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L^p-theory for second-order elliptic operators with unbounded drift

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1. Introduction and result

In this paper we deal with the second-order elliptic operators of the form

$$Au(x) := -\mathrm{div}(a(x)\nabla u(x)) + F(x)\cdot\nabla u(x) + V(x)u(x), \quad x\in\mathbb{R}^N,$$

where $N\in\mathbb{N}$ and the coefficients a, F, V are assume to be satisfy the following condition:

(A1) ${}^t a = a \in C^1(\mathbb{R}^N;\mathbb{R}^{N\times N})$, $F\in C^1(\mathbb{R}^N;\mathbb{R}^N)$, $V\in L^\infty_{\mathrm{loc}}(\mathbb{R}^N;\mathbb{R})$ and $a(x)$ is positive-definite for every $x\in\mathbb{R}^N$, that is, $\langle a(x)\xi,\xi\rangle>0$ for every $x\in\mathbb{R}^N$, $\xi\in\mathbb{C}^N\setminus\{0\}$.

Here $\langle\cdot,\cdot\rangle$ is the usual Hermitian product. The boundedness of a, F, V is **not required**. Under condition **(A1)** we define the minimal and maximal realization of A in $L^p(\mathbb{R}^N)$ $(1<p<\infty)$ respectively as

$$A_{p,\min}u := Au, \quad D(A_{p,\min}) := C_0^\infty(\mathbb{R}^N),$$
$$A_{p,\max}u := Au, \quad D(A_{p,\max}) := \{u\in L^p(\mathbb{R}^N)\cap W^{2,p}_{\mathrm{loc}}(\mathbb{R}^N)\,;\, Au\in L^p(\mathbb{R}^N)\}.$$

Our interest is the following properties of $A_{p,\min}$ and $A_{p,\max}$:

- essential m-accretivity of $A_{p,\min}$ (generation of a contraction semigroup $\{e^{-t\tilde{A}_{p,\min}}\}$ by $\tilde{A}_{p,\min}$, the closure of $A_{p,\min}$);
- m-accretivity of $A_{p,\max}$ (generation of a contraction semigroup $\{e^{-tA_{p,\max}}\}$) and coincidence $A_{p,\max}=\tilde{A}_{p,\min}$ ($C_0^\infty(\mathbb{R}^N)$ is a core for $A_{p,\max}$);
- m-sectoriality of $A_{p,\max}$ (analyticity of $\{e^{-tA_{p,\max}}\}$).

These properties of second-order elliptic operators with unbounded coefficients are closely related to those of Kolmogorov and Schrödinger operators.

In particular, if $A=-\Delta+V$ with singular potentials (i.e., $a=(\delta_{jk})$ and $F\equiv 0$, where δ_{jk} is the Kronecker delta), then there are many investigations dealing with the (essential) selfadjointness in $L^2(\mathbb{R}^N)$ and m-accretivity in $L^p(\mathbb{R}^N)$ (see, e.g., Kato [**9**], [**12**], Simon [**20**], Semenov [**19**], Okazawa [**16**] and others). More generally, the operators of the form $-\mathrm{div}(a\nabla)+V$ are considered in Kato [**10**, Section 2] and Kovalenko-Semenov [**13**]. The quasi-m-accretivity of Schrödinger operators with vector potential $(i\nabla+b)^2+F\cdot\nabla+V$ in $L^2(\mathbb{R}^N)$ is dealt with in Kato [**11**] and Okazawa-Yokota [**18**].

Metafune-Pallara-Prüss-Schnaubelt [**14**] obtained the m-accretivity (and sectoriality) of $A_{p,\max}$ when the symmetric diffusion $a\in C^1(\mathbb{R}^N;\mathbb{R}^{N\times N})$ satisfies

$$\langle a(x)x,x\rangle \le K(|x|^2\log|x|)^2, \quad x\in\mathbb{R}^N\ (|x|\ge R),$$

(the generality of which is explained in Eberle [**6**, Theorem 2.3]) and there exists a positive auxiliary function $U \in C^1(\mathbb{R}^N)$ such that

$$(1.1) \qquad 0 < U \le V \le cU, \quad \langle a\nabla U, \nabla U\rangle^{1/2} \le \gamma U^{3/2} + K$$

with some additional inequalities. Their proof of the m-accretivity is based on that of $-\mathrm{div}(a\nabla) + V$ by regarding $F \cdot \nabla$ as a perturbation. Since their argument depends on the so-called separation property

$$\|\mathrm{div}(a\nabla u)\|_{L^p(\mathbb{R}^N)} + \|F \cdot \nabla u\|_{L^p(\mathbb{R}^N)} + \|Vu\|_{L^p(\mathbb{R}^N)} \le C\|u + Au\|_{L^p(\mathbb{R}^N)},$$

it seems that condition (1.1) and the others are necessary.

Recently, Metafune-Pallara-Rabier-Schnaubelt [**15**] succeeded in proving the m-accretivity of $A_{p,\max}$ and identity $A_{p,\max} = \tilde{A}_{p,\min}$ under some conditions weaker than those in [**14**]. For example, they assume that there exist $\rho \in C^N(\mathbb{R}^N)$ and $s > 0$ such that $\nabla\rho \neq 0$ a.e. on $\mathbb{R}^N$ and

$$(1.2) \qquad V - \frac{\mathrm{div}F}{p} + s\Big[F \cdot \nabla\rho - \Big(1 - \frac{2}{p}\Big)\mathrm{div}(a\nabla\rho)\Big] - s^2\langle a\nabla\rho, \nabla\rho\rangle \ge 0.$$

However, condition (1.2) and the others in [**15**] seem to be rather unnatural as a generalization of $p = 2$ to $p \neq 2$.

