axis contained in $\mathbb{H}$. The collection of standard slit domains is denoted by $\mathcal{D}$.

We fix $D\in\mathcal{D}$ and consider a Jordan arc

$$\gamma:[0,t_\gamma)\rightarrow\overline{D},\quad \gamma(0)\in\partial\mathbb{H},\ \gamma(0,t_\gamma)\subset D, 0<t_\gamma\leq\infty. \tag{1.1}$$

For each $t\in[0,t_\gamma)$, let

$$g_t: D\setminus\gamma[0,t]\rightarrow D_t \tag{1.2}$$

be the unique conformal map from $D\backslash\gamma[0,t]$ onto some $D_t=\mathbb{H}\backslash\bigcup_{k=1}^{N}C_k(t)\in\mathcal{D}$ satisfying a *hydrodynamic normalization*

$$g_t(z)=z+\frac{a_t}{z}+o(1),\qquad z\rightarrow\infty. \tag{1.3}$$

a_t is strictly increasing in t with $a_0=0$, that is called *half-plane capacity.* We also define

$$\xi(t)=g_t(\gamma(t))\ (\in\partial\mathbb{H}),\quad 0\leq t<t_\gamma. \tag{1.4}$$

For a Borel set $A \subset \overline{\mathbb{H}}$, we use $\partial_p A$ to denote the boundary of A with respect to the topology induced by the path distance in $\mathbb{H} \setminus A$. For instance, when $A \subset \mathbb{H}$ is a horizontal line segment, then $\partial_p A$ consists of the upper part A^+ and the lower part A^- of the line segment A.

In §8 of [CFR], the following continuity properties of those quantities mentioned above are established:

(P.1) For every $0 < s < t_\gamma$, $g_t(z)$ is jointly continuous in $(t,z) \in [0,s] \times ((D \times \partial_p K \cup \partial\mathbb{H}) \setminus \gamma[0,s])$, where $K = \bigcup_{k=1}^N C_k$.

(P.2) a_t is continuous in $t \in [0, t_\gamma)$ so that the arc γ can be reparametrized in a way that $a_t = 2t, \ 0 \le t < t_\gamma$, which is called *half-plane capacity parametrization.*

(P.3) $\xi(t) \in \partial\mathbb{H}$ is continuous in $t \in [0, t_\gamma)$.

(P.4) $D_t \in \mathcal{D}$ is continuous in $t \in [0, t_\gamma)$ with respect to the topology in $\mathcal{D}$ described in the beginning of §3.

Historically $g_t(z)$ has been obtained by solving the extremal problem to maximize the coefficient a_t among all univalent functions on $D \setminus \gamma[0,t]$ with the hydrodynamic normalizaion. it follows that a_t is strictly increasing. But, in order to prove the above continuity properties, we need to use the next probabilistic representation of $g_t(z)$ shown in §7 of [CFR]:
Let $Z^{\mathbb{H},*} = (Z_t^{\mathbb{H},*}, \mathbb{P}_z^{\mathbb{H},*}), \ z \in D^*$, be the *Brownian motion with darning (BMD)* for D and let $F_t = \gamma[0,t], \quad \Gamma_r = \{z = x + iy : y = r\}, \ r > 0$. Then

$$\Im\, g_t(z) = \lim_{r \to \infty} r \cdot \mathbb{P}_z^{\mathbb{H},*}(\sigma_{\Gamma_r} < \sigma_{F_t}), \tag{1.5}$$

which was first obtained in [L] for *Excursion reflected Brownian motion* formulated there in place of BMD.

It is proved in [CFR, Theorem 9.9] that the family $g_t(z)$ satisfies the *Komatu-Loewner equation* under the half-plane capacity parametrization of γ:

$$\frac{dg_t(z)}{dt} = -2\pi\Psi_t(g_t(z), \xi(t)), \quad g_0(z) = z \in (D \cup \partial_p K) \setminus \gamma[0, t_\gamma), \quad 0 \le t < t_\gamma, \tag{1.6}$$

where $\Psi_t(z,\xi), \ z \in D_t, \ \xi \in \partial\mathbb{H}$, is the *BMD-complex Poisson kernel* for D_t, namely, the unique analytic function in z vanishing at ∞ whose imaginary part is the Poisson kernel of the BMD for the standard slit domain D_t.

The ODE (1.6) has been obtained in [BF2] and in its original version by Y. Komatu [K], but only in the sense of left derivative with respect to t. The differentiability of $g_t(z)$ in t is established in [CFR] by combining **(P.1)**,

(P.3), **(P.4)** with a Lipschitz continuity of the BMD complex Poisson kernel $\Psi(z,\xi)$ of $D \in \mathcal{D}$.

This is a résumé of a part of my joint work with Zhen-Qing Chen.

2 Bauer-Friedrich equation of slit motion

For a standard slit domain $D = \mathbb{H} \setminus \bigcup_{k=1}^{N} C_k$, the left and right endpoints of the k-th-slit C_k are denoted by $z_k = x_k + iy_k$ and $z'_k = x'_k + iy'_k$, respectively. The Jordan arc γ will be parametrized by the half-plane capacity which is possible by **(P.2)**. For $t \in [0, t_\gamma)$, the conformal map g_t from $D \setminus \gamma[0,t]$ onto D_t can be extended analytically to $\partial_p K$ in the following manner.

We fix $1 \le j \le N$. C_j^0 denotes $C_j \setminus \{z_j,\ z'_j\}$. We consider the open rectangles

$$R_+ = \{z : x \in (x_j, x'_j),\ y \in (y_j, y_j+\delta)\},\ R_- = \{z : x \in (x_j, x'_j),\ y \in (y_j-\delta, y_j)\},$$

and $R = R_+ \cup C_j^0 \cup R_-$ for $\delta > 0$ with $R_+ \cup R_- \subset D \setminus \gamma[0, t_\gamma)$. Since $\Im g_t(z)$ takes a constant value at C_j, g_t can be extended to an analytic function g_t^+ (resp. g_t^-) from R_+ (resp. R_-) to R across C_j^0 by the Schwarz reflection.

We next take $\varepsilon > 0$ with $\varepsilon < \frac{x'_j - x_j}{2}$ so that $B(z_j, \varepsilon) \setminus C_j \subset D \setminus \gamma[0, t_\gamma]$. Then $\psi(z) = (z - z_j)^{1/2}$ maps $B(z_j, \varepsilon) \setminus C_j$ conformally onto $B(0, \sqrt{\varepsilon}) \cap \mathbb{H}$. As in the proof of [CFR, Theorem 7.4], $f_t^\ell(z) = g_t \circ \psi^{-1}(z) = g_t(z^2 + z_j)$ can be extended to be analytic in $z \in B(0, \sqrt{\varepsilon})$ by the Schwarz reflection and by noting that the origin 0 is a removable singularity for f_t^ℓ. Analogously we can induce an analytic function f_t^r on $B(0, \sqrt{\varepsilon})$ from g_t on $B(z'_j, \varepsilon) \setminus C_j$.

Theorem 2.1 *The endpoints $z_j(t) = x_j(t) + iy_j(t)$, $z'_j(t) = x'_j(t) + iy_j(t)$, of the slit $C_j(t)$ satisfy the following equations for $1 \le j \le N$:*

$$\frac{d}{dt} y_j(t) = -2\pi \Im \Psi_t(z_j(t), \xi(t)), \tag{2.1}$$

$$\frac{d}{dt} x_j(t) = -2\pi \Re \Psi_t(z_j(t), \xi(t)), \tag{2.2}$$

$$\frac{d}{dt} x'_j(t) = -2\pi \Re \Psi_t(z'_j(t), \xi(t)), \tag{2.3}$$

If

$$g_t(z_j) = z_j(t), \quad g_t(z'_j) = z'_j(t), \quad t \in (0, t_\gamma),\ 1 \le j \le N, \tag{2.4}$$

then Theorem 2.1 is merely a special case of the Komatu-Loewner equation (1.6) with $z = z_j,\ z = z'_j,\ 1 \le j \le N$. But we do not know the validity of (2.4) in advance so that Theorem 2.1 requires a proof.

Its proof can be carried out by using the analytic extensions of the map g_t to $\partial_p K$ as are described in the paragraph preceding Theorem 2.1. Very roughly speaking, the derivative '$\frac{d}{dx} g_t(z)$' is then shown to be a C^1-function in two variables $t \ge 0$ and $z \in \partial_p K$. Further, by a complex analytic argument,the pre-image $\widetilde{z}_j(t) \in \partial_p C_j$ of $z_j(t)$ under g_t is proved to satisfy $\frac{d}{dz} g_t(\widetilde{z}_j(t)) = 0,\ \frac{d^2}{dz^2} g_t(\widetilde{z}_j(t)) \neq 0$, and an implicit function theorem yields Theorem 2.1.

We can now combine Theorem 2.1 with a local uniqueness of the solution of (1.6) as will be described in Proposition 4.3 below to conclude that (2.4) is actually the case.