The first purpose of this paper is to propose **Key Identity** (see below) for the operator A which behaves like a sesquilinear form over $L^p \times L^{p'}$ established in [**21**]. **Key Identity** plays a fundamental role in proving our three theorems for general coefficients. The second is to present a **simple and natural condition** (see **(A2)** below) for the m-accretivity of $\tilde{A}_{p,\min}$ and $A_{p,\max}$ (Theorems 1.1 and 1.2). Actually, we can improve the result in [**15**] to the effect that (1.2) is fairly simplified as (1.7) stated below. The third purpose is to establish the m-sectoriality of $A_{p,\max}$ under some stronger condition (Theorem 1.3). As stated above, the m-sectoriality was shown in [**14**] under the additional condition (1.1), while condition (1.1) is completely removed in our result. To clarify the simplicity (and sharpness in a sense) of our criterion, we give two typical and important examples in Section 3. One is Kato's example in [**10**, Appendix 2] transplanted into $L^p(\mathbb{R}^N)$ from $L^2(\mathbb{R}^N)$ and the other is Arendt-Metafune-Pallara's example in [**2, 3**] concerning $Au = -u'' + x^3u' + c|x|^\gamma u$ in $L^p(\mathbb{R})$.

Now we state **Key Identity** for the operator A.

Key Identity ([**21**]). *Assume that* **(A1)** *is satisfied. Then for every* $1 \le q \le \infty$, $w \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N)$ *and* $\psi \in C_0^\infty(\mathbb{R}^N)$,

$$(1.3) \qquad \int_{\mathbb{R}^N} (A\psi)\overline{w}\,dx = \int_{\mathbb{R}^N} \Big[\langle a\nabla\psi, \nabla w\rangle + \Big(V - \frac{\mathrm{div}F}{q}\Big)\psi\overline{w}\Big]\,dx + \int_{\mathbb{R}^N} \Big[\frac{1}{q'}\langle \overline{w}\nabla\psi, F\rangle - \frac{1}{q}\langle \psi\nabla\overline{w}, F\rangle\Big]\,dx,$$

where q' is the Hölder conjugate of q.

We see that if $w = \psi|\psi|^{q-2}$, the duality map of ψ on $L^q(\mathbb{R}^N)$ to $L^{q'}(\mathbb{R}^N)$ (with multiplying some constant), then the real part of the second term on the right-hand side of (1.3) vanishes. In fact, we can compute it as

$$\begin{aligned}\mathrm{Re}\int_{\mathbb{R}^N}\left[\frac{1}{q'}\langle\overline{w}\nabla\psi, F\rangle - \frac{1}{q}\langle\psi\nabla\overline{w}, F\rangle\right]dx &= \mathrm{Re}\Big(i\int_{\mathbb{R}^N}\langle\mathrm{Im}(\overline{\psi}\nabla\psi)|\psi|^{q-2}, F\rangle\,dx\Big)\\ &= 0.\end{aligned}$$

By virtue of this property, **Key Identity** with $q = p$ plays a crucial role in proving both the accretivity of $A_{p,\min}$ and maxmality of $\tilde{A}_{p,\min}$ and even identity $A_{p,\max} = \tilde{A}_{p,\min}$. This point of view enables us to remove the conditions like (1.1) and (1.2).

Next we state the assumption which will be used to estimate the respective terms on the right-hand side of (1.3).

To state our assumption we introduce the following class of functions $\mathcal{F}_R$ for $R > 0$:

$$\mathcal{F}_R := \Big\{f \in C([R,\infty);\mathbb{R})\ ;\ f > 0 \text{ on } [R,\infty),\quad \int_R^\infty \frac{1}{f(s)}\,ds = \infty\Big\}. \tag{1.4}$$

(A2) There exist constants $\alpha, \beta > 0, r \geq 2, R_0 > 0$ and $f \in \mathcal{F}_{R_0}$ and a nonnegative auxiliary function $\Psi_p \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N;\mathbb{R})$ such that

$$\frac{\langle a(x)x, x\rangle}{|x|^2} \leq \alpha(1 + \Psi_p(x))^{1-\frac{2}{r}} f(|x|)^2 \quad \text{a.a. } x \in \mathbb{R}^N \setminus B_{R_0}; \tag{1.5}$$

$$\frac{\langle F(x), x\rangle}{|x|} \leq \beta(1 + \Psi_p(x))^{1-\frac{1}{r}} f(|x|) \quad \text{a.a. } x \in \mathbb{R}^N \setminus B_{R_0}; \tag{1.6}$$

$$V - \frac{\mathrm{div} F}{p} \geq \Psi_p \quad \text{a.e. on } \mathbb{R}^N, \tag{1.7}$$

where B_R is the N-dimensional ball with center at the origin and radius R. The optimality of **(A2)** with $\Psi_p \equiv 0$ is essentially described in Davies [**5**] (see also [**6**, Remark of Theorem 2.3]).

Remark 1.1 (A class $\mathcal{F}_R$). For example, the function $f_0(s) := s\log s$ $(s \geq e)$ belongs to the class $\mathcal{F}_R$ with $R = e$. Theorem 1.1 with $f = f_0$ is proved in Sobajima [**21**]. Here we describe the general case $f \in \mathcal{F}_R$.

Now we are in a position to state our main result based on **Key Identity**. The first theorem asserts the essential m-accretivity of $A_{p,\min}$.

Theorem 1.1 (Essential m-accretivity). *Let $1 < p < \infty$. Assume that* **(A1)** *and* **(A2)** *are satisfied. Then $A_{p,\min}$ is essentially m-accretive in $L^p(\mathbb{R}^N)$, that is,*

$$\mathrm{Re}\int_{\mathbb{R}^N}(A_{p,\min}u)\overline{u}|u|^{p-2}\,dx \geq 0 \ \ \forall\, u \in D(A_{p,\min}), \qquad \overline{R(1 + A_{p,\min})} = L^p(\mathbb{R}^N),$$

where $R(1 + A_{p,\min})$ is the range of $1 + A_{p,\min}$.

Applying Theorem 1.1 to A and the formal adjoint of A defined as

$$Bv := -\mathrm{div}(a\nabla v) - F\cdot\nabla v + (V - \mathrm{div}F)v, \tag{1.8}$$

we obtain the identity $A_{p,\max} = \tilde{A}_{p,\min}$ with the aid of $(B_{p',\min})^*$, the adjoint operator of $B_{p',\min}$. In this case we need to control the minus sign in front of the first order term in B. This explains the difference of (1.9) from (1.6).

Theorem 1.2 (m-accretivity). *Let $1 < p < \infty$. Assume that* **(A1)** *and* **(A2)** *are satisfied with* (1.6) *replaced with a stronger condition:*

$$\frac{|\langle F(x), x\rangle|}{|x|} \le \beta(1+\Psi_p(x))^{1-\frac{1}{r}} f(|x|) \quad \text{a.a. } x \in \mathbb{R}^N \setminus B_R. \tag{1.9}$$

Then $A_{p,\max}$ is m-accretive in $L^p(\mathbb{R}^N)$, that is,

$$\mathrm{Re}\int_{\mathbb{R}^N} (A_{p,\max}u)\overline{u}|u|^{p-2}\,dx \ge 0 \ \ \forall\, u \in D(A_{p,\max}), \qquad R(1+A_{p,\max}) = L^p(\mathbb{R}^N).$$

Moreover, $C_0^\infty(\mathbb{R}^N)$ is a core for $A_{p,\max}$.

The Hille-Yosida theorem implies that $-A_{p,\max}$ in Theorem 1.2 generates a contraction semigroup $\{e^{-tA_{p,\max}}\}$ on $L^p(\mathbb{R}^N)$. Next we describe the result for analyticity of $\{e^{-tA_{p,\max}}\}$. To show that $\{e^{-zA_{p,\max}}\}$ is an analytic contraction semigroup of type $S(c^{-1})$ (i.e., $\{e^{-zA_{p,\max}}\}$ is analytic and contractive in $S(c^{-1})$), it suffices by [**8**, Theorem 1.5.9] to prove that $A_{p,\max}$ is m-sectorial of type $S(c)$ for some $0 \le c < \infty$, that is, $A_{p,\max}$ is m-accretive and

$$\int_{\mathbb{R}^N} (A_{p,\max}u)\overline{u}|u|^{p-2}\,dx \in S(c) \quad \forall\, u \in D(A_{p,\max}),$$

where $S(c)$ is the closed sector $S(0) := [0,\infty)$, $S(\infty) := \{z \in \mathbb{C}\,;\ \mathrm{Re}\, z \ge 0\}$ and

$$S(c) := \{z \in \mathbb{C}\,;\ |\mathrm{Im} z| \le c\mathrm{Re} z\} \quad (0 < c < \infty).$$

For m-sectoriality we reinforce the assumption further. Roughly speaking, we assume that the first order term can be completely controlled by the diffusion and potential. Then we establish the third theorem which asserts the m-sectoriality of $A_{p,\max}$.

Theorem 1.3 (m-sectoriality). *Let $1 < p < \infty$. Assume that* **(A1)** *and* **(A2)** *are satisfied with* (1.6) *replaced with a stronger condition:*

$$|\langle F(x), \xi\rangle| \le \beta\Psi_p(x)^{\frac{1}{2}}\langle a(x)\xi, \xi\rangle^{\frac{1}{2}}, \quad \text{a.a. } x \in \mathbb{R}^N, \xi \in \mathbb{C}^N. \tag{1.10}$$

Then $A_{p,\max}$ is m-sectorial of type $S(c_{p,\beta})$. In other words, $\{e^{-tA_{p,\max}}\}$ is extended to an analytic contraction semigroup of type $S(c_{p,\beta}^{-1})$ on $L^p(\mathbb{R}^N)$, where

$$c_{p,\beta} := \sqrt{\frac{|p-2|^2}{4(p-1)} + \frac{\beta^2}{4}}.$$

Remark 1.2 (Essential m-accretivity). Even if $V \equiv 0$, we can choose $\Psi_p \not\equiv 0$ in the special case where $-\mathrm{div}F$ is nonnegative and unbounded. This observation means that Theorem 1.1 improves [**6**, Theorem 2.3] which dealt with the case where $V \equiv 0$ and $\Psi_p \equiv 0$.

Remark 1.3 (m-accretivity). Theorem 1.2 is applicable to rapidly oscillating diffusions. For example, we consider the one-dimensional operator A with the following coefficients:

$$a(x) = 2 + \sin(x^3), \quad F(x) = x(\log(1 + x^2))^2, \quad V(x) = (\log(1 + x^2))^2 + \frac{4}{p-1}.$$

These coefficients satisfy the assumption in Theorem 1.2 with $R_0 = e$ and

$$f(s) = s \log s, \quad \Psi_p(x) := \frac{p-1}{2p}(\log(1 + x^2))^2.$$

However, if $p \neq 2$, then because of the singular behavior of $a'(x)$ it is difficult to construct the auxiliary function ρ satisfying (1.2).

Remark 1.4 (m-sectoriality). (**a**) If $F \equiv 0$, then Theorem 1.3 asserts that $A_{p,\max} = -\mathrm{div}(a\nabla) + V$ is m-sectorial of type $S(c_{p,0})$, where the constant $c_{p,0} = |p-2|/(2\sqrt{p-1})$ is already determined in Okazawa [**17**]. By this fact we can regarded Theorem 1.3 as a natural generalization of the result for Schrödinger operators in $L^p(\mathbb{R}^N)$.
(**b**) Kato gave an example in [**10**, Appendix 2] which showed that the Schrödinger operator $-\mathrm{div}(a\nabla) + V$ with unlimited growth diffusion a at infinity is selfadjoint in $L^2(\mathbb{R}^N)$ if the potential grows fast enough at infinity. His viewpoint can be explained from ours. Actually, an L^p-generalization of this fact is described in detail in Section 3.