We call (2.1)-(2.3) the *Bauer-Friedrich equation* as it first appeared in [BF1, BF2].

3 Randomized curve γ and induced process W

3.1 Random curve with domain Markov property and conformal invariance

Let $\mathcal{D}$ be the collection of all (labelled) standard slits domains.
For $D, \widetilde{D} \in \mathcal{D}$, define the distance $d(D, \widetilde{D})$ by

$$d(D, \widetilde{D}) = \max_{1 \le k \le N} (|z_k - \widetilde{z}_k| + |z'_k - \widetilde{z}'_k|).$$

We define an open subset S of the Euclidean space $\mathbb{R}^{3N}$ by

$$\begin{aligned} S \;=\; & \{(\mathbf{y}, \mathbf{x}, \mathbf{x}') \in \mathbb{R}^{3N} :\ \mathbf{y},\ \mathbf{x},\ \mathbf{x}' \in \mathbb{R}^N,\ \mathbf{y} > \mathbf{0},\ \mathbf{x} < \mathbf{x}', \\ & \text{either } x'_j < x_k \text{ or } x'_k < x_j \text{ whenever } y_j = y_k,\ j \neq k\}. \end{aligned}$$

The space $\mathcal{D}$ can be identified with S as a topological space. We write $\mathbf{s}(D)$ (resp. $D(\mathbf{s})$) the element in S (resp. $\mathcal{D}$) corresponding to $D \in \mathcal{D}$ (resp. $\mathbf{s} \in S$).

A set $F \subset \mathbb{C}$ is called a compact $\mathbb{H}$-hull if $\overline{F}$ is a compct continuum, $F = \overline{F} \cap \mathbb{H}$ and $\mathbb{H} \setminus F$ is simply connected. We let

$$\widehat{\mathcal{D}} = \{\widehat{D} = D \setminus F : D \in \mathcal{D},\ F \text{ compact } \mathbb{H}\text{−hull},\ F \cap \mathbb{H} \subset D\}.$$

For $\widehat{D} \in \widehat{\mathcal{D}}$, let

$$\begin{aligned} \Omega(\widehat{D}) \;=\; & \{\gamma = \{\gamma(t) : 0 \le t < t_\gamma\} : \text{Jordan arc}, \\ & \gamma(0, \infty) \subset \widehat{D},\ \gamma(0) \in \partial(\mathbb{H} \setminus F),\ 0 < t_\gamma \le \infty\Big\}. \end{aligned}$$

Two curves $\gamma,\ \widetilde{\gamma} \in \Omega(\widehat{D})$ are regarded to be equivalent if $\widetilde{\gamma}$ is obtained from γ by a reparametrization. $\dot{\Omega}(\widehat{D})$ will designate the family of the equivalence classes of $\Omega(\widehat{D})$.

Given $\gamma \in \Omega(\widehat{D})$, the associated conformal map g_t from $\widehat{D} \setminus \gamma[0,t]$ to $D_t \in \mathcal{D}$ (for $t \in [0, t_\gamma)$) is required to satisfy the hydrodynamic normalization (1.3). Due to **(P.2)**, the curve γ admits its half-plane capacity reparametrization.

Each $\dot{\gamma} \in \dot{\Omega}(\widehat{D})$ will be represented by a curve (denoted by $\dot{\gamma}$ again) belonging to this class parametrized by half-plane capacity. We conventionally adjoin an extra point Δ to $\overline{\mathbb{H}}$ and define $\dot{\gamma}(t) = \Delta$ for $t \geq t_{\dot{\gamma}}$ so that $\dot{\gamma}$ can be regarded as a map from $[0,\infty]$ to $\overline{\mathbb{H}} \cup \{\Delta\}$. We then introduce σ-fields of subsets of $\dot{\Omega}(\widehat{D})$ by

$$\dot{\mathcal{G}}_t(\widehat{D}) = (\sigma\{\dot{\gamma}(s):\ 0 \leq s \leq t\}) \cap \{t < t_{\dot{\gamma}}\},\ t \geq 0, \quad \dot{\mathcal{G}}(\widehat{D}) = \sigma\{\dot{\gamma}(s) : s \geq 0\}.$$

For each $\widehat{D} \setminus F \in \widehat{\mathcal{D}}$ and $z \in \partial(\mathbb{H} \setminus F)$, we consider a probability measure $\mathbb{P}_{\widehat{D},z}$ on $(\dot{\Omega}(\widehat{D}), \dot{\mathcal{G}}(\widehat{D}))$ satisfying

$$\mathbb{P}_{\widehat{D},z}(\{\dot{\gamma}(0) = z\}) = 1. \tag{3.1}$$

and further **(DMP)** and **(CI)** stated below.