The plan of this paper is as follows. Theorems 1.1, 1.2 and 1.3 are proved in Section 2. To illustrate the simplicity and sharpness of our criterion, we discuss two typical examples which are considered respectively in [**10**, Appendix 2] and [**2**, Section 6] (and also [**3**]), are given in Section 3.

2. Proofs of theorems via Key Identity

For selfcontainedness we start with

Proof of Key Identity. Using integration by parts, we obtain

$$\int_{\mathbb{R}^N} (-\mathrm{div}(a\nabla\psi))\overline{w}\, dx = \int_{\mathbb{R}^N} \langle a\nabla\psi, \nabla w\rangle\, dx,$$

$$\begin{aligned}\int_{\mathbb{R}^N} (F \cdot \nabla\psi)\overline{w}\, dx &= \frac{1}{q'}\int_{\mathbb{R}^N} \langle \overline{w}\nabla\psi, F\rangle\, dx + \frac{1}{q}\int_{\mathbb{R}^N} \langle \nabla\psi, wF\rangle\, dx \\ &= \frac{1}{q'}\int_{\mathbb{R}^N} \langle \overline{w}\nabla\psi, F\rangle\, dx - \frac{1}{q}\int_{\mathbb{R}^N} \langle \psi\nabla\overline{w}, F\rangle\, dx \\ &\quad - \frac{1}{q}\int_{\mathbb{R}^N} (\mathrm{div}F)\psi\overline{w}\, dx.\end{aligned}$$

Combining the above equalities yields (1.3). □

Now we prove Theorem 1.1 via **Key Identity**.

Proof of Theorem 1.1. It is easy to show that $A_{p,\min}$ is accretive in $L^p(\mathbb{R}^N)$. In fact, if $2 \le p < \infty$, then for every $u \in C_0^\infty(\mathbb{R}^N)$, taking the real part of **Key Identity** with $q = p$, $w = |u|^{p-2}u$ and $\psi = u$, we have that

$$\begin{aligned}\mathrm{Re}\int_{\mathbb{R}^N}(Au)\overline{u}|u|^{p-2}\,dx = (p-1)&\int_{\mathbb{R}^N}|u|^{p-4}\langle a\mathrm{Re}(\overline{u}\nabla u),\mathrm{Re}(\overline{u}\nabla u)\rangle\,dx\\ &+\int_{\mathbb{R}^N}|u|^{p-4}\langle a\mathrm{Im}(\overline{u}\nabla u),\mathrm{Im}(\overline{u}\nabla u)\rangle\,dx\\ &+\int_{\mathbb{R}^N}\left(V-\frac{\mathrm{div}F}{p}\right)|u|^p\,dx.\end{aligned}$$

We see from **(A1)** and (1.7) that

$$\mathrm{Re}\int_{\mathbb{R}^N}(Au)\overline{u}|u|^{p-2}\,dx \ge 0.$$

Note that this inequality is justified even if $1 < p < 2$. This proves the accretivity of $A_{p,\min}$.

Next we prove that $R(1 + A_{p,\min})$ (the range of $1 + A_{p,\min}$) is dense in $L^p(\mathbb{R}^N)$. Let $v \in L^{p'}(\mathbb{R}^N)$. Suppose that for every $\varphi \in C_0^\infty(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} v(\varphi + A\varphi)\,dx = 0. \tag{2.1}$$

Then it suffices to prove that $v = 0$ a.e. on $\mathbb{R}^N$. We may assume without loss of generality that v is real-valued. Since the diffusion matrix a satisfies the condition **(A1)**, we see from the elliptic regularity (see e.g., Agmon [**1**, Lemma 5.1]) that $v \in H^1_{\mathrm{loc}}(\mathbb{R}^N) \cap C(\mathbb{R}^N)$. Using **Key Identity** with $q = p$, $w = v$ and $\psi = \varphi$, we can rewrite (2.1) as the equality

$$\int_{\mathbb{R}^N}\left[\langle a\nabla v,\nabla\varphi\rangle + \frac{1}{p'}\langle F, v\nabla\varphi\rangle - \frac{1}{p}\langle F,\varphi\nabla v\rangle + \left(1+V-\frac{\mathrm{div}F}{p}\right)v\varphi\right]dx = 0. \tag{2.2}$$

By the density argument, (2.2) remains true even if $\varphi \in H^1(\mathbb{R}^N)$ has a compact support.

To manage the property $\|v\|_{L^{p'}(\mathbb{R}^N)} < \infty$, we approximate the duality map (with multiplying some constant) $|v|^{p'-2}v$ by the following procedure.

Case (i): $2 \le p' < \infty$ $(1 < p \le 2)$. We introduce a sequence of the cut-off functions $\{\zeta_n\}_n \subset W^{1,\infty}(\mathbb{R}^N)$ defined as

$$\zeta_n(x) := \begin{cases} 1 & \text{if } |x| \le R_0 \text{ or } \displaystyle\int_{R_0}^{|x|}\frac{1}{f(s)}\,ds < n,\\ 0 & \text{if } \displaystyle\int_{R_0}^{|x|}\frac{1}{f(s)}\,ds > n+1,\\ n+1-\displaystyle\int_{R_0}^{|x|}\frac{1}{f(s)}\,ds & \text{otherwise}\end{cases}$$

for $n \in \mathbb{N}$ and $x \in \mathbb{R}^N$. We see from the standard argument (see e.g., [**4**, Propositions IX.4 and IX.5]) that for every $n \in \mathbb{N}$, $\zeta_n^r |v|^{p'-2} v \in H^1(\mathbb{R}^N)$ has a compact support and

$$\nabla(\zeta_n^r |v|^{p'-2} v) = r\zeta_n^{r-1} |v|^{p'-2} v \nabla\zeta_n + (p'-1)\zeta_n^r |v|^{p'-2} \nabla v.$$