For each $D \in \mathcal{D}$ and $t \geq 0$, define the shift operator $\dot{\theta}_t : \dot{\Omega}(D) \cap \{t < t_{\dot{\gamma}}\} \mapsto \dot{\Omega}(D \setminus \dot{\gamma}[0,t])$ by $(\dot{\theta}_t \dot{\gamma})(s) = \dot{\gamma}(t+s),\ s \in [0, t_{\dot{\gamma}} - t)$.

(DMP) (domain Markov property): for any $t \geq 0$ and any $D \in \mathcal{D}$,

$$\mathbb{P}_{D,z}\left(\dot{\theta}_t^{-1}\Lambda | \dot{\mathcal{G}}_t(D)\right) = \mathbb{P}_{D \setminus \dot{\gamma}[0,t], \dot{\gamma}(t)}(\Lambda), \quad \forall \Lambda \in \dot{\mathcal{G}}(D \setminus \dot{\gamma}[0,t]), \quad \forall z \in \partial\mathbb{H}. \tag{3.2}$$

(CI) (conformal invariance): for any $\widehat{D} = D \setminus F \in \widehat{\mathcal{D}}$ and any conformal map f from $\widehat{D}$ onto $f(\widehat{D}) \in \widehat{\mathcal{D}}$,

$$\mathbb{P}_{f(\widehat{D}), f(z)} = f_* \cdot \mathbb{P}_{\widehat{D},z}, \quad \forall z \in \partial(\mathbb{H} \setminus F). \tag{3.3}$$

3.2 Markov property, Brownian scaling property and homogeneity of W

For each $D \in \mathcal{D}$, $\dot{\gamma} \in \dot{\Omega}(D)$ and $t \in [0, t_{\dot{\gamma}})$, $\dot{\gamma}$ induces the conformal map g_t from $D \setminus \dot{\gamma}[0,t]$ onto $D_t = g_t(D) \in \mathcal{D}$, which sends $\dot{\gamma}(t)$ to $\xi(t)$. Let $\{\mathbf{s}(t) = \mathbf{s}(D_t),\ t \in [0, t_{\dot{\gamma}})\}$ be the induced slit motion, where D_0 denotes D. We then consider a joint process

$$\mathbf{W}_t = \begin{cases} (\xi(t), \mathbf{s}(t)) \in \mathbb{R} \times S \subset \mathbb{R}^{3N+1}, & 0 \leq t < t_{\dot{\gamma}}, \\ \delta, & t \geq t_{\dot{\gamma}}, \end{cases}$$

where δ is an extra point conventionally adjoined to $\mathbb{R} \times S$.
We shall occasionally write $\mathbf{s}(t)$ as $g_t(\mathbf{s})$ for $\mathbf{s} = \mathbf{s}(D)$.

For $\xi \in \mathbb{R}$ and $\mathbf{s} \in S$, define a probability measure $\mathbb{P}_{(\xi,\mathbf{s})}$ on $(\dot{\Omega}(D(\mathbf{s})), \dot{\mathcal{G}}(D(\mathbf{s})))$ by

$$\mathbb{P}_{(\xi,\mathbf{s})} = \mathbb{P}_{D(\mathbf{s}),(\xi,0)}.$$

Theorem 3.1 (time homogeneous Markov property of $(\mathbf{W}_t, \mathbb{P}_{(\xi,\mathbf{s})})$)
$\{\mathbf{W}_t\}$ is $\{\dot{\mathcal{G}}_t(D(\mathbf{s}(0))\}$-adapted. It holds for any $\xi \in \mathbb{R}, \mathbf{s} \in S$ that

$$\mathbb{P}_{(\xi,\mathbf{s})}(\mathbf{W}_0 = (\xi, \mathbf{s})) = 1, \tag{3.4}$$

$$\mathbb{P}_{(\xi,\mathbf{s})}\left(\mathbf{W}_{t+s} \in B \mid \dot{\mathcal{G}}_t(D(\mathbf{s}))\right) = \mathbb{P}_{\mathbf{W}_t}(\mathbf{W}_s \in B), \quad t,\ s \geq 0,\ B \in \mathcal{B}(\mathbb{R} \times S). \tag{3.5}$$

Theorem 3.2 (Brownian scaling property of $(\mathbf{W}_t,\ \mathbb{P}_{(\xi,\mathbf{s})})$)
For $\mathbf{s} \in S$, $\xi \in \mathbb{R}$ and any $c > 0$

$$\{c^{-1}\mathbf{W}_{c^2t},\ t \geq 0\} \text{ under } \mathbb{P}_{(c\xi,c\mathbf{s})} \ \sim\ \{\mathbf{W}_t,\ t \geq 0\} \text{ under } \mathbb{P}_{(\xi,\mathbf{s})}. \tag{3.6}$$

For $\eta \in \mathbb{R}$, denote by $\widehat{\eta}$ the $3N$-vector with the first N-entries 0 and the next $2N$-entries η. Notice that

$$\mathbf{s}(D + \eta) = \mathbf{s}(D) + \widehat{\eta}, \quad \text{for} \quad D \in \mathcal{D},\ \eta \in \mathbb{R}.$$

Theorem 3.3 (Homogeneity of $(\mathbf{W}_t,\ \mathbb{P}_{(\xi,\mathbf{s})})$ in x-dirction)
For $\mathbf{s} \in S$, $\xi \in \mathbb{R}$ and any $\eta \in \mathbb{R}$

$$\{(\xi(t)-\eta, \mathbf{s}(t)-\widehat{\eta}),\ t \geq 0\} \text{ under } \mathbb{P}_{(\xi+\eta,\mathbf{s}+\widehat{\eta})} \ \sim\ \{(\xi(t), \mathbf{s}(t)),\ t \geq 0\} \text{ under } \mathbb{P}_{(\xi,\mathbf{s})}. \tag{3.7}$$

3.3 Stochastic differential equation for W

We write $\mathbf{w} = (\xi, \mathbf{s}) \in \mathbb{R} \times S$. We have shown by Theorem 3.1 that $(\mathbf{W}_t, \mathbb{P}_{\mathbf{w}})$ is a time homogeneous Markov process taking value in $\mathbb{R} \times S \subset \mathbb{R}^{3N+1}$. Its sample path is continuous up to the life time $t_{\dot{\gamma}} \leq \infty$ owing to **(P.3)** and **(P.4)**. Denote by P_t its transition semigroup defined as $P_t f(\mathbf{w}) = \mathbb{E}_{\mathbf{w}}[f(W_t)],\ t \geq 0,\ \mathbf{w} \in \mathbb{R} \times S$.

Denote by $C_\infty(\mathbb{R} \times S)$ the space of all continuous functions on $\mathbb{R} \times S$ vanishing at infinity. In this section, we shall assume that $\{P_t; t > 0\}$ satisfies the following property:

(C) $P_t(C_\infty(\mathbb{R}\times S))\subset C_\infty(\mathbb{R}\times S),\ t>0,\quad C_c^\infty(\mathbb{R}\times S)\subset\mathcal{D}(L),$

where L is the infinitesimal generator of $\{P_t,\ t>0\}$ defined by

$$\begin{aligned} Lf(\mathbf{w}) &= \lim_{t\downarrow 0}\frac{1}{t}(P_tf(\mathbf{w})-f(\mathbf{w})),\ \mathbf{w}\in\mathbb{R}\times S,\\ \mathcal{D}(L) &= \{f\in C_\infty(\mathbb{R}\times S):\ \text{the right hand side above}\\ &\qquad \text{converges uniformly in } \mathbf{w}\in\mathbb{R}\times S\}. \end{aligned} \tag{3.8}$$

Then $(\mathbf{W}_t,\mathbb{P}_{\mathbf{w}})$ is a *Feller-Dynkin diffusion* in the sense of [RW]. In view of [RW, III, (13.3)], the restriction $\mathcal{L}$ of L to $C_c^\infty(\mathbb{R}\times S)$ is a second order elliptic partial differential operator expresed as

$$\mathcal{L}f(\mathbf{w})=\frac{1}{2}\sum_{i,j=1}^{3N+1}a_{ij}(\mathbf{w})f_{w_iw_j}(\mathbf{w})+\sum_{i=1}^{3N+1}b_i(\mathbf{w})f_{w_i}(\mathbf{w})+k(\mathbf{w})f(\mathbf{w}),\ \mathbf{w}\in\mathbb{R}\times S, \tag{3.9}$$

where a is a non-negative definite symmetric matrix valued continuous function, b is a vector valued continuous function and k is a non-poisitive continuous function.