Setting $K_n := \operatorname{supp} \zeta_n$ for $n \in \mathbb{N}$ and choosing $\varphi = \zeta_n^r |v|^{p'-2} v$ in (2.2), we deduce that

$$\begin{aligned} (2.3) \qquad (p'-1) \int_{K_n} \zeta_n^r \langle a\nabla v, \nabla v \rangle |v|^{p'-2} \, dx + r \int_{K_n \setminus K_{n-1}} \zeta_n^{r-1} \langle a\nabla v, \nabla \zeta_n \rangle |v|^{p'-2} v \, dx \\ + \frac{r}{p'} \int_{K_n \setminus K_{n-1}} \zeta_n^{r-1} \langle F, \nabla \zeta_n \rangle |v|^{p'} \, dx + \int_{K_n} \zeta_n^r \left(1 + V - \frac{\operatorname{div} F}{p} \right) |v|^{p'} \, dx = 0. \end{aligned}$$

By the Cauchy-Schwarz and Young inequalities, we have

$$\begin{aligned} (2.4) \qquad \int_{K_n} \zeta_n^r \left(1 + V - \frac{\operatorname{div} F}{p} \right) |v|^{p'} \, dx \le \frac{r^2}{4(p'-1)} \int_{K_n \setminus K_{n-1}} \zeta_n^{r-2} \langle a\nabla\zeta_n, \nabla\zeta_n \rangle |v|^{p'} \, dx \\ - \frac{r}{p'} \int_{K_n \setminus K_{n-1}} \zeta_n^{r-1} \langle F, \nabla\zeta_n \rangle |v|^{p'} \, dx. \end{aligned}$$

On the other hand, note that

$$\nabla\zeta_n(x) = \begin{cases} \dfrac{-x}{|x| f(|x|)} & \text{if } x \in K_n \setminus K_{n-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Thus it follows from (1.5), (1.6) of the condition **(A2)** and Young's inequality that there exist constants $C_1, C_2 > 0$ such that for every $n \in \mathbb{N}$,

$$\begin{aligned} \zeta_n^{r-2} \langle a\nabla\zeta_n, \nabla\zeta_n \rangle &= \frac{\zeta_n^{r-2} \langle a(x)x, x \rangle}{|x|^2 f(|x|)^2} \\ &\le \alpha \zeta_n^{r-2} (1 + \Psi_p)^{1-\frac{2}{r}} \\ &\le C_1 + \frac{2(p'-1)}{r^2} \zeta_n^r (1 + \Psi_p), \\ -\zeta_n^{r-1} \langle F, \nabla\zeta_n \rangle &= \frac{\zeta_n^{r-1} \langle F(x), x \rangle}{|x| f(|x|)} \\ &\le \beta (1 + \Psi_p)^{1-\frac{1}{r}} \zeta_n^{r-1} \\ &\le C_2 + \frac{p'}{2r} \zeta_n^r (1 + \Psi_p). \end{aligned}$$

Therefore, combining (2.4), (1.7) and the above estimates, we have

$$\int_{K_n} \zeta_n^r (1 + \Psi_p) |v|^{p'} \, dx \le (C_1 + C_2) \int_{K_n \setminus K_{n-1}} |v|^{p'} \, dx + \int_{K_n \setminus K_{n-1}} \zeta_n^r (1 + \Psi_p) |v|^{p'} \, dx.$$

Consequently, we see that

$$\int_{K_{n-1}} |v|^{p'} \, dx \le (C_1 + C_2) \int_{K_n \setminus K_{n-1}} |v|^{p'} \, dx.$$

Finally, by Lebesgue's dominated convergence theorem we obtain

$$\int_{\mathbb{R}^N} |v|^{p'}\,dx \le 0.$$

This implies that $v = 0$ a.e. on $\mathbb{R}^N$, that is, $R(1 + A_{p,\min})$ is dense in $L^p(\mathbb{R}^N)$.

Case (ii): $1 < p' < 2$ $(2 < p < \infty)$. To verify (2.3) we introduce the function

$$G_\varepsilon(s) := (s^2 + \varepsilon)^{(p'-2)/2} s \quad \text{for } s \in \mathbb{R},\ \varepsilon > 0.$$

Then $\zeta_n^r G_\varepsilon(v) \in H^1(\mathbb{R}^N)$ has a compact support and

$$\nabla[\zeta_n^r G_\varepsilon(v)] = r\zeta_n^{r-1} v_\varepsilon^{p'-2} v \nabla\zeta_n + \zeta_n^r v_\varepsilon^{p'-4}\big((p'-1)v^2 + \varepsilon\big)\nabla v,$$

where $v_\varepsilon := \sqrt{v^2 + \varepsilon}$ for $\varepsilon > 0$. Taking $\varphi = \zeta_n^r G_\varepsilon(v)$ in (2.2), we derive that

$$\begin{aligned}
0 &= \int_{K_n} \zeta_n^r \langle a\nabla v, \nabla v\rangle v_\varepsilon^{p'-4}\big((p'-1)v^2 + \varepsilon\big)\,dx \\
&\quad + r\int_{K_n\setminus K_{n-1}} \zeta_n^{r-1}\langle a\nabla v, \nabla\zeta_n\rangle v_\varepsilon^{p'-2} v\,dx \\
&\quad + \frac{r}{p'}\int_{K_n\setminus K_{n-1}} \zeta_n^{r-1}\langle F, \nabla\zeta_n\rangle v_\varepsilon^{p'-2} v^2\,dx \\
&\quad - \frac{p'-2}{p'}\int_{K_n} \varepsilon\zeta_n^r\langle F, \nabla v\rangle v_\varepsilon^{p'-4} v\,dx \\
&\quad + \int_{K_n} \zeta_n^r\left(1 + V - \frac{\operatorname{div}F}{p}\right) v_\varepsilon^{p'-2} v^2\,dx \\
&= (\mathrm{I}_\varepsilon) + (\mathrm{II}_\varepsilon) + (\mathrm{III}_\varepsilon) + (\mathrm{IV}_\varepsilon) + (\mathrm{V}_\varepsilon).
\end{aligned}$$

The integrands of $(\mathrm{II}_\varepsilon)$–$(\mathrm{V}_\varepsilon)$ are respectively estimated as follows:

$$|\zeta_n^{r-1}\langle a\nabla v, \nabla\zeta_n\rangle v_\varepsilon^{p'-2} v| \le |\nabla v||a\nabla\zeta_n||v|^{p'-1}; \tag{2.5}$$

$$|\zeta_n^{r-1}\langle F, \nabla\zeta_n\rangle v_\varepsilon^{p'-2} v^2| \le |F||\nabla\zeta_n||v|^{p'}; \tag{2.6}$$

$$|\varepsilon\zeta_n^r\langle F, \nabla v\rangle v_\varepsilon^{p'-4} v| \le |F||\nabla v||v|^{p'-1}; \tag{2.7}$$

$$\left|\zeta_n^r\left(1 + V - \frac{\operatorname{div}F}{p}\right) v_\varepsilon^{p'-2} v^2\right| \le \left|1 + V - \frac{\operatorname{div}F}{p}\right| |v|^{p'}. \tag{2.8}$$

Letting $\varepsilon \to 0$, we have

$$\begin{aligned}
\zeta_n^{r-1}\langle a\nabla v, \nabla\zeta_n\rangle v_\varepsilon^{p'-2} v &\to \zeta_n^{r-1}\langle a\nabla v, \nabla\zeta_n\rangle |v|^{p'-2} v; \\
\zeta_n^{r-1}\langle F, \nabla\zeta_n\rangle v_\varepsilon^{p'-2} v^2 &\to \zeta_n^{r-1}\langle F, \nabla\zeta_n\rangle |v|^{p'}; \\
\varepsilon\zeta_n^r\langle F, \nabla v\rangle v_\varepsilon^{p'-4} v &\to 0; \\
\zeta_n^r\left(1 + V - \frac{\operatorname{div}F}{p}\right) v_\varepsilon^{p'-2} v^2 &\to \zeta_n^r\left(1 + V - \frac{\operatorname{div}F}{p}\right) |v|^{p'}.
\end{aligned}$$

Since all the functions on the right-hand side of (2.5)–(2.8) belong to $L^1(K_n)$ and converge as above, Lebesgue's dominated convergence theorem yields that

$$(\mathrm{II}_\varepsilon) \to r\int_{K_n\setminus K_{n-1}} \zeta_n^{r-1}\langle a\nabla v, \nabla\zeta_n\rangle |v|^{p'-2}v\,dx \qquad (\varepsilon\to 0); \tag{2.9}$$

$$(\mathrm{III}_\varepsilon) \to \frac{r}{p'}\int_{K_n\setminus K_{n-1}} \zeta_n^{r-1}\langle F, \nabla\zeta_n\rangle |v|^{p'}\,dx \qquad (\varepsilon\to 0); \tag{2.10}$$

$$(\mathrm{IV}_\varepsilon) \to 0 \qquad (\varepsilon\to 0); \tag{2.11}$$

$$(\mathrm{V}_\varepsilon) \to \int_{K_n} \zeta_n^{r}\left(1+V-\frac{\mathrm{div}F}{p}\right)|v|^{p'}\,dx \qquad (\varepsilon\to 0). \tag{2.12}$$

Moreover, the integrand of (I_ε) has a lower bound:

$$\zeta_n^r\langle a\nabla v, \nabla v\rangle v_\varepsilon^{p'-4}\big((p'-1)v^2+\varepsilon\big) \ge (p'-1)\zeta_n^r\langle a\nabla v, \nabla v\rangle v_\varepsilon^{p'-4}v^2. \tag{2.13}$$

The function on the right-hand side of (2.13) is positive and monotone increasing with respect to ε. Noting that (I_ε) is bounded by virtue of (2.9)–(2.12), we see that

$$(\mathrm{I}_\varepsilon) \to (p'-1)\int_{K_n} \zeta_n^r\langle a\nabla v, \nabla v\rangle |v|^{p'-2}\,dx \quad (\varepsilon\to 0).$$

In conclusion, we have (2.3). Proceeding similarly as in the case where $2\le p'<\infty$, we obtain $v=0$ a.e. on $\mathbb{R}^N$. This completes the proof of Theorem 1.1. □

In view of Theorem 1.1 we can complete

Proof of Theorem 1.2. We consider the minimal realization of B (introduced in (1.8)) in the dual space $L^{p'}(\mathbb{R}^N)$:

$$B_{p',\min}v = -\mathrm{div}(a\nabla v) - F\cdot\nabla v + (V-\mathrm{div}F)v, \quad D(B_{p',\min}) = C_0^\infty(\mathbb{R}^N).$$

Then by virtue of condition (1.9) the triplet $(\tilde a, \tilde F, \tilde V)$ defined as

$$\tilde a := a, \quad \tilde F := -F, \quad \tilde V := V - \mathrm{div}F$$

satisfies the assumption of Theorem 1.1 in $L^{p'}(\mathbb{R}^N)$. Therefore we see from the duality argument that $\tilde A_{p,\min}$ coincides with $(B_{p',\min})^*$. Namely, the domain of $\tilde A_{p,\min}$ is characterized as

$$\{u\in L^p(\mathbb{R}^N)\,;\ Au\in L^p(\mathbb{R}^N) \text{ in the sense of distribution}\}. \tag{2.14}$$

By virtue of Theorem 1.1, it suffices to show that $D(A_{p,\max})$ (defined in Section 1) coincides with (2.14). The density of $C_0^\infty(\mathbb{R}^N)$ and the Calderón-Zygmund estimate

$$\|u\|_{W^{2,p}(B_R)} \le C\big(\|Au\|_{L^p(B_{2R})} + \|u\|_{L^p(B_{2R})}\big)$$

(see e.g., Gilbarg-Trudinger [**7**, Theorem 9.11]) yield that the domain (2.14) is contained in $W^{2,p}_{\mathrm{loc}}(\mathbb{R}^N)$. We finish the proof of Theorem 1.2. □

Remark 2.1. If we regard the operator $A_{p,\max}$ as the distributional sense, then the characterization for the domain of $\tilde{A}_{p,\min}$ is already shown in the step of proving (2.14).