A real funcion $u(\mathbf{w})=u(\xi,\mathbf{s})$ on $\mathbb{R}\times S$ is called *homogeneous with degree* 0 (resp. -1) if

$$u(c\mathbf{w})=u(\mathbf{w})\quad(\text{ resp. } u(c\mathbf{w})=\frac{1}{c}u(\mathbf{w})\ )\quad\text{for any}\quad c>0.$$

The same definition of the homogeneity is in force for a real function $u(\mathbf{s})$ on S.

Lemma 3.4 (i) *$a_{ij}(\mathbf{w})$ is a homogenous function of degree 0 for every $0\le i,j\le 3N+1$, while $b_i(\mathbf{w})$ is a homogenous function of degree -1 for every $1\le i\le 3N+1$. $k(\mathbf{w})$ vanishes identically.*
(ii) *For every $1\le i,j\le 3N+1$,*

$$a_{ij}(\xi+\eta,\mathbf{s}+\widehat{\eta})=a_{ij}(\xi,\mathbf{s}),\ \ b_i(\xi+\eta,\mathbf{s}+\widehat{\eta})=b_i(\xi,\mathbf{s}), \tag{3.10}$$

for any $\xi\in\mathbb{R},\ \mathbf{s}\in S,\ \eta\in\mathbb{R}$.

Now (3.8) implies that

$$P_tf(\mathbf{w})-f(\mathbf{w})=\int_0^t P_s(\mathcal{L}f)(\mathbf{w})ds,\ t\ge 0,\ \mathbf{w}\in\mathbb{R}\times S,\ f\in C_c^\infty(\mathbb{R}\times S). \tag{3.11}$$

We denote by $W_t^{(j)}$ the j-th coordinate of the process $\mathbf{W}_t$ so that

$$W_t^{(1)} = \xi(t), \quad (W_t^{(2)}, \cdots, W_t^{(3N+1)}) = \mathbf{s}(t).$$

On account of [RY, VII,(2.4)], (3.9) and (3.11) imply that the process

$$M_t^j = W_t^{(j)} - W_0^{(j)} - \int_0^t b_j(\mathbf{W}_s)ds, \quad t \geq 0, \ 1 \leq j \leq 3N+1,$$

are local martingales with

$$\langle M^j, M^k \rangle_t = \int_0 a_{jk}(\mathbf{W}_s)ds, \quad t \geq 0, \quad 1 \leq j, k \leq 3N+1. \tag{3.12}$$

Recall that, for $\mathbf{s} = (\mathbf{y}, \mathbf{x}, \mathbf{x}')$. $\mathbf{y}, \mathbf{x}, \mathbf{x}' \in \mathbb{R}^N$, $z_j = x_j + iy_j$, $z_j' = x_j' + iy_j$ are the endpoints of the slit C_j in $D(\mathbf{s}) \in \mathcal{D}$ $1 \leq j \leq N$. For $\mathbf{s} \in S$, let $\Psi_{\mathbf{s}}(z, \xi)$ be the complex Poisson kernel of the Brownian motion with darning (BMD) on $D(\mathbf{s})$. Then the Bauer-Friedich equation (2.1)-(2.3) established in §2 reads

$$\mathbf{s}_j(t) - \mathbf{s}_j(0) = \int_0^t d_j(\mathbf{W}(s))ds, \quad t \geq 0, \tag{3.13}$$

for the function $d_j(\mathbf{w}) = d_j(\xi, \mathbf{s})$ defined by

$$d_j(\mathbf{w}) = \begin{cases} -2\pi \Im \Psi_{\mathbf{s}}(z_j, \xi), & 1 \leq j \leq N, \\ -2\pi \Re \Psi_{\mathbf{s}}(z_j, \xi), & N+1 \leq j \leq 2N, \\ -2\pi \Re \Psi_{\mathbf{s}}(z_j', \xi), & 2N+1 \leq j \leq 3N. \end{cases} \tag{3.14}$$

In particular, we are left with one martingale M^1:

$$M^j = 0, \quad 2 \leq j \leq 3N+1. \quad \langle M^1, M^1 \rangle_t = \int_0^t a_{11}(\mathbf{W}_s)ds., \ t \geq 0.$$