Finally, we prove Theorem 1.3 as an application of Theorem 1.2.

Proof of Theorem 1.3. By virtue of Theorem 1.2, it suffices to prove that $A_{p,\min}$ is sectorial of type $S(\omega)$ for some $\omega \geq 0$, that is, we shall show that for every $u \in C_0^\infty(\mathbb{R}^N)$,

$$\left|\mathrm{Im}\int_{\mathbb{R}^N}(Au)\overline{u}|u|^{p-2}\,dx\right| \leq \omega \mathrm{Re}\int_{\mathbb{R}^N}(Au)\overline{u}|u|^{p-2}\,dx. \tag{2.15}$$

Applying **Key Identity** with $w = |u|^{p-2}u$ and $\psi = u$, we have

$$\begin{aligned}\mathrm{Re}\int_{\mathbb{R}^N}(Au)\overline{u}|u|^{p-2}\,dx &= (p-1)\int_{\mathbb{R}^N}|u|^{p-4}\langle a\mathrm{Re}(\overline{u}\nabla u),\mathrm{Re}(\overline{u}\nabla u)\rangle\,dx\\&\quad+\int_{\mathbb{R}^N}|u|^{p-4}\langle a\mathrm{Im}(\overline{u}\nabla u),\mathrm{Im}(\overline{u}\nabla u)\rangle\,dx\\&\quad+\int_{\mathbb{R}^N}\left(V-\frac{\mathrm{div}F}{p}\right)|u|^p\,dx,\\\mathrm{Im}\int_{\mathbb{R}^N}(Au)\overline{u}|u|^{p-2}\,dx &= (p-2)\int_{\mathbb{R}^N}|u|^{p-4}\langle a\mathrm{Re}(\overline{u}\nabla u),\mathrm{Im}(\overline{u}\nabla u)\rangle\,dx\\&\quad+\int_{\mathbb{R}^N}|u|^{p-2}\langle\mathrm{Im}(\overline{u}\nabla u),F\rangle\,dx.\end{aligned}$$

Setting

$$\begin{aligned}X &:= \int_{\mathbb{R}^N}|u|^{p-4}\langle a\mathrm{Re}(\overline{u}\nabla u),\mathrm{Re}(\overline{u}\nabla u)\rangle\,dx,\\Y &:= \int_{\mathbb{R}^N}|u|^{p-4}\langle a\mathrm{Im}(\overline{u}\nabla u),\mathrm{Im}(\overline{u}\nabla u)\rangle\,dx,\\Z &:= \int_{\mathbb{R}^N}\Psi_p|u|^p\,dx\end{aligned}$$

and using (1.7), (1.10) and the Cauchy-Schwarz inequality yield that

$$\begin{aligned}\mathrm{Re}\int_{\mathbb{R}^N}(Au)\overline{u}|u|^{p-2}\,dx &\geq (p-1)X+Y+Z,\\\left|\mathrm{Im}\int_{\mathbb{R}^N}(Au)\overline{u}|u|^{p-2}\,dx\right| &\leq |p-2|\sqrt{XY}+\beta\sqrt{YZ}.\end{aligned}$$

Therefore setting $\omega \geq 0$ as

$$\omega^2 = c_{p,\beta}^2 = \frac{|p-2|^2}{4(p-1)}+\frac{\beta^2}{4},$$

we can obtain the sectorial estimate (2.15). Consequently, $A_{p,\max}$ is m-sectorial of type $S(c_{p,\beta})$. $\square$

3. Examples

3.1. Kato's example

Here we deal with the operator of the form

$$Au = -\mathrm{div}(a\nabla u) + Vu, \tag{3.1}$$

where a and V (and $F \equiv 0$) satisfy **(A1)** and

$$\frac{\langle a(x)x, x\rangle}{|x|^2} \le c_0(1+|x|)^\rho, \quad V(x) \ge c_1|x|^\ell \tag{3.2}$$

with $\rho, \ell, c_0, c_1 > 0$. Kato proved in [**10**, Appendix 2] that $A_{2,\max}$ is selfadjoint under the assumptions $\rho \le 2+\ell$ if $N = 1$ and $\rho < 2+\ell$ if $N \ge 2$. This means that the operator $-\mathrm{div}(a\nabla) + V$ with the unlimited growth diffusion a at infinity is selfadjoint in $L^2(\mathbb{R}^N)$ if the potential grows fast enough at infinity. Recently, the remaining case where $\rho = 2+\ell$ and $N \ge 2$ was solved by [**15**]. They also tries to generalize to the L^p-setting. However, when $p \neq 2$, they require an extra restriction to the derivatives of diffusion a.

So we consider the m-sectoriality of $A_{p,\max}$ in $L^p(\mathbb{R}^N)$ as an L^p-generalization of nonnegative selfadjointness in $L^2(\mathbb{R}^N)$ under the original conditions in [**10**].

Applying Theorem 1.3 to the coefficients of A defined as (3.1), we obtain the following theorem showing that **(A1)** and (3.2) are essential if $\rho < 2+\ell$. In fact, the critical case $\rho = 2+\ell$ is still open under **(A1)** and (3.2). It seems that some endpoint technique is required when one succeeds to deal with the critical case.