Theorem 3.5 (i) *The diffusion* $\mathbf{W}_t = (\xi(t), \mathbf{s}(t))$ *satisfies under* $\mathbb{P}_{(\xi,\mathbf{s})}$ *the following stochastic differential equation:*

$$\xi(t) = \xi + \int_0^t \alpha(\mathbf{s}(s) - \widehat{\xi}(s))dB_s + \int_0^t d(\mathbf{s}(s) - \widehat{\xi}(s))ds \tag{3.15}$$

$$\mathbf{s}_j(t) = \mathbf{s}_j + \int_0^t d_j(\xi(s), \mathbf{s}(s))ds, \quad t \geq 0, \quad 1 \leq j \leq 3N, \tag{3.16}$$

for a non-negative homogeneous function $\alpha(\mathbf{s})$ *of* $\mathbf{s} \in S$ *with degree* 0, *a homogeneous function* $d(\mathbf{s})$ *of* $\mathbf{s} \in S$ *with degree* -1 *and the functions*

$d_j((\xi,\mathrm{s}))$, $1 \le j \le 3N$, *given by* (3.14). *Here* B_t *is a one-dimensional standard Brownian motion and* $\widehat{\xi}(s)$ *denotes the* $3N$*-vector with the first* N*-entries* 0 *and the next* $2N$*-entries* $\xi(s)$.
(ii) $d_j(0,\mathrm{s})$ *is a homogeneous function of* s *with degree* -1 *and*

$$d_j(\xi+\eta,\mathrm{s}+\widehat{\eta}) = d_j(\xi,\mathrm{s}), \quad \xi \in \mathbb{R},\ \mathrm{s} \in S,\ \eta \in \mathbb{R},\ 1 \le j \le 3N. \tag{3.17}$$

4 Stochastic Komatu-Loewner evolution

4.1 Solving the SDE for given coefficients (α, d)

We consider the following condition for a real function $f = f(\mathrm{s})$ on S:

(L) For any $\mathrm{s}_0 \in S$ and any finite open interval $J \subset \mathbb{R}$, there exist a neighborhood $U(\mathrm{s}_0)$ of s_0 in S and a constant $L > 0$ such that

$$|f(\mathrm{s}_1 - \widehat{\xi}) - f(\mathrm{s}_2 - \widehat{\xi})| \le L\,|\mathrm{s}_1 - \mathrm{s}_2|, \quad \mathrm{s}_1,\ \mathrm{s}_2 \in U(\mathrm{s}_0), \quad \xi \in J, \tag{4.1}$$

where $\widehat{\xi}$ is the $3N$-vector with the first N-entries 0 and the next $2N$-entries ξ.

Recall that the coefficient $d_j(\xi,\mathrm{s})$ in the equation (3.16) is defined by (3.14) and satisfies

$$d_j(\xi,\mathrm{s}) = \widetilde{d}_j(\mathrm{s}-\widehat{\xi}), \quad \text{for} \quad \widetilde{d}_j(\mathrm{s}) = d_j(0,\mathrm{s}), \quad \mathrm{s} \in S,\ \xi \in \mathbb{R},\ 1 \le j \le 3N, \tag{4.2}$$

by virtue of (3.17).

Lemma 4.1 (i) *The function* $\widetilde{d}_j(\mathrm{s})$, $\mathrm{s} \in S$, *satisfies condition* **(L)** *for every* $1 \le j \le 3N$.
(ii) *If a function* f *on* S *satisfies the condition* **(L)**, *then it holds for any* $\mathrm{s}_1,\ \mathrm{s}_2 \in U(\mathrm{s}_0)$ *and for any* $\xi_1,\ \xi_2 \in J$ *that*

$$|f(\mathrm{s}_1 - \widehat{\xi}_1) - f(\mathrm{s}_2 - \widehat{\xi}_2)| \le L\left(|\mathrm{s}_1 - \mathrm{s}_2| + \sqrt{2N}|\xi_1 - \xi_2|\right). \tag{4.3}$$

In this and the next sections, we assume that we are given a non-negative homogeneous function $\alpha(\mathrm{s})$ of $\mathrm{s} \in S$ with degree 0 and a homogeneous function $d(\mathrm{s})$ of $\mathrm{s} \in S$ with degree -1 both satisfying the condition **(L)**.

Theorem 4.2 *The SDE* (3.15), (3.16) *admits a unique strong solution* $\mathbf{W}_t = (\xi(t),\mathrm{s}(t))$, $t \in [0,\zeta)$, *where* ζ *is the time when* $\mathbf{W}_t$ *approaches the point at infinity of* $\mathbb{R} \times S$.

4.2 Stochastic Komatu-Loewner evolution

Let us consider a solution $\mathbf{W}_t = (\xi(t), \mathbf{s}(t)),\ t \in [0, \zeta)$, of the SDE (3.15) and (3.16) obtained in Theorem 4.2. We write $D_t = d(\mathbf{s}(t)) \in \mathcal{D},\ t \in [0, \zeta)$. D_0 is denoted by D.

We substitute $(\xi(t), s(t))$ into the Komatu-Loewner equation

$$\frac{d}{dt} z(t) = -2\pi \Psi_{\mathbf{s}(t)}(z(t), \xi(t)). \tag{4.4}$$

We consider solutions $z(t)$ of (4.4) with the initial condition

$$z(\tau) = z_0 \in D_\tau \cup \partial_p K(\tau) \cup (\mathbb{H} \setminus \xi(\tau)), \tag{4.5}$$

for any initial time $\tau \in [0, \zeta)$ and any initial position z_0.

For each $1 \le j \le N$, $\partial_p C_j^0 = C_j^{0,+} \cup C_j^{0,-}$ will denote the set $\partial_p C_j$ with its two endpoints being removed. We further let $\partial_p K^0 = \cup_{j=1}^N \partial_p C_j^0$.

Proposition 4.3 *Take any* $\tau \in [0, \zeta)$.
(i) *For each* $1 \le j \le N$ *and for* $z_0 = z_j(\tau)$ (resp. $z_0 = z_j'(\tau)$), $\{z_j(t),\ t \in [0, \zeta)\}$ (resp. $\{z_j'(t),\ t \in [0, \zeta)\}$) *is the unique solution of* (4.4) *satisfying* $z(\tau) = z_0$.
(ii) *For each* $1 \le j \le N$ *and for* $z_0 \in C_j^{0,+}(\tau)$ (resp. $z_0 \in C_j^{0,-}(\tau)$), *there exists a unique solution* $\{z(t),\ t \in [0, \zeta)\}$ *of* (4.4) *satisfying* $z(\tau) = z_0$. *It satisfies that* $z(t) \in C_j^{0,+}(t)$ (resp. $z(t) \in C_j^{0,-}(t)$) *for every* $t \in [0, \zeta)$.
(iii) *For* $z_0 \in \partial\mathbb{H} \setminus \xi(\tau)$, *there exists a unique solution* $\{z(t),\ t \in (t_{\tau,z_0}^-, t_{\tau,z_0}^+)\}$ *of* (4.4) *satisfying* $z(\tau) = z_0$. *It satisfies that* $z(t) \in \partial\mathbb{H}$ *for every* $t \in (t_{\tau,z_0}^-, t_{\tau,z_0}^+)$. *Here*

$$\begin{cases} t_{\tau,z_0}^- = \inf\{t \in [0, \tau) : \inf_{s \in [t,\tau)} |z(s) - \xi(s)| > 0\}, \\ t_{\tau,z_0}^+ = \sup\{t \in (\tau, \zeta) : \inf_{s \in (\tau,t]} |z(s) - \xi(s)| > 0\}. \end{cases}$$

(iv) *For* $z_0 \in D_\tau$, *there exists a unique solution* $\{z(t),\ t \in [0, t_{\tau,z_0})\}$ *of* (4.4). *It satisfies that* $z(t) \in D_t$ *for every* $t \in [0, t_{\tau,z_0})$. *Here*

$$t_{\tau,z_0} = \sup\{t \in (\tau, \zeta) : \inf_{s \in (\tau,t]} |z(s) - \xi(s)| > 0\}. \tag{4.6}$$

By Proposition 4.3 (iv), we see that, for each $z \in D$, there exists a unique solution $z(t) \in D_t,\ t \in [0, t_z)$, of the equation (4.4) with initial condition $z(0) = z$. Here

$$t_z = \sup\{t \in (0, \zeta) : \inf_{s \in (0,t]} |z(s) - \xi(s)| > 0\}. \tag{4.7}$$

We let

$$F_t = \{z \in D : t_z \leq t\}, \quad t > 0. \tag{4.8}$$

Theorem 4.4 (i) *There exists a unique solution $g_t(z)$, $t \in [0, t_z)$, of the equation*

$$\frac{d}{dt} g_t(z) = -2\pi \Psi_{\mathbf{s}(t)}(g_t(z), \xi(t)), \quad g_0(z) = z \in D. \tag{4.9}$$

g_t is a one-to-one map from $D \setminus F_t$ onto D_t for each $t > 0$.
(ii) *F_t is a bounded closed subset of $\mathbb{H}$. $\mathbb{H} \setminus F_t$ is simply connected.*
For each $t > 0$, g_t is a conformal map from $D \setminus F_t$ onto D_t.
(iii) *$g_t(z)$ satisfies the hydrodynamic normalization condition at infinity.*
(iv) *F_t is strictly increasing in t.*

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