Theorem 3.1. *Let $1 < p < \infty$. Let $F \equiv 0$ and let a and V satisfy* **(A1)** *and* (3.2). *Assume that $0 < \rho < 2+\ell$. Then $A_{p,\max}$ is m-sectorial of type $S(c_p)$, where $c_p := |p-2|/(2\sqrt{p-1})$. Moreover, $C_0^\infty(\mathbb{R}^N)$ is a core for $A_{p,\max}$.*

Proof. If $0 < \rho \le 2$, then **(A1)** is automatically satisfied with $f(r) = r$ and $\Psi_p = 0$. On the other hand, if $2 \le \rho < \ell+2$, then in view of (1.7) with $F \equiv 0$ and $V \ge 0$, we can take the auxiliary function as $\Psi_p := c_1|x|^\ell$. Thus (1.5) in **(A2)** is rewritten as

$$\frac{\langle a(x)x, x\rangle}{|x|^2} \le \alpha(1+c_1|x|^\ell)^{1-\frac{2}{r}} f(|x|)^2 \quad \text{a.a. } x \in \mathbb{R}^N \setminus B_R.$$

Setting $R_0 := 1$, $f(s) := s$ and the exponent r as

$$r := 2 + \frac{2(\rho-2)}{\ell+2-\rho} \tag{3.3}$$

and using (3.2), we can compute that for every $x \in \mathbb{R}^N \setminus B_1$,

$$\begin{aligned}\frac{\langle a(x)x, x\rangle}{|x|^2} &\le c_0(1+|x|)^\rho \\ &\le 2^\rho c_0|x|^{\rho-2}|x|^2 \\ &\le 2^\rho c_0 c_1^{-\frac{\rho-2}{\ell}}(1+c_1|x|^\ell)^{1-\frac{2}{r}} f(|x|)^2,\end{aligned}$$

and hence (1.5) is satisfied. In both cases, noting again that $F \equiv 0$, we see that the coefficients of (3.1) satisfy (1.10) with $\beta = 0$. Therefore, the assumption of Theorem 1.3 is satisfied. □

3.2. Arendt-Metafune-Pallara's example

In [**2**, Section 6] (and [**3**, Section 3]) they introduced a typical and important example

$$Au = -u'' + x^3u' + c|x|^\gamma u \tag{3.4}$$

in $L^p(\mathbb{R})$ where $\gamma \geq 0$ and $c > 0$. In our notation the triplet (a, F, V) is determined as

$$a(x) := 1, \quad F(x) := x^3, \quad V(x) := c|x|^\gamma; \tag{3.5}$$

note that the triplet (a, F, V) in (3.5) automatically satisfies **(A1)** and (1.5) in **(A2)**. They precisely characterized the properties of A depending on the parameter γ and c as follows.

Proposition 3.1 ([**2**, Propositions 6.1, 6.3 and 6.4]). *Let $1 < p < \infty$ and let (a, F, V) be as in* (3.5). *Then one has the following assertions*:
(i) *If $\gamma > 2$ or $\gamma = 2$ and $c > 3/p$, then $-A_{p,\max}$ generates a C_0-semigroup $\{e^{-tA_{p,\max}}\}$ on $L^p(\mathbb{R})$ and $C_0^\infty(\mathbb{R}^N)$ is a core for $A_{p,\max}$.*
(ii) *If $\gamma \geq 6$, then $\{e^{-tA_{p,\max}}\}$ is an analytic semigroup on $L^p(\mathbb{R})$.*

We shall prove Proposition 3.1 only in the case where $\gamma > 2$ by our criterion instead of theirs. Note that the case where $\gamma = 2$ and $c > 3/p$ is an endpoint of our criterion $r = \infty$. Therefore we excluded this case.

Proof of Proposition 3.1. (i) It suffices to show that the triplet $(a, F, V+\lambda)$ is applicable to Theorem 1.2. First note again that the triplet $(a, F, V + \lambda)$ in (3.5) satisfies **(A1)** and (1.5) in **(A2)**. On the other hand, we see by the Young inequality that

$$\begin{aligned} V(x) - \frac{F'(x)}{p} &= c|x|^\gamma - \frac{3|x|^2}{p} \\ &\geq \frac{c}{2}|x|^\gamma - \left(1 - \frac{2}{\gamma}\right)\left(\frac{3}{p}\right)^{\frac{\gamma}{\gamma-2}}\left(\frac{c}{2}\right)^{-\frac{2}{\gamma-2}} =: \frac{c}{2}|x|^\gamma - \lambda_0. \end{aligned}$$

Thus we set $\lambda := \lambda_0 + c/2$ and choose the auxiliary function as

$$\Psi_p(x) := \frac{c}{2}(|x|^\gamma + 1).$$

Then we have (1.7) for $(a, F, V + \lambda)$:

$$(V(x) + \lambda) - \frac{F'(x)}{p} \geq \Psi_p(x).$$

Moreover, taking $f(s) := s$, we see that (1.9) is also satisfied: for every $x \in \mathbb{R} \setminus B_e$,

$$\begin{aligned} \frac{|F(x) \cdot x|}{|x|} &= |x|^3 \\ &\leq |x|^2 f(|x|) \\ &\leq \left(\frac{c}{2}\right)^{-\frac{2}{\gamma}} (1 + \Psi_p(x))^{\frac{2}{\gamma}} f(|x|). \end{aligned}$$

Consequently, taking

$$r := \max\left\{2, \frac{\gamma}{\gamma-2}\right\},$$

we conclude that the triplet $(a, F, V+\lambda)$ satisfies the assumption of Theorem 1.2.
(ii) If $\gamma \geq 6$, then we obtain that $(a, F, V+\lambda)$ satisfies (1.10): for $x \in \mathbb{R}$ and $\xi \in \mathbb{C}$,

$$\begin{aligned} |F(x)\cdot\xi| &\leq |x|^3|\xi| \\ &\leq (1+|x|^\gamma)^{\frac{3}{\gamma}}\langle a(x)\xi,\xi\rangle^{\frac{1}{2}} \\ &\leq \left(\frac{C}{2}\right)^{-\frac{3}{\gamma}} \Psi_p(x)^{\frac{1}{2}}\langle a(x)\xi,\xi\rangle^{\frac{1}{2}}. \end{aligned}$$

Therefore Theorem 1.3 yields that $\{e^{-\lambda t}e^{-tA_{p,\max}}\}$ can be extended to an analytic contraction semigroup on $L^p(\mathbb{R})$. This completes the proof. □

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